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H-FUNCTION EVOLUTION DURING VIOLENT RELAXATION

NOAM SOKER
Mathematics-Physics, University of Haifa at Oranim, Tivon 36006, Israel; soker@phys1.technion.ac.il
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ABSTRACT

We examine the evolution of H-functions during the process of violent relaxation of collisionless self-gravitating systems. Our arguments follow a few, relatively recent, works which dealt with this subject. We point out that the functional that is extremized through violent relaxation does not necessarily increase at all times and that it may decrease if the system has specific, far from equilibrium, conditions. We propose to distinguish between two phases of violent relaxation. In one phase the H-function may decrease with time, while in the other it is always a nondecreasing function of time. The phase during which the H-function may decrease is characterized by the conversion of kinetic to gravitational energy, and it happens before the central-dense and outer-dilute equilibrium structure is formed. We propose initial conditions to be imposed on N-body numerical simulations, which can be performed in order to test our claims.

Subject headings: celestial mechanics, stellar dynamics — methods: analytical — methods: numerical

1. INTRODUCTION

Three decades have passed since Lynden-Bell (1967) first suggested the process of violent relaxation, through which collisionless self-gravitating systems rapidly relax. The quest for better understanding of violent relaxation has continued since then (e.g., Shu 1978, 1987; Saslaw 1985; Madsen 1987). In particular, there has been a great interest in criteria for violent relaxation (e.g., Saslaw 1970; Kadomtsev & Pogutse 1970) and the functional that is extremized through violent relaxation (Lynden-Bell 1967; Shu 1978; Stiavelli & Bertin 1987; Spergel & Hernquist 1992). The form of the functional that is extremized is

$$H = -\int C(F)\, d^3x\, d^3v\,, \tag{1.1}$$

and is usually called the H-function. It should be noted that in statistical mechanics H is usually defined with the opposite sign. $C(F)$ is a convex function ($d^2C/dF^2 > 0$) of the coarse-grained distribution function with $C(0) = 0$. For an ideal gas, the entropy is given by $S = k_B H$, where k_B is the Boltzmann constant, and $C(F) = F \ln F$. For collisionless self-gravitating systems $C(F)$ should have a different form.

In the last decade, several papers brought the H-function of collisionless self-gravitating systems back to the scientific literature. The arguments started with an incorrect statement (which was in any case not crucial to the main conclusions of their paper) made by Tremaine, Henon, & Lynden-Bell (1986), who treated the evolution of the H-function in a self-gravitating system during the violent relaxation process. They use the coarse-grained distribution function F, which is obtained from the fine-grained distribution function f by dividing phase space into equal volume macrocells and averaging f over the volume of each macrocell. They assume that at some initial time, $t = t_1$, the coarse-grained distribution function and the fine-grained distribution function are equal, and they show that for any later time, $t_2 > t_1$, the inequality $H(t_2) \geq H(t_1)$ holds. Although they claim to show that H is a nondecreasing function of time, Sridhar (1987), who demonstrates how H can decrease in 1 d.o.f., and Kandrup (1987a) point out that the first authors actually show the weaker inequality only. Kandrup argues that a general expression for the evolution of the H-function at arbitrary time must explicitly include the form of the potential, which, for self-gravitating systems without external potential, is given by $\nabla^2 U(\boldsymbol{x}, t) = 4\pi G\rho$, where U is the potential and ρ is the mass density. Kandrup ends his discussion by claiming that whether H is a monotonically increasing function of time during the violent relaxation process is still an open question.

Starting from where Kandrup (1987a) ends, Soker (1990) derives an expression for the time derivative of a general H-function in which the potential appears explicitly. Soker applies the expression to self-gravitating systems and argues that the condition for having a nondecreasing H-function for all coarse-grained distribution functions is that, on the average, the high-density regions contract and the low-density regions expand. Although Soker derives his results under several assumptions, he conjectures that the results are more general. In a relatively recent paper, Spergel & Hernquist (1992) propose a functional that is extremized through violent relaxation. Their elegant and convincing derivation is based on two *Ansätze*, which imply the existence of a detailed balance, from which it can be shown that their functional is a nondecreasing function of time.

The goal of this paper is to show that the assumptions of Spergel & Hernquist (1992) are generally valid only when the violently relaxing system is not too far from the final equilibrium, in the sense that a structure of central dense region together with an outer dilute region is already defined. In order to do so, we compare in § 2 their *Ansätze* with the criterion for violent relaxation proposed by Saslaw (1970) and with the results of Soker (1992). It is well known that the H-theorem, which claims that the H-function is a nondecreasing function of time, holds only after "initial correlations have died away," as Kandrup (1987b) puts it. The paper by Kandrup contains an excellent, very relevant discussion of properties of the entropy as well as the H-theorem. In § 3 we propose a set of initial conditions to be used in N-body numerical simulations of gravitating systems, which can be performed in order to check the validity of our claims regarding H-function evolution during violent relaxation. We follow Jaynes (1971) when we propose initial conditions where kinetic energy is converted into gravitational energy. This may lead to the decrease of the

H-function at early stages, although its final value will be higher than its initial value. We summarize in § 4.

2. ON CRITERIA FOR VIOLENT RELAXATION

We start by giving the expression for the time variation of H that was derived by Soker (1992). We will use this expression for dH/dt to propose a way to characterize violent relaxation, at least in what we call a "vigorous phase." We give the expression for a system in which the systematic variation is along one coordinate axis x^k (eq. [3.10] of Soker 1992),

$$\dot{H} = \frac{1}{3}\int C''(\langle f\rangle)[\nabla^2 U(\Delta s)^2 - (\Delta v)^2] \times \left(\frac{\partial f}{\partial v^k}\right)\left(\frac{\partial f}{\partial x^k}\right) d^3x\, d^3v\,. \tag{2.1}$$

The different variables are defined as follows. The phase space is divided into macrocells of sizes $2\Delta s$ and $2\Delta v$, along the coordinates x^k and v^k, respectively. We denote by $\langle f\rangle$ the coarse-grained distribution function obtained from the fine-grained distribution function f by averaging over the phase space macrocells $2\Delta s \times 2\Delta v$. C is a convex function, $d^2C/d\langle f\rangle^2 > 0$ with $C(0) = 0$, and U is a general potential. Equation (2.1) is accurate to the third power in Δv and Δs. As for the other assumptions and constraints which have been made in the derivation of equation (2.1), see Soker (1990).

For self-gravitating systems $\nabla^2 U = 4\pi G\rho$ and is thus nonnegative. Therefore, the two terms inside the square brackets in equation (2.1) have an opposite contribution in each macrocell. In order to obtain $\dot{H} > 0$ for arbitrary Δs and Δv, it is necessary that, on average, $(\partial f/\partial \boldsymbol{v})\nabla f$ be positive in the high-density regions and negative in the low-density regions. In general, this implies that the H-function increases when, on average, the high-density regions contract and the low-density regions expand. During relaxation of stellar systems via the violent relaxation mechanism, for example, this must not be the case. In this process groups of objects are formed and destroyed until the system relaxes (Saslaw 1985). In this process the high-density regions can expand and the low-density regions can contract. It is possible, therefore, that during the violent relaxation process, if we take macrocells with $\Delta s \ll \tau\,\Delta v$, we obtain an opposite sign for $\dot{H}$ than by taking macrocells with $\Delta s \gg \tau\,\Delta v$, at any specific time. Here τ is a characteristic timescale, i.e., the same τ required by the time coarse-graining of Saslaw (1970). This means that a coarse-grained distribution function for which the H-function decreases momentarily may be found, although it will increase for all possible coarse-grained distribution functions over a long enough period of time.

Until now we have summarized the derivation and discussion of Soker (1992). We shall now go further. Based on the behavior of $\dot{H}$ discussed above, we propose the following classification for a violently relaxing system when it is far from equilibrium conditions. A collisionless self-gravitating system is in the phase of a "vigorous" violent relaxation if, at any given time during this phase, a coarse-grained distribution function for which momentarily $\dot{H} \le 0$ can be found. We do not specify here which form should be selected for $C(F)$, but it is probably appropriate to choose the $C(F)$ that the system maximizes.

Criteria for characterizing systems during the violent relaxation process, without direct reference to the evolution of the H-function, have been proposed by Saslaw (1970) and Kadomtsev & Pogutse (1970) (see the discussion in Saslaw 1985). Saslaw (1970) gives a criterion by using coarse graining in time, while Kadomtsev & Pogutse (1970) use coarse graining in phase space. The way by which the second criterion relates to the first has already been discussed by Saslaw (1985). The physics behind the criterion given by Saslaw (1970) is that the largest contribution to the potential at a specific point comes from a distant mass whose mean field time fluctuations is uncorrelated with the variation of the local value of f. Saslaw shows his criterion to be sufficient, but he leaves open the question of whether the criterion is also necessary.

With the goal of understanding structures of galaxies from fundamental statistical mechanics, Spergel & Hernquist (1992) consider functional extremization. In their search for a functional that is extremized in violent relaxation, Spergel & Hernquist start with two *Ansätze*. First *Ansatz*: The effects of time-varying fields attending violent dynamical processes on individual particles can be represented as a sequence of discrete, impulsive scattering events. Second *Ansatz*: The states accessible to a particle undergoing a scattering event are determined fully by the instantaneous values of the particles' integrals of motion at the time of the scattering.

Spergel & Hernquist (1992) then assume that the scattering is a local process and that the density of state available is $g(E) \propto (E - U)^{1/2}$, where E is the particle total energy, and $E - U$ is its kinetic energy. According to the model of Spergel & Hernquist, most of the scatterings occur near perigalacticon. This model contains the assumption that a group of gravitating point masses, which is the scatterer, is already defined near the center of the system.

From these *Ansätze* and assumptions Spergel & Hernquist propose that violent relaxation maximizes a functional K, in which the density of states $\psi(E)$ is replaced by the local density of states $g(E)$. This is equivalent to replacing $\ln f$ by $\ln(f\psi/g)$ in the expression for $C(F)$ in equation (1.1). The functional K is a nondecreasing function of time.

Two comments regarding the functional K should be made. First, the idea of changing $\ln f$ into $\ln(fh)$, in the expression for the functional, where h is a "weight" function of microcells, is not new. Stiavelli & Bertin (1987), for example, take h to be the radial period of orbits with specified energy and angular momentum. Thus, the weight function depends on the properties of individual stars.

Second, the transition from $\psi(E)$, which depends on the number density of particles near E, to $g(E)$, which depends only on the individual star energy E, resembles the transition from the Gibbs H-function to the Boltzmann H-function (Jaynes 1965). With our sign in the definition of the H-function, the Boltzmann H-function is equal to or larger than the Gibbs H-function. The Gibbs H-function is equal to the Boltzmann H-function if and only if the distribution function factors into a product of single-particle functions (e.g., Jaynes 1965). It is the Gibbs H-function that is actually used in the current paper.

How are the assumptions and derivation of the extremized functional of Spergel & Hernquist related to previous criteria and results? The second *Ansatz* of Spergel & Hernquist (1992) appears to be in accord with the criterion of Saslaw (1970), which asserts that the mean field time fluctuations of a distance mass is uncorrelated with the variation of the local value of f. However, Spergel & Hernquist's first *Ansatz* and the assumptions that follow it, concerning discrete scattering by a local potential, appear to be in disagreement with the criterion proposed by Saslaw (1970), which claims that the largest contribution to the potential at a specific point comes from a distant mass.

The expression for the local density of states $g(E) \propto (E - U)^{1/2}$ indicates that particles are more likely to be scattered into states with higher total and kinetic energy. Energy conservation implies that the total energy of the scattering group of point masses decreases during the interaction. The group of gravitating point masses that scatter the particle will contract when its total energy decreases, while the scattered particles' apgalacticon will increase. Therefore, the scenario presented by Spergel & Hernquist (1992) is compatible with the condition derived by Soker (1990) for the increase of the H-function (eq. [2.1]).

The result of the above considerations is that the assumptions of Spergel & Hernquist (1992), although reasonable and appearing to lead to the right extremized functional at the end of the violent relaxation, are not necessarily fulfilled at the beginning of the relaxation process. Their assumptions seem to characterize the phase of the violent relaxation when the high-density and low-density regions are already well defined. If, indeed, our claims above hold, then it is reasonable to define two phases of violent relaxation. An early phase, the "vigorous phase," where the criteria of Saslaw (1970; see also Kadomtsev & Pogutse 1970), as well as the one suggested here, following equation (2.1), apply; and a second phase, the "calm phase" where the assumptions of Spergel & Hernquist (1992) hold. The second phase starts when the high-density and low-density regions begin to form, although they are not yet relaxed. The calm phase corresponds, in the more general case, to the phase at which initial correlations have already decayed, and therefore a general H-theorem can hold (Kandrup 1987b). It is not necessary that the first phase occur. In the process of galaxies merger, for example, where the high-density regions are well defined before the collision, the first phase appears not to exist. A numerical test to the above proposed scenario of H-function evolution is described next.

3. PROPOSED NUMERICAL SIMULATION TESTS

Jaynes (1971) notes that in order to have the H-function decrease (increase with the plus sign in his definition of H), the kinetic energy should decrease. He discusses a gas obeying the van der Waals equation of state, where the attractive van der Waals forces convert kinetic to potential energy after an explosion starts. He then quantitatively considers the violeting of the H-theorem when a volume constraint is removed.

In our case, it is the gravitational attraction that will convert kinetic to potential energy. Since we are interested in suggesting a numerical test and not in rigorously proving the decrease of the H-function, we can make several simplifying assumptions. We assume that the temperature T of the point masses system can be approximately defined, even though the system is not in equilibrium, by $3Nk_B T/2 = E_k$, where E_k is the kinetic energy, k_B is Boltzmann constant, and N is the total number of particles. We take all particles to be of equal mass and take the H-function of an ideal gas as an approximation to our collisionless self-gravitating system (Shu 1978):

$$H = A + N \ln V + \tfrac{3}{2} N \ln T , \tag{3.1}$$

where V is the volume and A is a constant. The condition for the decrease of H, $dH/dt < 0$ when the volume increases is (Jaynes 1971)

$$\left(\frac{\partial T}{\partial V}\right)_E < -\frac{2T}{3V} . \tag{3.2}$$

We take a spherical system of mass M and radius R, so that the potential energy is $E_G \simeq GM^2/R = GM^2(4\pi/3V)^{1/3}$, where $V = 4\pi R^3/3$ is the volume. The total energy is $E \simeq E_k - GM^2/R$. Substituting for the kinetic energy in the expression for the temperature gives

$$T \simeq \frac{2}{3}\frac{1}{Nk_B}\left(E + \frac{GM^2}{R}\right) . \tag{3.3}$$

Substituting $R \propto V^{-1/3}$ and differentiating gives

$$\left(\frac{\partial T}{\partial V}\right)_E \simeq -\frac{2}{9}\frac{1}{Nk_B}\frac{GM^2}{VR} . \tag{3.4}$$

By substituting equations (3.3) and (3.4) in condition (3.2), we find that the conditions on the total energy and kinetic energy, in order for the H-function to decrease when the volume increases, are

$$E < -\frac{1}{2}\frac{GM^2}{R} , \quad \text{and} \quad E_K < \frac{1}{2}\frac{GM^2}{R} . \tag{3.5}$$

Thus, the kinetic energy at the beginning of the process should be less than half the magnitude of the gravitational energy. Since we start by increasing the volume, the kinetic energy will further decrease. The conversion of kinetic to potential energy is a necessary condition for violating the H-theorem (Jaynes 1971). However, when the system finally relaxes and reaches equilibrium, it is virialized and $E_K = |E_G|/2$. Thus, in the overall relaxation process, potential energy will be converted into kinetic energy.

Even though we do not expect the temperature to be well defined initially, and the gravitational energy may somewhat differ from the $V^{-1/3}$ dependence, we can use the considerations given above to propose a numerical test to study the decrease of the H-function. We suggest the following initial conditions and calculations of the H-function evolution in a spherical N-body numerical simulation. The numerical simulations may also be performed for nonspherical, nonequal mass particles systems, of course. First, the kinetic energy should be less, preferentially much less, than half the gravitational potential energy magnitude: $E_K \ll |E_G|/2$. This initial condition of a "cold system" is compatible with systems violently relaxing into states resembling distribution of elliptical galaxies (Tremaine et al. 1986).

Second, the system should expand outward, mimicking an explosion. That is, the initial largest component of the velocities of most particles, or even all particles, is the positive radial component. In order to calculate the H-function as a function of time, the phase-space volume should be divided into a set of macrocells, each having a volume $\Delta\mu = \Delta \boldsymbol{x}\, \Delta \boldsymbol{v}$, in a suitable coordinate system. Since the velocity distribution is not isotropic in our proposed simulations, the distribution function may not be coarse grained along energy hypersurfaces, as done by Hernquist, Spergel, & Heyl (1993), for example.

The coarse-grained distribution function is calculated after every time step by counting the number of particles in each macrocell $N_j(t)$. Since the particles are assumed to be of equal mass, the distribution function in macrocell j is given by $F_j(t) = N_j(t)/\Delta\mu$. A convex function $C(F)$ must be assumed. It is reasonable to compare several different functions that have been proposed (Lynden-Bell 1967; Shu 1978; Stiavelli & Bertin 1987; Spergel & Hernquist 1992). The H-function is given then by summing over all macrocells $H = -\Delta\mu\Sigma_j C(F_j)$. It should be noted that the violent relaxation is an incomplete process, so that the search for a maximum of the H-function should be done up to a prescribed precision. A good discussion on the incompleteness of violent relaxation, on some properties and

limitations of coarse- and fine-grained distribution functions, and on other assumptions and characters of violent relaxation may be found in the two back-to-back papers by Madsen (1987) and Shu (1987), and in Stiavelli & Bertin (1987).

We note that even in systems which are already in equilibrium there are statistical fluctuations in the number of objects in the macrocells, which cause statistical fluctuations in the H-function. These fluctuations can cause H to decrease for a short period of time. However, the timescales of these fluctuations are expected to be of the order of the particles crossing times of the macrocells, which are shorter than the duration for which H will decrease as a result of transferring kinetic to potential energy. Also, by increasing the number of particles, the fluctuations will be smaller, while the change in H due to relaxations will not be much influenced.

4. SUMMARY

Our main results are summarized as follows.

1. A coarse-grained H-function (or functional), which is maximized through violent relaxation of collisionless self-gravitating systems, does not necessarily increase at all times during the relaxation.

2. The coarse-grained H-function is likely to decrease when kinetic energy is converted to gravitational energy, before the central-dense and outer-dilute structure is well defined.

3. This behavior of the H-function leads to a possible distinction between two phases of violent relaxation. The "vigorous phase," during which the H-function may decrease, and the "calm phase," during which the H-function always increases. The vigorous phase will occur only in systems starting with specific initial conditions. We claim that the assumptions of Spergel & Hernquist (1992) are appropriate for the calm phase. The relaxation will always end through the calm phase, and therefore the functional which Spergel & Hernquist (1992) propose to be extremized seems to be correct. The calm phase corresponds in the general case to the phase at which initial correlations have already decayed (Kandrup 1987b).

4. We proposed a set of initial conditions to be imposed on N-body numerical simulations in order to test our claims.

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