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LINEAR THEORY OF MAGNETIZED, VISCOUS, SELF-GRAVITATING GAS DISKS

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ABSTRACT

Motivated by the idea that gravitational instability in the gaseous disk of the galaxy may form giant molecular clouds, we consider the linear theory of viscous, magnetized, self-gravitating gas disks. The effective viscosity of the interstellar medium in the solar neighborhood is of order 0.8 km s^{-1} kpc. Viscous, self-gravitating disks are known to be unstable, and for the solar neighborhood the growth rate of the viscous instability is $\simeq 10^8$ yr^{-1}. After defining a quantitative measure of the nonaxisymmetric responsiveness of the disk R, we show that R declines as viscosity is increased. Magnetized, inviscid self-gravitating disks in solid-body rotation are also known to be subject to an instability that is similar to the viscous instability. We show that this instability is not present in differentially rotating disks, and that magnetic fields also tend to reduce nonaxisymmetric responsiveness.

Subject headings: galaxies: ISM — galaxies: kinematics and dynamics — galaxies: structure — ISM: magnetic fields

1. INTRODUCTION

It has been proposed that individual giant molecular clouds (GMCs), or groups of GMCs, form by gravitational instability in the gas layers of disk galaxies (Elmegreen 1979, 1987, 1989, 1991; Cowie 1981; Jog & Solomon 1984; Balbus & Cowie 1985; see also the discussion of Goldreich & Lynden-Bell 1965, hereafter GL, and the review of Elmegreen 1992). This idea finds observational support in the work of Kennicutt (1989), who showed that in the star-forming regions of disk galaxies the gas layer is close to gravitational instability, according to the criterion of Toomre (1964). While there is now a well-developed linear theory of disks that incorporates many of the important physical effects, the purpose of this paper is to incorporate two pieces of physics whose qualitative effects have not been fully appreciated: magnetic fields and viscosity. We consider these effects together because both may be important for the formation of GMCs, and their dynamics are closely connected.

The galactic disks familiar to the observer are composed of a stellar component with several populations, and a multiphase gaseous component. At the outset we shall adopt an isothermal fluid model for the interstellar medium (ISM), identifying the velocity dispersion of clouds in the ISM with the sound speed in our isothermal model. The appropriateness of this model has been extensively discussed and criticized elsewhere (see, e.g., Levinson & Roberts 1981; Cowie 1980; Scalo & Struck-Marcell 1984; Tomisaka 1987; Elmegreen 1989). Our view is that this model represents the pressure tensor of the ISM in a crude average sense, and thus can be used to understand the interplay between pressure, rotation, and self-gravity in disks.

We shall also neglect the gravitational interaction of the stars and gas. Considered separately, the stellar and gaseous components each have a characteristic wavelength λ_c where that component is most responsive to perturbations. In the thin-disk limit for a component with surface density Σ, and Toomre's parameter $Q \equiv c_s \kappa / G\Sigma$ (here κ is the epicyclic frequency and c_s is the velocity dispersion), we have $\lambda_c \simeq 2\pi^2 Q^2 G\Sigma/\kappa^2$. In the solar neighborhood $\Sigma \simeq 41\ M_\odot\ \mathrm{pc}^{-2}$ for the stars, $\Sigma \simeq 13\ M_\odot\ \mathrm{pc}^{-2}$ for the gas (Gould 1990; Kuijken & Gilmore 1989), $\kappa \simeq 36$ km s^{-1} kpc^{-1}, the radial velocity dispersion of the stars is $\simeq 45$ km s^{-1} (Binney & Tremaine 1987), and the velocity dispersion of the gas is $\simeq 6$ km s^{-1} (Crovisier 1978). Then for the gaseous component $Q \simeq 1.2$, while for the stellar component, $Q \simeq 2.8$ (these estimates are highly uncertain), so $\lambda_c(\mathrm{stars})/\lambda_c(\mathrm{gas}) \simeq 17$. Thus the characteristic scales for the gas and stars are widely separated. In this limit the gas and stars are close to being decoupled gravitationally, although at λ_c(gas), for example, the stellar background can still exert a substantial destabilizing influence on the gas (see, e.g., Jog & Solomon 1984). In order to focus on the effects of magnetic fields and viscosity, however, we shall consider only the gaseous component and neglect the effect of the stellar background.

As a further approximation we shall assume the gas layer is infinitesimally thin. This approximation may be evaluated by noting that the scale height z_h of a self-gravitating sheet is $\pi Q^2 G\Sigma/\kappa^2$, hence $z_h/\lambda_c = 1/2\pi$, so the approximation is fair. The correction factor for finite thickness can be calculated exactly for a self-gravitating gaseous sheet (see GL). Finite thickness tends to make the self-gravitating sheet more stable than its thin counterpart, lowering the velocity dispersion of the marginally stable disk by 30% at fixed surface density.

The plan for the remainder of the paper is as follows. In § 2 we develop the linearized equations for evolution of a viscid, magnetized disk. In § 3 we evaluate the effect of finite viscosity, and in § 4 we consider the effect of magnetic fields. Section 5 contains a summary and conclusions.

2. LINEAR THEORY OF VISCOUS, MAGNETIZED DISKS

The equations governing the evolution of a thin gas layer are the two-dimensional continuity equation

$$\frac{D \ln \Sigma}{Dt} = -\nabla \cdot \boldsymbol{V}, \tag{1}$$

the Euler equations

$$\frac{D\boldsymbol{V}}{Dt} = \boldsymbol{f}, \tag{2}$$

and the Poisson equation

$$\nabla^2 \Phi = 4\pi G \Sigma\, \delta(z). \tag{3}$$

Here D/Dt is the connective derivative, V is the velocity measured in an inertial frame, and

$$f \equiv -\nabla(\Phi + \Phi_0 + \Phi_i + c_s^2 \ln \Sigma) + \frac{(B \cdot \nabla)B}{4\pi\rho} - \frac{\nabla B^2}{8\pi\rho} + \nu\nabla^2 V + \left(\frac{\nu}{3}\right)\nabla(\nabla \cdot V) . \quad (4)$$

Here Φ_0 is a fixed axisymmetric potential, Φ is the rest of the potential associated with the disk, and Φ_i is any imposed external potential. The density $\rho = \Sigma/H$, where H is a (constant) scale height. The viscosity ν is assumed to be independent of density and position.

The magnetic field is assumed to lie entirely in the plane of the disk. It can then be obtained from a vector potential $A_z \hat{z}$, where $\hat{z}$ is normal to the plane of the disk. The evolution of A_z is given by

$$\frac{DA_z}{Dt} = 0 , \quad (5)$$

so that the vector potential is carried with the fluid.

Equations (1) and (2) can be combined by taking the divergence of equation (2) and D/Dt of equation (1) to obtain

$$\frac{D^2 \ln \Sigma}{Dt^2} = -\nabla \cdot f + \left(\nabla \cdot \frac{DV}{Dt} - \frac{D}{Dt} \nabla \cdot V\right) . \quad (6)$$

The term in parentheses can be rewritten in terms of the rate-of-strain tensor,

$$e_{ij} \equiv \tfrac{1}{2}(\partial_j V_i + \partial_i V_j) \quad (7)$$

and the "potential vorticity"

$$\xi \equiv \left(\frac{\nabla \times V}{\Sigma}\right) . \quad (8)$$

Since $\nabla \times V$ has only one component, normal to the plane of the disk, we regard it as a scalar. The result is Hunter's equation:

$$\frac{D^2 \ln \Sigma}{Dt^2} = -\nabla \cdot f + e_{ij} e_{ij} - \frac{1}{2} \xi^2 \Sigma^2 \quad (9)$$

(Hunter 1964; GL).

Now consider a fiducial point in the disk that corotates with the gas at, in standard cylindrical coordinates, $R = R_0$ and $\phi = \phi_0 + \Omega_0 t$ (we assume $\Omega_0 < 0$). Erect a local Cartesian coordinate frame $x. y$ with $x \equiv R - R_0$ and $y \equiv R_0(\phi - \phi_0 - \Omega_0 t)$. Velocities measured in this frame, which rotates with frequency Ω_0, are denoted v. Assuming that departures from the fiducial point are small, $|\delta x|/R_0 \sim \epsilon \ll 1$, one can expand the equations of motion (eq. [2]), or Hunter's equation, through first order in ϵ to obtain a *local model* of the disk.

In the local model the equilibrium consists of a constant surface density Σ, constant sound speed c_s, and rectilinear shear $v = 2Ax\hat{y}$, where $A \equiv (r/2)(d\Omega/dr) > 0$ is one of Oort's constants. This model neglects the curvature of the orbits, but it includes the essential local elements of disk dynamics: shear, tides, and Coriolis force.

Notice that in the absence of viscosity and magnetic fields, the local model is fully characterized by a single dimensionless parameter: Toomre's Q. It is then convenient to scale all other physical quantities in terms of the characteristic length $G\Sigma/\kappa^2$ ($\simeq$45 pc in the solar neighborhood), characteristic time $1/\kappa$ ($\simeq 2.7 \times 10^7$ yr), and characteristic mass $G^2\Sigma^3/\kappa^4$ ($\simeq 2.5 \times 10^4\, M_\odot$). These can be combined to form a characteristic velocity $G\Sigma/\kappa$ ($\simeq$1.6 km s^{-1}) and a characteristic viscosity $G^2\Sigma^2/\kappa^3$ ($\simeq$0.07 km s^{-1} kpc).

In the local model the potential vorticity is

$$\xi = \frac{\nabla \times v + 2\Omega}{\Sigma} . \quad (10)$$

For the equilibrium local model, $\xi_0 = 2(\Omega + A)/\Sigma \equiv 2B/\Sigma$, where B is one of Oort's constants. In the absence of viscosity and magnetic fields, $D\xi/Dt = 0$ in an isothermal fluid. Otherwise,

$$\frac{D\xi}{Dt} = \frac{\nabla \times f}{\Sigma} \quad (11)$$

(e.g. Pedlosky 1987). When one transforms from an inertial frame to a rotating frame, all the terms in Hunter's equation are unchanged, except the potential vorticity.

Now consider the small perturbations to the equilibrium local model. We set $\Sigma = \Sigma_0 + \delta\Sigma$, $v \to 2Ax\hat{y} + \delta v$, $\xi = 2B/\Sigma + \delta\xi$, $A_z = A_{z0} + \delta A_z$. The linearized form of Hunter's equation, the continuity equation, the induction equation, and the potential vorticity equation are, respectively,

$$\frac{D^2}{Dt^2}\frac{\delta\Sigma}{\Sigma_0} = -\nabla \cdot \delta f + 2A\left(\frac{\partial\, \delta v_x}{\partial y} + \frac{\partial\, \delta v_y}{\partial x}\right) - 2B(\xi_0\, \delta\Sigma + \Sigma_0\, \delta\xi) , \quad (12a)$$

$$\frac{D}{Dt}\frac{\delta\Sigma}{\Sigma_0} = -\nabla \cdot \delta v , \quad (12b)$$

$$\frac{D}{Dt}\, \delta A_z = -(\delta v \cdot \nabla)A_z , \quad (12c)$$

$$\frac{D}{Dt}\, \delta\xi = \frac{\nabla \times \delta f}{\Sigma_0} , \quad (12d)$$

where

$$\delta f = -\nabla(\delta\phi + c_s^2\, \delta\Sigma/\Sigma_0) + \frac{B \times \nabla^2 \delta A_z \hat{z}}{4\pi\rho} + \nu\nabla^2\, \delta v + (4\nu/3)\nabla(\nabla \cdot \delta v) . \quad (13)$$

Here $D/Dt = \partial/\partial t + 2Ax\, \partial/\partial y$ is the derivative moving with the unperturbed flow.

A convenient way of analyzing the linearized equations of motion in a shear flow, invented by Julian & Toomre (1966; JT) and GL, is to consider the evolution of perturbations of the form $\exp(i\boldsymbol{k} \cdot \boldsymbol{x})$, where $k_x = -2Ak_y t + k_{x0}$, and $k_y =$ constant. We shall refer to an individual Fourier component of this form as a "shearing wave." The representation of fluid variables in terms of shearing waves is complete (any perturbation can be decomposed into a sum of shearing waves), and it has the advantage that it eliminates explicit spatial dependences from the perturbation equations, since $(D/Dt)[f(t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x})] = (\partial f/\partial t) \exp(i\boldsymbol{k} \cdot \boldsymbol{x})$. If $k_y > 0$, and if the radial wavenumber is initially large and positive, then as it evolves it grows smaller, passes through zero, and then becomes large and negative. We shall decompose the perturbation into shearing waves and then use the governing equations to write ordinary differential equations for the evolution of the wave amplitudes.

From now on we denote the amplitudes of the shearing waves by $\delta\Sigma$, δA_z, etc., and drop the subscript zero on unperturbed variables. For each shearing wave, the solution to the Poisson equation is

$$\delta\phi = -\frac{2\pi G}{|\boldsymbol{k}|}\,\delta\Sigma\,. \tag{14}$$

One can then use the definition of potential vorticity and the continuity equation to eliminate $\delta\boldsymbol{v}$. Hunter's equation, the potential vorticity equation, and the induction equation, respectively, then reduce to

$$\begin{aligned}\frac{d^2\,\delta\Sigma}{dt^2} &+ \delta\Sigma\left(\kappa^2 - 2\pi G\Sigma|\boldsymbol{k}| + c_s^2|\boldsymbol{k}|^2 + 8AB\,\frac{k_y^2}{|\boldsymbol{k}|^2}\right)\\ &+ \frac{d\,\delta\Sigma}{dt}\left[4A\,\frac{k_y k_x}{|\boldsymbol{k}|^2} + |\boldsymbol{k}|^2\left(\frac{4\nu}{3}\right)\right]\\ &+ \delta\xi\Sigma^2\left(2\Omega + 4A\,\frac{k_y^2}{|\boldsymbol{k}|^2}\right)\\ &- i\,\delta A_z|\boldsymbol{k}|^2\Sigma\,\frac{(\boldsymbol{k}\times\boldsymbol{B})}{4\pi\rho} = \Sigma|\boldsymbol{k}|^2\,\delta\phi_i\,,\end{aligned} \tag{15a}$$

$$\frac{d\,\delta\xi}{dt} - i\,\delta A_z|\boldsymbol{k}|^2\,\frac{(\boldsymbol{k}\cdot\boldsymbol{B})}{4\pi\rho\Sigma} + \nu|\boldsymbol{k}|^2\left(\delta\xi + \xi_0\,\frac{\delta\Sigma}{\Sigma}\right) = 0\,, \tag{15b}$$

$$\begin{aligned}\frac{d\,\delta A_z}{dt} &- \frac{i}{|\boldsymbol{k}|^2\Sigma}\\ &\times\left(\frac{d\,\delta\Sigma}{dt}(\boldsymbol{k}\cdot\boldsymbol{B}) + \delta\xi\Sigma^2(\boldsymbol{k}\cdot\boldsymbol{B}) + \delta\Sigma 2B(\boldsymbol{k}\cdot\boldsymbol{B})\right) = 0\,.\end{aligned} \tag{15c}$$

Here $\delta\phi_i$ is the Fourier coefficient of any external potential, and we have abbreviated $(\boldsymbol{k}\cdot\boldsymbol{B})\cdot\hat{z}$ as $\boldsymbol{k}\times\boldsymbol{B}$, since the cross product has only one component.

One can now evolve an arbitrary linear amplitude disturbance in a viscous self-gravitating disk by Fourier decomposing it, evolving the Fourier coefficients with equations (15a)–(15c), and then reassembling the Fourier components at the desired time.

Parts of equations (15a)–(15c) have been considered by other authors. Hunter & Horak (1983) considered the evolution of shearing waves in a viscous disk. The equations they wrote down are nearly equivalent to the viscous part of our equations (15a)–(15c). Similarly, shearing waves in magnetized disks have been considered by Elmegreen (1987), who wrote down a set of equations equivalent but not identical to the magnetic parts of equations (15a)–(15c).

3. VISCOUS, UNMAGNETIZED DISKS

For clarity, we have separated our discussions of viscosity and magnetic fields. This section considers disks with viscosity but no fields, while the next section considers disks with magnetic fields but no viscosity.

What is the effective viscosity of the local ISM? In an ordinary gas, viscosity ν is approximately the mean free path l multiplied by the sound speed c_s. If we regard the ISM as a "fluid" composed of dense clouds with mean column density $\langle N\rangle$, then the mean free path is $l = \langle N\rangle/\langle n\rangle$, where $\langle n\rangle$ is the mean hydrogen number density of the ISM.[1] For a typical diffuse cloud with $A_V = 0.2$, $\langle N\rangle \simeq 4\times10^{20}$ cm^{-2} (Spitzer 1978), while $\langle n\rangle \simeq 1$ cm^{-3}, so $l = 130$ pc. The velocity dispersion of clouds is $c_s \simeq 6$ km s^{-1} (Crovisier 1978). If the mean free path becomes as large as the cloud epicyclic amplitude c_s/κ, then the viscosity is modified (see, e.g., Goldreich & Tremaine 1978; also Yuan 1984, and Steiman-Cameron & Durisen 1988). Here $c_s/\kappa \simeq 170$ pc, so this effect is not a large correction. The viscosity is then $\nu \simeq lc_s \simeq 0.8$ km s^{-1} kpc, comparable to the estimates of Yuan (1984) and Yuan & Cheng (1991).

How does the viscosity scale with Galactocentric radius? For a self-gravitating sheet, the central density is $\rho_0 = \kappa^2/2\pi Q^2 G$. Thus if the mean cloud column density, cloud velocity dispersion, and Q are constant with radius, $\nu \sim 1/\kappa^2$. In comparison to the characteristic viscosity $\nu_c \equiv G^2\Sigma^2/\kappa^3$, however, we have $\nu/\nu_c = \langle N\rangle 2\pi^2 Q^3\mu/\Sigma$, where μ is the mean molecular weight. Then if Q, $\langle N\rangle$, and c_s are constant with radius, $\nu/\nu_c \sim 1/\Sigma$, and the viscosity again decreases inward.

3.1. *Axisymmetric Modes*

The dispersion relation for the axisymmetric modes of a viscous, unmagnetized disk can be recovered from equations (15a)–(15c) by setting $k_y = 0$, $\boldsymbol{B} = 0$, and assuming the perturbed quantities scale as e^{st}. Then the waves do not shear, i.e., k_x = constant. The dispersion relation is

$$\begin{aligned}&[s^2 + \kappa^2 - 2\pi G\Sigma|k_x| + c_s^2 k_x^2 + s(4\nu/3)k_x^2]\\ &\qquad\times(s + k_x^2\nu) - \nu k_x^2\kappa^2 = 0\,.\end{aligned} \tag{16}$$

Two of the three branches of this dispersion relation are damped versions of the usual density waves. In the limit of small viscosity, the damping rate is

$$\mathrm{Re}\,(s) \simeq -\frac{2}{3}\,\nu k_x^2\left(1 + \frac{3\kappa^2}{4\omega^2}\right), \tag{17}$$

where $\omega^2 \equiv [\mathrm{Im}\,(s)]^2 \simeq \kappa^2 - 2\pi G\Sigma|k_x| + c_s^2 k_x^2$ is the dispersion relation for the usual density waves.

Consider the dispersion relation for gaseous density waves in the solar neighborhood. Using the solar neighborhood parameters given in § 1, the estimated viscosity is $11G^2\Sigma^2/\kappa^3$, and $Q = 1.2$. Turning off the viscosity, the minimum of ω^2 (where the disk is most responsive to perturbations) lies at $k = \pi G\Sigma/c_s^2 = 4.9$ kpc^{-1}, where $\omega = \pm 0.55\kappa$. Turning on the viscosity, the minimum of ω^2 lies at slightly lower wavenumber, but now $\omega = (\pm 0.69 - i0.8)\kappa$. Evidently the viscosity is large enough to overdamp density waves.

In the limit of small viscosity, the third branch of equation (16) becomes

$$s \simeq \frac{\nu k_x^2(2\pi G\Sigma|k_x| - c_s^2 k_x^2)}{\kappa^2 - 2\pi G\Sigma|k_x| + c_s^2 k_x^2}\,. \tag{18}$$

The growth rate is positive for small $|k_x|$, so viscous, self-gravitating disks are unstable. This recovers the result of Lynden-Bell & Pringle (1974). For our standard solar

[1] This implies that the viscosity is inversely proportional to density, contrary to what we have assumed in writing eq. (4). For the local model, however, the linearized equations do not change even if ν depends on the density, since the equilibrium shear stress is constant in space.

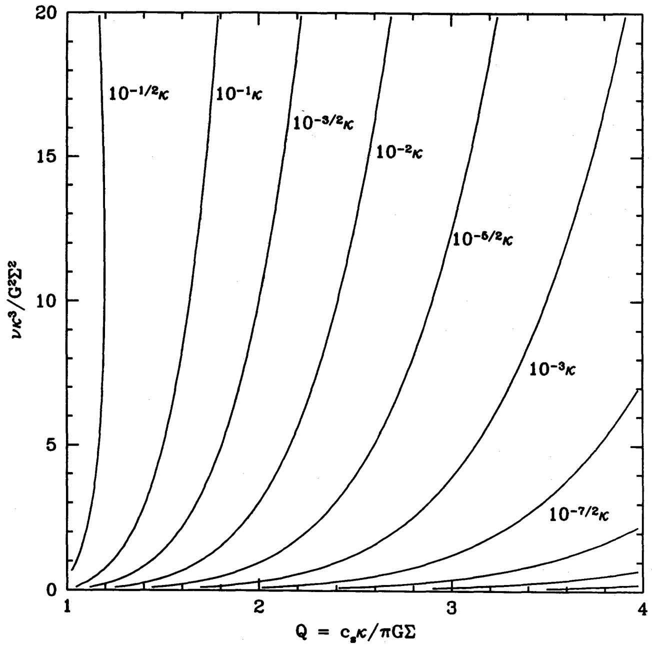

FIG. 1.—Maximum (over k_x) growth rate of the viscous instability of self-gravitating disks, as a function of viscosity ν and Toomre's Q. The contours are at logarithmic intervals of $10^{1/2}$.

neighborhood parameters, the maximum over k_x of the growth rate is 0.32κ. Figure 1 shows the maximum of the growth rate over k_x for various values of Q and ν. Evidently for $\nu \gtrsim G^2\Sigma^2/\kappa^3$ and $Q \lesssim 2$ the growth rate is a substantial fraction of κ.

The physical origin of the viscous instability lies in the connection between rotational support and potential vorticity noted by Lynden-Bell (1966) and Hunter (1964). This relation is most clearly seen in Hunter's equation (eq. [9]):

$$\frac{D^2 \ln \Sigma}{Dt^2} = -\nabla \cdot f + e_{ij} e_{ij} - \frac{1}{2} \xi^2 \Sigma^2 . \tag{19}$$

Recall that D/Dt is the convective derivative, f is the acceleration on each fluid element, and $e_{ij} \equiv (\partial_j V_i + \partial_i V_j)/2$ is the rate-of-strain tensor. Negative-definite terms on the right-hand side of equation (19) are stabilizing in the sense that they tend to decrease fluctuations in the surface density. The potential vorticity enters as a negative-definite term on the right, and is the only place where rotation appears in Hunter's equation. Thus rotation supports the local model against gravitational instability through the potential vorticity and only through the potential vorticity. Processes that cause the potential vorticity to evolve, such as viscosity, or magnetic fields, can induce instability by compromising rotational support.

3.2. *Nonaxisymmetric Response of Viscous Disk*

GMCs form a concentration of mass in the disk on a timescale of a few times 10^7 yr, and this concentration of mass causes a response in the surrounding material. In the case of a point mass in a stellar disk, JT have shown that the mass involved in this response, or wake, can greatly exceed the mass of the perturber if the disk is close to gravitational instability. The presence of the wake will influence the amount of mass incorporated into the GMC as it forms and the rate at which that mass accumulates. It is therefore of interest to understand how viscosity affects the response of the disk.

First we shall define a quantitative measure of "responsiveness." Because the formation of the wake around a GMC-like perturber is a predominantly nonaxisymmetric phenomenon, and because nonaxisymmetric shearing waves have a complicated time evolution, one must solve a representative initial-value problem to characterize the response.

The initial-value problem we have chosen is the response to a localized concentration of mass that is suddenly introduced at $t = 0$. A convenient functional form for this perturbation is

$$\delta\Sigma_p = \frac{aM_p}{2\pi(a^2 + r^2)^{3/2}} , \tag{20}$$

which has total mass M_p (arbitrary, but set to $1G^2\Sigma^3/\kappa^4 \simeq 2.5 \times 10^4\ M_\odot$ in this case) and characteristic scale a, which we have set to $2G\Sigma/\kappa^2 \simeq 90$ pc. We also assume the rotation curve is flat, so that $A = -\Omega/2$. A typical response of an inviscid disk to this perturbing surface density after a time $4/\kappa \simeq 1.1 \times 10^8$ yr is shown in Figure 2*a*.

We now define the *responsiveness* R as

$$R(\nu, Q; t, a) \equiv \frac{1}{2M_p} \int d^2x\, |\delta\Sigma(x, t)| , \tag{21}$$

where the factor of $\frac{1}{2}$ prevents double counting of the responding fluid elements.

How do we expect the responsiveness to depend on ν? Viscosity alters the evolution of shearing waves in two ways. First, it causes the potential vorticity of the disk to evolve. Second, it has a damping effect, as can be seen in equations (15a)–(15c), where it multiplies a term proportional to $d\,\delta\Sigma/dt$. Which effect dominates? On a long enough timescale, the evolution of potential vorticity will dominate, since the purely unstable viscous modes will grow exponentially, whereas individual nonaxisymmetric shearing waves eventually decay away. We are motivated by our interest in GMC formation, however, to consider the response of the disk on timescales of a few times $1/\kappa$, comparable to the timescale for GMC formation. Comparison of Figure 2*a* ($\nu = 0$) and Figure 2*b* ($\nu = 11G^2\Sigma^2/\kappa^3$) shows that viscosity can indeed inhibit the responsiveness of the disk.

More generally we have evaluated R for a variety of ν and Q. The results are shown in Figure 3. The response declines as ν increases for all values of Q; thus, viscous damping dominates over potential vorticity evolution. This result is not sensitive to our choice of t_f, and holds at least for $t_f < 12/\kappa$. over which period R grows approximately as t^2. For $Q = 1$ in an inviscid disk, the perturber provokes a response that contains 21 times its own mass. This confirms the extreme sensitivity of self-gravitating disks noted in the context of thin stellar disks by JT. The response is gradually reduced as ν is increased, and begins to drop significantly for $\nu \gtrsim G^2\Sigma^2/\kappa^3$. Thus in the linear regime, and on timescales of a few orbital periods, viscosity tends to reduce the responsiveness of the disk.

4. MAGNETIZED, INVISCID DISKS

Magnetic fields and viscosity share the property that they cause potential vorticity to evolve. In viscous disks this

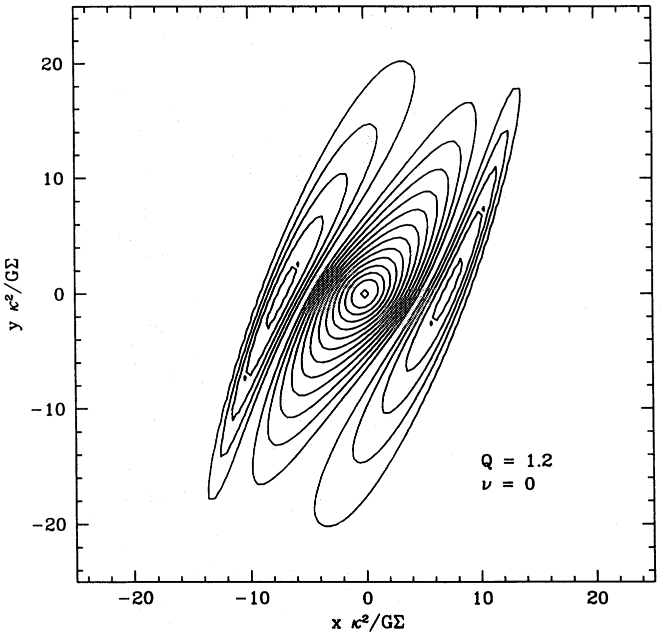

FIG. 2a

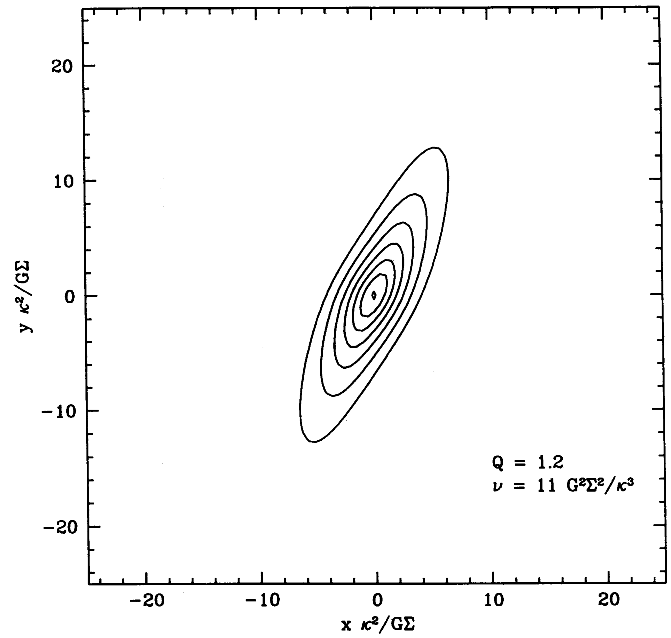

FIG. 2b

FIG. 2.—Typical response of a viscous, unmagnetized disk to a perturbing mass placed at $x = y = 0$. The disk has $Q = 1.2$, the perturber is introduced at $t = 0$, and the response is shown at $t = 4/\kappa$. The contours are at intervals of $\delta\Sigma/\Sigma = 0.01$. Dashed contours represent negative surface density perturbations. (*a*) $\nu = 0$. (*b*) $\nu = 11G^2\Sigma^2/\kappa^3$ (solar neighborhood value).

compromises the rotational support of the disk, leading to instability. Magnetized disks in solid-body rotation were considered by Lynden-Bell (1966), who showed that they are unstable at long wavelengths. Elmegreen has calculated the evolution of nonaxisymmetric waves in a differentially rotating thin disk (Elmegreen 1987; Elmegreen 1991 also considers nonaxisymmetric waves in a disk of finite thickness, approximately incorporating the Parker instability). Elmegreen showed that under some circumstances growth of nonaxisymmetric waves is enhanced, and in others it is inhibited by the presence of the magnetic field. Our work below builds on Elmegreen's results.

First, however, what is the magnetic field in the solar neighborhood? Pulsar rotation measures and many other lines of evidence tell us that the Galaxy has a magnetic field with a significant ordered components on scales $\gtrsim 100$ pc. Locally, the ordered component of the magnetic field is $\approx 1.6\ \mu$G and is oriented in the azimuthal direction (Rand & Kulkarni 1989). The mean density of the ISM is $\simeq 3 \times 10^{-24}$ g cm^{-3}, implying $V_{A\phi} \simeq 2.6$ km s^{-1}, and $V_{Ar} \approx 0$. This should be compared to the local velocity dispersion of the ISM, $\simeq 6$ km s^{-1}, so $V_{A\phi}/c_s \simeq 0.4$.

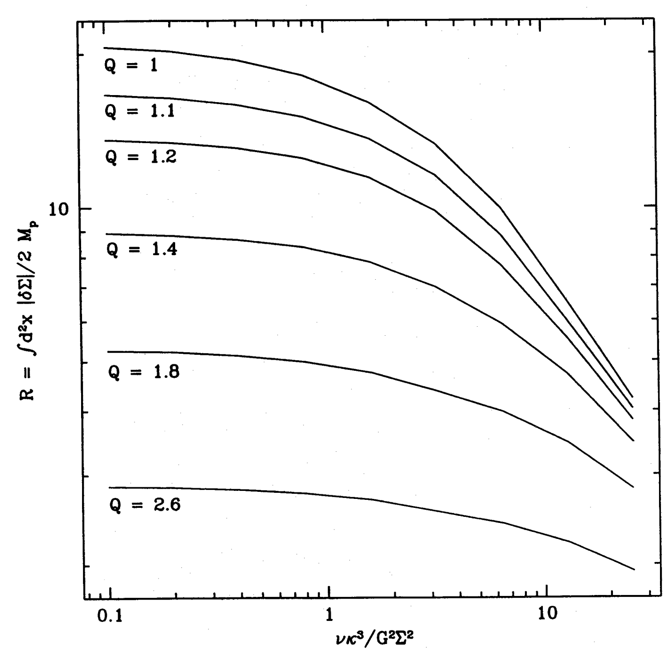

FIG. 3.—Responsiveness $R \equiv \int d^2x\,|\delta\Sigma|/2M_p$ (see § 3.2) of a viscous, unmagnetized disk as a function of viscosity ν and Toomre's Q.

4.1. *Axisymmetric Modes*

The dispersion relation for axisymmetric modes in a magnetized, inviscid disk can be obtained from equations (15a)–(15c) by setting $k_y = 0$, $\nu = 0$, and letting the perturbation scale as e^{st}. In terms of the Alfvén velocity $\boldsymbol{V}_A \equiv \boldsymbol{B}/(4\pi\rho)^{1/2}$, the dispersion relation is

$$s^4 + s^2[\kappa^2 - 2\pi G\Sigma|k_x| + c_s^2 k_x^2 + (\boldsymbol{k} \times \boldsymbol{V}_A)^2 + (\boldsymbol{k} \cdot \boldsymbol{V}_A)^2] + 2As(\boldsymbol{k} \times \boldsymbol{V}_A)(\boldsymbol{k} \cdot \boldsymbol{V}_A) + (\boldsymbol{k} \cdot \boldsymbol{V}_A)^2 \times (c_s^2 k_x^2 - 2\pi 2\pi G\Sigma|k_x|) = 0\ . \tag{22}$$

If $A \neq 0$ and $V_{Ax} \neq 0$, then the shear will tend to wrap up the unperturbed field and $\boldsymbol{k} \times \boldsymbol{V}_A$ will be time dependent. Equation (22) is valid in the WKB sense only in the limit that $\boldsymbol{k} \times \boldsymbol{V}_A$ is slowly varying.

To recover Lynden-Bell's (1966) instability, we set the shear rate $2A = 0$. The growth rate of the unstable modes can then be obtained analytically in the limit $V_A \ll c_s$. The result is

$$s^2 \simeq \frac{(\boldsymbol{k} \cdot \boldsymbol{V}_A)^2(2\pi G\Sigma|k_x| - c_s^2 k_x^2)}{\kappa^2 - 2\pi G\Sigma|k_x| + (c_s^2 + V_A^2)k_x^2}\ . \tag{23}$$

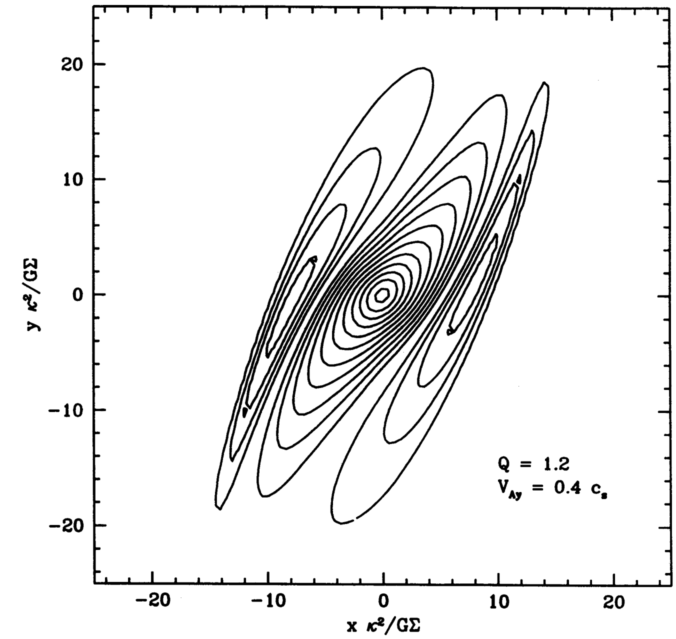

FIG. 4a

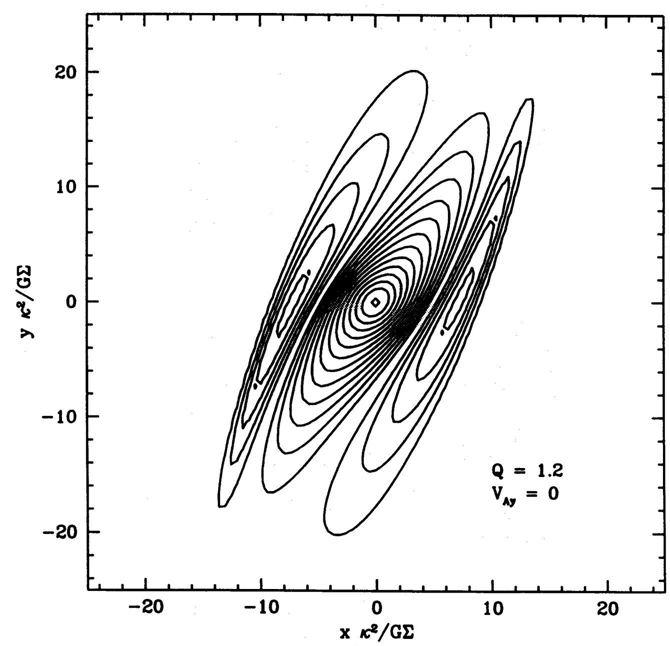

FIG. 4b

FIG. 4.—Typical response of a magnetized, inviscid disk to a perturbing mass placed at $x = y = 0$. The disk has $Q = 1.2$, the perturber is introduced at $t = 0$, and the response is shown at $t = 4/\kappa$. The contours are at intervals of $\delta\Sigma/\Sigma = 0.01$. Dashed contours represent negative surface density perturbations. (*a*) $V_{Ax} = 0$, $V_{Ay} = 0.4c_s$ (solar neighborhood values). (*b*) $V_{Ax} = V_{Ay} = 0$.

Notice the remarkable similarity between equation (23) and equation (18), the dispersion relation for the viscous instability. This reflects the similarity in the underlying physics causing the instabilities: in both cases potential vorticity is no longer conserved, compromising rotational support.

Now consider a differentially rotating disk ($A \neq 0$). Suppose the magnetic field is purely radial at $t = 0$. Then $\boldsymbol{k} \times \boldsymbol{V}_A = 0$ and $\boldsymbol{k} \cdot \boldsymbol{V}_A = k_x V_{Ax}$. Because differential rotation appears as purely linear shear in the local model, the magnetic field evolves as $V_{Ax} =$ constant, $V_{Ay} = 2AtV_{Ax}$. Then $\boldsymbol{k} \cdot \boldsymbol{V}_A =$ constant, and $\boldsymbol{k} \times \boldsymbol{V}_A = k_x V_{Ay} = k_x 2AtV_{Ax}$. Evidently when $|t| \lesssim 1/A$, the field is rapidly changing, in the sense that $|dV_A/dt|/|V_A| \sim \kappa$, so the WKB approximation is invalid. When $|t| \gg 1/A$ the field changes more slowly but V_{Ax} is small in comparison to V_{Ay}.

In the limit $V_{Ay} \gg V_{Ax}$, when the WKB approximation is valid, the general dispersion relation (22) becomes

$$s^2[s^2 + \kappa^2 - 2\pi G\Sigma|k_x| + (c_s^2 + V_A^2)k_x^2] = 0 . \quad (24)$$

The nontrivial solutions to this dispersion relation are essentially the familiar density waves of inviscid, unmagnetized spiral structure theory, except that the sound speed has been replaced by the magnetosonic speed. These modes are *more* stable than they would be without the magnetic field. The Lynden-Bell instability is hidden in the trivial solution $s^2 = 0$. The growth rates of these modes are of order $\kappa V_{Ax}/V_{Ay}$; numerical integrations confirm that they grow only weakly. We conclude that Lynden-Bell's instability is not present in differentially rotating disks.

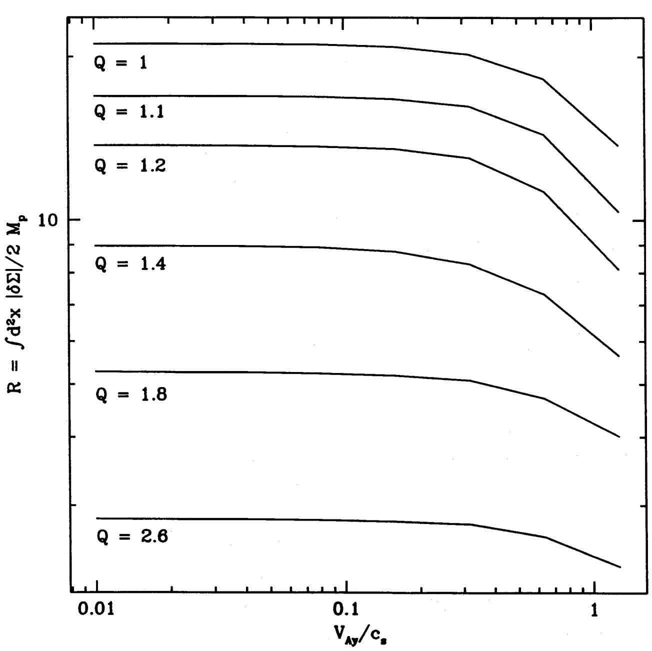

FIG. 5.—Responsiveness $R \equiv \int d^2x|\delta\Sigma|/2M_p$ (see § 3.2) of a magnetized, inviscid disk as a function of the azimuthal Alfvén velocity V_{Ay} and Toomre's Q. The radial Alfvén velocity $V_{Ax} = 0$.

4.2. *Responsiveness of a Magnetized Disk*

While axisymmetric modes are stabilized by the magnetic field, it is possible that nonaxisymmetric waves are not. To evaluate this possibility, we have measured the responsiveness R (see eq. [21] for the definition of R) of magnetized disks. Thus we integrate the initial-value problem consisting of an imposed perturbing surface density introduced at $t = 0$, and evaluate the mass involved in the response at $t = 4/\kappa$.

A typical response is shown in Figure 4. Figure 4*a* shows the response of a model with $V_{Ax} = 0$, $V_{Ay} = 0.4c_s$, and $Q = 1.2$ (appropriate for the solar neighborhood). A comparison panel shows the response at $t = 4/\kappa$ for a model with $V_{Ay} = 0$. The presence of this subthermal magnetic field reduces the amplitude of the response somewhat.

We have evaluated the responsiveness for various values of V_{Ay} and Q. The results are shown in Figure 5. As might be anticipated, the response is a monotonically declining function of both V_{Ay} and Q. Magnetic fields do not destabilize a differentially rotating disk; rather they increase stability by providing new support against self-gravity in the form of magnetic pressure. The stabilizing effect of the field becomes significant once the magnetic pressure becomes comparable to the gas pressure.

5. DISCUSSION

For viscous, self-gravitating disks, we have shown the following. (1) The effective viscosity of the gas layer in the solar neighborhood is of order 0.8 km s^{-1} kpc, similar to the estimates of Yuan (1984) and Yuan & Cheng (1991). It is likely that the viscosity decreases inward in the galaxy. (2) For viscosities this large, the axisymmetric viscous instability of self-gravitating disks (Lynden-Bell & Pringle 1974) is present with a growth rate of $\approx 0.3\kappa \approx 10^8$ yr^{-1}. (3) Viscosity obviously causes the usual density waves to damp. At the characteristic scale for density waves in the solar neighborhood the waves are overdamped. (4) While viscosity induces an axisymmetric instability, it also has a damping effect. The sum of these two effects is such that if we introduce a perturbing mass into the disk at $t = 0$, and measure the amplitude R of the response (defined in § 3) a few times $1/\kappa$ later, R declines with increasing viscosity. The reduction in responsiveness is large for $\nu \gtrsim G^2\Sigma^2/\kappa^3$. This surprising victory of the damping effects of viscosity is only temporary, of course, since on long enough timescales (many e-folding times), the instability will dominate.

Our work on viscosity is most directly comparable to that of Hunter & Horak (1983). They considered a model that has no structure in the direction parallel to the rotation vector (ρ = constant, independent of z), rather than a thin disk. Aside from this difference (which implies that the Poisson equation is solved differently), the equations they considered are equivalent to ours. They evaluated the behavior of the nonaxisymmetric shearing waves using a WKB approximation, and found strong damping of density waves and strong growth of "vortices," equivalent to the third, unstable mode in our dispersion relation. These authors were principally interested in applications to circumstellar disks.

For magnetized self-gravitating disks, we have shown that (1) the magnetic instability of self-gravitating disks in solid-body rotation discovered by Lynden-Bell (1966) is not present in differentially rotating disks, and (2) the net effect of adding a magnetic field is to reduce the nonaxisymetric responsiveness R of the disk. The reduction becomes large when the Alfvén speed approaches the sound speed.

Our work on magnetized disks is most directly comparable with that of Elmegreen (1987). He wrote down and integrated a set of equations that are equivalent to ours, and our numerical results are consistent with his. Elmegreen found that for disks that are almost in solid-body rotation (i.e., $A \ll \kappa$), the magnetic field enhanced the growth of some nonaxisymmetric waves. This can be understood as a manifestation of Lynden-Bell's (1966) instability, which may be relevant to regions within spiral arms where the rotation curve is locally solid body. For disks with a locally flat rotation curve, however, we have shown that magnetic fields reduce the nonaxisymmetric responsiveness. Our work cannot be compared directly with Elmegreen (1989), which includes heating and cooling, or with Elmegreen (1991), which incorporates vertical buoyancy.[2] Both these effects can be destabilizing, but we have not considered them here

In conclusion, both viscosity and magnetic fields can significantly alter the linear development of perturbations in a self-gravitating disk. The effects of viscosity appear to be somewhat more important in the solar neighborhood. These results suggest that it is not a good approximation to neglect either effect, particularly in studies of the nonlinear development of gravitational instability in galactic disks.

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[2] This later paper includes a discussion of the stabilizing effect of magnetic fields for flat rotation curves in the presence of cooling.

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