The Astrophysical Journal, 461:L45–L49, 1996 April 10


# SPECTRAL SIMILARITY OF FAST-EXPANDING SCATTERING-DOMINATED ENVELOPES

Shlomi Pistinner
Department of Physics, Ben-Gurion University, Beer Sheba 84105, Israel; shlomip@bgumail.bgu.ac.il

and

Giora Shaviv
Asher Space Research Institute and Department of Physics, Technion Israel Institute of Technology, Haifa 32000, Israel; gioras@physics.technion.ac.il
*Received 1995 June 16; accepted 1996 February 1*

## ABSTRACT

We discuss recent numerical and analytical results that show that different sets of input parameters for supernova/nova model atmospheres yield synthetic spectra that are indistinguishable from one another. The underlying physical reason for this phenomenon is that the atmospheres are Thomson scattering–dominated, and have a gray scattering coefficient. Implications for distance measurements by means of the expanding photosphere method are briefly discussed.

*Subject headings:* cosmology: theory — novae, cataclysmic variables — supernovae: general

## 1. INTRODUCTION

Supernovae (SNs) have been used as distance indicators for over two decades. The idea for this application is attributed to Zwicky; however, its first implementation was carried out by Branch & Patchett (1973) and Kirshner & Kwan (1974). The extensive, ongoing research in SNs as distance indicators is strongly motivated by the desire to obtain the value of the Hubble constant and the deceleration parameter. However, model atmospheres are also expected to provide information which it is hoped will shed some light on the supernova phenomenon itself. Pioneering attempts to use SNs as distance indicators have invoked the assumption that SNs can be treated as standard candles. In addition, the expanding photosphere method was used, in which it is assumed that the photosphere radiates like a blackbody. Wagoner (1981) was the first to point out the difficulties associated with these assumptions. He studied the consequences of the fact that the continuum extinction coefficient in Type II SNs is Thomson scattering–dominated and showed that each Type II SN should be calibrated individually. This calibration was to be done using the theory of stellar atmospheres. (The same remarks have been found to apply to other types of SNs, but we only consider Type II SNs in this paper.) A large number of groups and individual researchers have embarked on this project (Shaviv, Wehrse, & Wagoner 1985; Hershokwitz, Linder, & Wagoner 1986; Höflich, Wehrse, & Shaviv 1986; Lucy 1987a, b; Höflich 1987, 1990, 1994a, b; Eastman & Kirshner 1989; Schmutz et al. 1990; Hauschildt 1992a, b, 1993; Eastman & Pinto 1993; Schmidt, Kirshner, & Eastman 1992; Schmidt et al. 1994; Hauschildt & Ensman 1994; Baron, Hauschildt, & Branch 1994; Nugent et al. 1995). The problems associated with the use of the expanding photosphere method have been reviewed recently by Branch & Tamann (1992) and by Wagoner & Montes (1993).

It is a common practice today, at least for Type II SNs, to calibrate each SN individually. The essence of this technique is to compare a synthetic spectrum to an observed one. The comparison provides an absolute energy scale for the observed spectrum and hence a distance. Other quantities which are of interest, e.g., heavy-elemental abundances, are also obtained.

The assumption underlying the synthetic spectroscopic procedure is that for practical purposes there is a one-to-one relation between a set of input parameters which uniquely determine an atmosphere and the spectrum it yields. Thus, a unique set of input parameters and hence a unique atmosphere can be determined from spectroscopy.

This spectroscopic uniqueness assumption (SUA), to be defined more precisely in § 2, lacks a general physical basis and results from the experience gained in modeling normal stellar atmospheres. In modeling normal stellar atmospheres there is a small likelihood that two different sets of input data to the model atmospheres would yield an identical spectrum. However, the one-to-one correspondence between a set of input parameters and a spectrum is not strictly valid even in the case of the Sun (Muchmore 1986). Muchmore (1986) showed that the steady state model atmosphere equations with the standard boundary conditions (such as luminosity, radius, etc.) have at least two different solutions. These different solutions were found by starting the iteration for the solution with different initial approximations. The resulting (converged) solutions of the equations with identical boundary conditions were different and possessed two different spectra. Muchmore (1986) traced the nonuniqueness of the solution to the CO molecular opacity.

The other extreme case, namely, when different input data yield the same final results, was discovered in the modeling of Wolf-Rayet atmospheres. There exists a family of input parameters that yields identical equivalent widths for the emission lines (Hamann et al. 1992).

There is, in fact, no physical basis for the SUA in SN and nova atmosphere modeling. On the contrary, Hafner, Baschek, & Wehrse (1994) have refuted the validity of the SUA for Type II SNs by finding two different sets of input parameters which yield models with extremely close spectra. These authors did not provide a theoretical explanation for this phenomenon. The findings of Hafner et al. (1994) confirm the insensitivity to the model atmosphere input parameters reported by several authors (Eastman & Kirshner 1989; Höflich 1990). Pistinner et al. (1995, hereafter P95) found similar problems for novae.

There are good reasons to expect the SUA to lack general

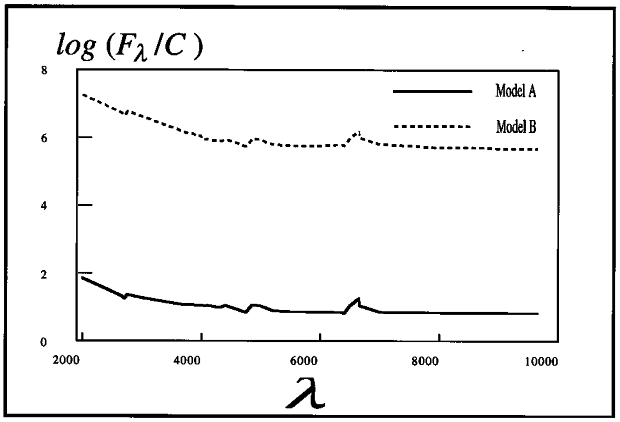


FIG. 1.—Spectra of two similar models. Parameters of the similar models are given in Table 1.

validity in some cases. As an elementary example, in scattering-dominated atmospheres where the last absorption of radiation by the matter occurs at great depths and sufficiently high temperatures, the visible range may become the Rayleigh-Jeans part of the Planck distribution and hence practically independent of the parameters of the atmosphere. The expected consequence is therefore that a large range of input parameters yields model atmospheres with the same visible synthetic spectrum. Inversely, the fit between the observed and synthetic spectra can be carried out for a wide range of input parameters. We will show in this paper that the SUA can fail in the visible range in SNs and novae even before the Rayleigh-Jeans limit is closely approached.

Supernovae and novae seem to differ at first glance; in particular, the luminosities and photospheric radii of these two classes of objects differ by orders of magnitude. However, SN and nova atmospheres possess similar physical properties; in particular, the extinction coefficients in these atmospheres are practically the same. This physical feature enables an equivalent theoretical discussion of the emerging spectra, but with some different input parameters. P95 developed an approximate analytical radiative transfer theory for continuum scattering–dominated expanding atmospheres and applied it to novae to predict the conditions under which the SUA breaks down. The approximate analytical theory is a general one for continuum scattering–dominated expanding atmospheres and therefore can be applied to Type II SNs. In this Letter we summarize the results of P95 for novae and discuss a preliminary application of the theory to Type II SNs (Pistinner et al. 1996, hereafter P96). We demonstrate the power of the theory in predicting failure of the SUA for the two spectra shown in Figure 1. These spectra were calculated numerically for the models with distinctly different sets of parameters. The model parameters are given in Table 1. (See §§ 3 and 4 for a description of the model parameters.) The ratio between the two spectra is shown in Figure 2. The flatness of the curve in

TABLE 1

MODEL PARAMETERS

| $R_{phot}$ ($10^{15}$ cm) | $T_{eff}$ (K) | $n$ | $v_{phot}$ (km $s^{-1}$) | $\langle\Lambda\rangle_P$ | $n_{e,(\tau=1)}$ ($10^{10}$ $cm^{-3}$) |
|---|---|---|---|---|---|
| 3........... | 10500 | 10 | 3500 | $10^{-2}$ | 0.3 |
| 10........... | 7500 | 10 | 3500 | $10^{-2}$ | 0.085 |

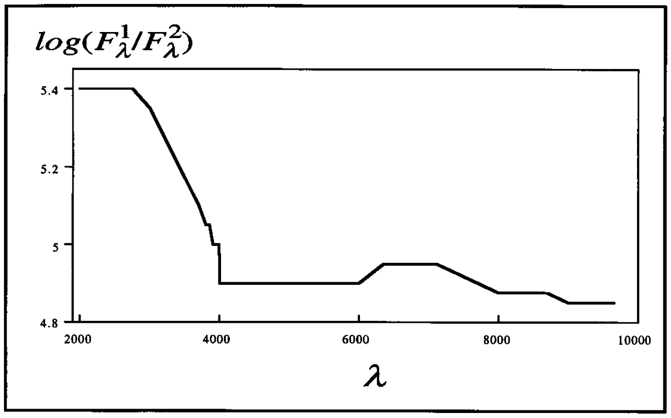


FIG. 2.—Ratio of the spectra of the two models shown in Fig. 1.

the visible demonstrates the insensitivity of the spectrum to the input parameters.

The structure of this Letter is the following. In § 2 we define spectral similarity. In § 3 we discuss the parameters that can be determined from SN and nova model atmospheres. In § 4 we discuss similarity wavelength bands and similarity parameters. We summarize our results in § 5.

## 2. THE INFORMATION CONTAINED IN THE SPECTRUM

The distance to a given star is derived from the ratio of the observed monochromatic flux $f_\lambda$ to the monochromatic luminosity $L_\lambda$ emitted by the star. The quantity $f_\lambda$ is a measurable quantity, whereas the quantity $L_\lambda$ is obtained by means of a theoretical procedure. An objective criterion is required to determine whether $L_\lambda$ and $f_\lambda$ are compatible with one another. Such a criterion can rely only on the variation of $f_\lambda$ and $L_\lambda$ with respect to $\lambda$ and must be independent of the magnitudes of $f_\lambda$ and $L_\lambda$.

We define

$$S(F_\lambda) = \frac{\partial \ln F_\lambda}{\partial \lambda} \tag{1}$$

to be the information function (IF) for $F_\lambda$ which can be any observed $f_\lambda$ or theoretically calculated $L_\lambda$. We choose to write the IF in this particular form to reflect the requirement that the magnitudes of $L_\lambda$ and $f_\lambda$ have no effect on the compatibility determination (but see Hafner et al. 1994 for a different definition). Irrespective of the IF definition, once the $S$ functions for $f_\lambda$ and $L_\lambda$ have been found to be identical within a given accuracy, one can use the magnitudes of $f_\lambda$ and $L_\lambda$ to obtain a distance.

The function $S(F_\lambda)$ depends explicitly on a number of input parameters $X_i$ of the model atmosphere and hence can be written as $S(\lambda, X_i)$. These are the parameters that one wishes to determine from the compatibility between the IFs. The SUA is that $S(\lambda; X_i)$ is a unique and single-valued function of $X_i$.

To demonstrate the breakdown of the SUA, one can use numerical experiments. Let $S^1(\lambda)$ and $S^2(\lambda)$ be two IFs obtained from two model atmospheres with different input parameters. Let us define the difference in spectra and input parameters as

$$\varepsilon(\lambda_1, \lambda_2) = \sum_{\lambda_j=\lambda_1}^{\lambda_2} \frac{S^1(\lambda_j) - S^2(\lambda_j)}{S^1(\lambda_j) + S^2(\lambda_j)} \tag{2}$$

and

$$\delta_i = \frac{X_i^1 - X_i^2}{X_i^1 + X_i^2}. \quad (3)$$

If one finds that $\delta_i \geq 1$ and $\varepsilon(\lambda_1, \lambda_2) \ll 1$ for some wavelength band, then for practical purposes the SUA does not hold for the two models in that band. We call the invalidity of the SUA "spectral similarity" and denote it by SS. [In the example shown in Figures 1 and 2 $\varepsilon$(3000 Å, 10,000 Å) = $10^{-2}$, $\varepsilon$(4000 Å, 10,000 Å) = $10^{-4}$, and $\varepsilon$(1000 Å, 3000 Å) ≈ 1. Clearly, SS exists over the range 4000–10,000 Å.] P95 showed that the SS in novae emerges from the prevailing physical conditions in these atmospheres.

P95 demonstrated that there can be a combination of input parameters that significantly affect an IF (of a spectrum or spectral band) only as arguments of one atmospheric variable. Thus, no matter how the combination members are varied, the IF stays the same as long as that particular variable stays the same. We define such a variable to be a SS parameter.

## 3. THE INPUT PARAMETERS

We describe now the input parameters for SN models that one tries to determine by fitting theoretical to observed spectra. A distinct difference between novae and SNs is that in a SN envelope the matter and the radiation are decoupled from a dynamical point of view (cf. Hauschildt & Ensman 1994 for numerical results and P96 for analytical ones). This dictates two different types of velocity laws for novae and SNs. Here we concentrate on the SN case. However, we note that this difference does not affect the SS.

When kinetic energy dominates the flow (i.e., supersonic flow), the velocity law is homologous and the atmosphere is in free expansion. Thus, a single parameter (i.e., the velocity at the photosphere) is sufficient to describe the velocity field at any given moment. The density viewed by any Lagrangian observer in the atmosphere decreases as the inverse cube of time, and thus the general shape of the density stratification of the expanding atmosphere preserves the shape it had at the time the explosion became homologous. The density is therefore given by a function of the following form: $\rho(\eta, t) = \rho_s(\eta)/t^3$, where $\eta$ is a Lagrangian coordinate—for example, the mass inside radius $r$. The density stratification of SN (nova) atmospheres $\rho_s(\eta)$ is described by a function which decreases strongly with radius. Two stratification laws are traditionally used, a power law $[\rho_s(\eta = \rho_s^0(\eta/\eta_0)^{-n}]$ and an exponential law $[\rho_s(\eta) = \rho_s^0 e^{-\eta/l_\eta}]$. The parameters for these laws are the power-law index $n$ and the $e$-folding distance $l_\eta$.

The luminosity (Lagrangian or Eulerian) at any given moment is an input parameter in all prescriptions. The temporal variation of the luminosity is derived from a fit of a synthetic light curve to an observed one. The synthetic light curve can be calculated from an optically thick model for the flow or, in the best case, by a reconstruction of a light curve from the synthetic spectra. The luminosity is sometimes input explicitly (Eastman & Kirshner 1989; Schmidt et al. 1992). It can also be input implicitly by defining a standard radius (loosely called a photospheric radius) at a total-extinction optical depth unity for a standard wavelength and assigning an effective temperature for that standard radius (cf. Höflich 1990; Hauschildt 1992b; Baron et al. 1994). The different parameterizations may be viewed as a transformation of the function $S_\lambda$ from one space of $X_i$ into another space, $X_i'$. The assumption is that this is a regular transformation but not necessarily a single-valued one.

TABLE 2

SPECTRAL SIMILARITY PARAMETERS IN TWO WAVELENGTH BANDS

| Wavelength Band (Å) | Physical Conditions | Similarity Parameters | Combination of Input Parameters |
|---|---|---|---|
| 0–3000 ............ | $\Lambda_\lambda \not\ll 1$ | Strength of metal lines | $\epsilon_{\rm UV}$, $z_i$, $\chi$ |
| 3000–10000 ........ | $\Lambda_\lambda \ll 1$ | $\Lambda_\lambda \approx \langle\Lambda_\lambda\rangle_P$ | $R_{\rm phot}$, $T_{\rm eff}$, $z_i$ |

The physical equivalence of the two approaches can be shown rather easily (P96). Although the two approaches are equivalent in principle, they differ in practice. P95 found it easier to predict the SS using the Höflich-Hauschildt-Baron approach, and for this reason we restrict the discussion in this Letter to this approach. The details of the approach will be discussed in the next section.

We stress here that each model atmosphere is a snapshot of the coasting atmosphere. This approximation is justified by the fact that the photon diffusion time is long relative to the expansion time. The SS is found when the atmosphere is Thomson scattering–dominated, a situation which appears prior to hydrogen recombination in Type II SN (nova) atmospheres.

## 4. SPECTRAL SIMILARITY BANDS AND INPUT PARAMETERS

The model input parameters $X_i$ are $X_1 = R_{\rm phot}$, the photospheric radius defined by $R(\tau_{\rm ext}^{5000\,\text{Å}} = 1)$; $X_2 = T_{\rm eff}$, the effective temperature which provides a representative value of the radiation energy density at the photosphere; $X_3 = v_{\rm phot}$, the matter velocity at the photosphere, $X_4 = v_{\rm phot}/v_e$ or $n$. The last parameter does not define uniquely the spatial extent of the atmosphere and one has to truncate the infinite atmosphere. The truncation of the atmosphere is carried out by an additional parameter $X_5 = \rho_{\rm out}$, which is the density at the outer edge of the atmosphere; $\rho_{\rm out}$ is defined in such away that the optical depth beyond it is sufficiently low. (This parameter has a negligible effect in SN atmospheres, but is important for novae, since these objects are much more extended and have $n \cong$ 2–3.) The element abundances $z_i$ are also important input parameters.

Additional parameters are required to get a good fit to the spectra. In particular, the UV range, which is heavily blanketed by metal lines, cannot be fitted without the parameters $\chi$, the microturbulent velocity width for lines, and $\epsilon_{\rm UV}$, the thermalization parameter of the UV lines (Baron et al. 1994). The $\epsilon_{\rm UV}$ parameter, at least, is not a physical requirement, but is only required by the fact that an adequate NLTE treatment of the UV line blanketing is not yet available.

For the models we are considering, the SS parameters for two wavelength bands are given in Table 2. Note that the combinations of input parameters which affect the SS parameters are different in the different bands.

The spectra shown in Figure 1 were numerically calculated for moving solar composition atmospheres with the Kurucz line list. We can, however, understand these spectra by applying our approximate analytical theory to an analog static pure hydrogen atmosphere without lines. In this case, the SS is merely a continuum similarity (CS). It can be shown (cf. P95, who used a power-law density, and P96, who used an exponential density law) that the following expression for the

Planck mean of the continuum thermalization parameter $\Lambda_\lambda$ (Wagoner 1981),

$$\langle \Lambda_\lambda \rangle_P(\tau = 1) = 3.2 \times 10^{-2} \left[ \frac{n_e(\tau = 1)}{10^{10}\ \mathrm{cm}^{-3}} \right] \left( \frac{T_{\mathrm{eff}}}{10^4\ \mathrm{K}} \right)^{-9/2}, \quad (4)$$

is a SS parameter for those regions in the spectra for which $\Lambda_\lambda \approx \langle \Lambda_\lambda \rangle_P \ll 1$ and for the combination of input parameters $R_{\mathrm{phot}}$, $T_{\mathrm{eff}}$, and $z_i$. Note that $n_e(\tau = 1)$ can be related to the photospheric radius if the density exponent is known and hydrogen in the atmosphere is assumed to be fully ionized. $\langle \Lambda_\lambda \rangle_P$ is evaluated at the photospheric radius and effective temperature. (The justification for the use of effective temperature instead of gas temperature is given in P95.) The condition $\Lambda_\lambda \ll 1$ is fulfilled for $\lambda > 3000$ Å. Thus models with different values for the combination of input parameters, but the same $\langle \Lambda_\lambda \rangle_P$, are predicted to have CS for $\lambda > 3000$ Å.

The spectra shown in Figure 1, although calculated for more complex atmospheres than those formally treated by the analytical theory, have the same value of $\langle \Lambda_\lambda \rangle_P$, i.e., $10^{-2}$. And, as predicted, they are nearly identical for $\lambda > 3000$ Å as Figure 2 shows. Thus, the models demonstrate the ability of the analytical theory developed by P95 to predict the appearance of CS. The physical reason for the similarity is that the atmosphere is Thomson scattering–dominated, and hence has a nearly gray scattering coefficient for $\lambda > 3000$ Å. This leads to the small $\langle \Lambda_\lambda \rangle_P$—a situation which emerges from the rather low density in SN and nova atmospheres as compared to otherwise similar (with respect to temperature) stellar atmospheres. In such cases, P95 have shown that $\langle \Lambda_\lambda \rangle_P$ controls the temperature structure about the thermalization layer (which may, of course, vary in optical depth with wavelength) for $\lambda > 3000$ Å. Since the continuum spectra are generally formed by a mixture of temperatures from about the thermalization layer, it follows that spectra from a wavelength band where the different atmospheres have similar thermalization layer temperature structures will generally be similar. We note, in particular, that the Rayleigh-Jeans limit in the visible does not need to be closely approached for this similarity to be obtained. The conclusion is that $\Lambda_\lambda \approx \langle \Lambda_\lambda \rangle_P$ is a SS parameter for $\lambda > 3000$ Å for the kind of atmospheres we are considering.

The fact that the calculated spectra are for a moving atmosphere, while real SNs and novae have moving atmospheres, and the analytical theory is for the static situation is not a problem of the above analysis. The analytical solution for a pure gray continuum scattering atmosphere in motion found by Pistinner & Shaviv (1994) was used by P95 to show that the CS is not affected by the motion. However, the effect of motion on lines is not yet clear, and an approximate analytical theory capable of handling it is currently under construction.

As with motion, there is also the concern that our calculated spectra and real spectra have hydrogen lines which are neglected in the analytical theory. Support for the conclusion that this is not a problem is given by the fact that numerical models of moving envelopes show the existence of the similarity also when hydrogen lines are present. Counterintuitively, the motion enhances the similarity (Hafner et al. 1994). Furthermore, the hydrogen lines in two different models have indistinguishable shapes provided the velocities are the same. This is so because the lines are strongly scattering and their extinction depends mainly on the radiation field. Thus, CS models of pure hydrogen atmospheres with lines are practically spectrally similar models.

The CS breaks down in the region $\lambda \leq 3000$ Å, where $\Lambda_\lambda \not\ll 1$. Thus, in principle, different input parameters should yield distinguishable spectra in this range. At first glance it seems that the UV range is ideal to make a distinction between various models which are at the early coasting phase. However, SN and nova atmospheres are strongly metal-line–blanketed in this region, and it is still impossible to fit an observed spectrum without adding input parameters that extend the possibility of combinations of parameters that lead to nonunique fits. Table 2 shows the combination we have found for the UV. The SS parameter in this case is the overall strength of the metal lines. Although the strength of metal lines is a somewhat vague parameter, it is useful in practical fitting exercises.

The problems in using the UV range to distinguish between different models will be discussed elsewhere.

## 5. SUMMARY

Static, low-density, pure hydrogen atmospheres have extinction coefficients which are dominated by Thomson scattering and hence are nearly gray. As a consequence, the thermalization parameter is a SS parameter for $\lambda > 3000$ Å. Models with the same SS parameter value, but quite different physical conditions, show the same spectra. The SS parameter value can be obtained from an approximate analytical theory (P95) and applied to predict the CS models. We stress that the CS exists for $\lambda > 3000$ Å.

We conclude that the spectral uniqueness assumption is not always valid in such atmospheres, and one needs to exercise caution in deciding which synthetic spectrum is compatible with the observed one. In other words, the atmosphere parameters which are the input parameters for the theoretical modeling may not be uniquely determined from the fit to the observed spectra. The addition of the element of motion to these atmospheres does not eliminate the CS. On the contrary, it has a tendency to wash away information from the spectra.

It is therefore, highly recommended to rely on spectra obtained after hydrogen recombination in the atmosphere. These spectra result from a radiation field which is thermally coupled to the matter, and are likely to be more sensitive to the model atmosphere input parameters. The recommendation is also justified by numerical experiments (Höflich 1990).

If the SN (nova) is too faint to obtain a good spectrum after hydrogen recombination, then fitting the UV and the optical simultaneously prior to hydrogen recombination is essential. The use of a time sequence is highly recommended.

We wish to thank B. Baschek and R. Wehrse for drawing our attention to the important numerical results which showed the similarity. Discussions with S. Starrfield and P. Hauschildt are gratefully acknowledged. Comments and criticism by the referee, David J. Jeffery, were instrumental in clarifying the issues presented in the paper.

## REFERENCES

Baron, E., Hauschildt, P. H., & Branch, D. 1994, ApJ, 426, 334
Branch, D., & Patchett, J. B. 1973, MNRAS, 161, 71
Branch, D., & Tamman, G. A. 1992, ARA&A, 30, 359
Eastman, R., & Kirshner, R. P. 1989, ApJ, 347, 771
Eastman, R., & Pinto, P. A. 1993, ApJ, 412
Hafner, M., Baschek, B., & Wehrse, R. 1994, MNRAS, 269, 764

Hamann, W. R., Leuenhagen, U., Koesterke, L., & Wessolowski, U. 1992, A&A, 255, 200
Hauschildt, P. H. 1992a, J. Quant. Spectrosc. Radiat. Transfer, 47, 433
———. 1992b, ApJ, 398, 224
———. 1993, J. Quant. Spectrosc. Radiat. Transfer, 50, 301
Hauschildt, P. H., & Ensman, L. M. 1994, ApJ, 424, 905
Hershokwitz, S., Linder, E., & Wagoner, R. 1986, ApJ, 303, 800
Höflich, P. 1987, Synthetic Spectra for the Early Stages of SN 1987A, in ESO Workshop on SN 1987A (Garching: ESO), 449
———. 1990, Habil. thesis, Heidelberg Univ.
———. 1994a, A&A, 246, 481
———. 1994b, ApJ, 443, 89
Höflich, P., Wehrse, R., & Shaviv, G. 1986, A&A, 163, 105
Kirshner, R. P., & Kwan, J. 1974, ApJ, 193, 27
Lucy, L. B. 1987a, Computed Spectra for SN1987A, in ESO Workshop on SN 1987A (Garching: ESO), 417
Lucy, L. B. 1987b, A&A, 182, 31
Muchmore, D. 1986, A&A, 155, 172
Nugent, P., Branch, D., Baron, E., Fisher, A., Vaughan, T., & Hauschildt, P. H. 1995, Phys. Rev. Lett., in press
Pistinner, S., & Shaviv, G. 1994, ApJ, 436, 837
Pistinner, S., Shaviv, G., Hauschildt, P. H., & Starrfield, S. 1995, ApJ, 451, 721 (P95)
———. 1996, preprint (P96)
Schmidt, B. P., Kirshner, R. P., & Eastman, R. G. 1992, ApJ, 395, 366
Schmidt, B. P., et al. 1994, AJ, 107, 1444
Schmutz, W., Abbott, D. C., Russel, R. S., Hamann, W. R., & Wessolowski, U. 1990, ApJ, 335, 253
Shaviv, G., Wehrse, R., & Wagoner, R. 1985, ApJ, 289, 198
Wagoner, V. R. 1981, ApJ, 250, L65
Wagoner, V. R., & Montes, M. 1993, Phys. Rev., 227, 205