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COSMIC MICROWAVE BACKGROUND COMPTONIZATION BY HOT INTRACLUSTER GAS

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ABSTRACT

The commonly used expressions for the cosmic microwave background intensity and temperature changes due to Compton scattering by hot gas in clusters of galaxies (the Sunyaev-Zel'dovich effect) are based on a solution to the Kompaneets nonrelativistic diffusion equation. These expressions are insufficiently accurate because of the high electron velocities and low scattering probability in the hot intracluster gas. The results of an improved relativistic treatment show significant deviations, at frequencies on the Wien side, from what is predicted from the Kompaneets-based calculation. Useful expressions are given which can be readily integrated to determine the exact magnitude of the effect at any frequency and gas temperature. An important feature of the relativistic spectral distribution of the intensity change is the dependence of the critical frequency—where the effect vanishes—on the gas temperature. This dependence must be taken into account in determinations of peculiar velocities of clusters from measurements of the kinematic Sunyaev-Zel'dovich effect.

Subject headings: cosmic microwave background — cosmology: theory — radiation mechanisms: nonthermal

1. INTRODUCTION

Compton scattering of the cosmic microwave background (CMB) radiation by hot intergalactic gas—the Sunyaev-Zel'dovich (S-Z) effect—is an important process whose spectral imprint on the CMB—a systematic shift of photons from the Rayleigh-Jeans (R-J) to the Wien side of the spectrum—can be used as a sensitive cosmological probe. Scattering of the radiation by hot intracluster (IC) gas is of particular interest: Measurements of the effect toward a cluster, when combined with X-ray measurements, can yield the value of the Hubble constant, the cluster peculiar velocity, as well as IC gas properties. (For reviews of the effect, see Sunyaev & Zel'dovich 1980a; Rephaeli 1990.)

Quantitative description of the scattering process—and the resulting simple expressions for the spectral intensity and temperature changes—were given by Zel'dovich & Sunyaev (1969) and Sunyaev & Zel'dovich (1972). The calculations in these papers are based on a solution to the Kompaneets (1957) equation, a nonrelativistic diffusion approximation to the full kinetic equation for the change of the photon distribution due to scattering. The expressions derived in these papers were used in virtually all the works on the S-Z effect in clusters. At high frequencies these expressions are, however, *insufficiently* accurate for some of their intended uses, such as the determination of the Hubble constant and peculiar velocities of clusters. There are two reasons for this. First, because of the low optical thickness to Compton scattering in clusters, $\tau \simeq 10^{-2}$, most photons are not scattered even once. A diffusion approximation would then hardly seem adequate. Second, with IC gas temperatures spanning the range 3–15 keV, electron velocities in the IC gas are near-relativistic. Even so, the Kompaneets-based treatment of Zel'dovich & Sunyaev (1969) yields an adequate description of the spectral change *at low frequencies*. A more accurate description of the spectral change due to scattering requires the calculation of the exact frequency redistribution, using the full expression for the scattering probability, and the correct relativistic form for the electron velocity distribution.

The development of experiments capable of making sensitive measurements of the S-Z effect at high microwave frequencies motivates the need for a more exact calculation of the intensity change due to Compton scattering. The MAX (Meinhold et al. 1993), MSAM (Cheng et al. 1994), and SUZIE (which has actually detected the effect in A2163; Wilbanks et al. 1994) experiments use detectors whose frequency passbands are well into the Wien region of the CMB spectrum. Measurements with these (and similar) experiments seem to offer good prospects for advanced investigations of the effect in its most interesting spectral region (Gould & Rephaeli 1978). The exact—to within a few percent—spectral distribution of the effect is also very important for optimal design of experiments to determine peculiar velocities of clusters from measurements of the kinematic S-Z effect (Sunyaev & Zel'dovich 1980b; Rephaeli & Lahav 1991).

The inadequacy of the Sunyaev & Zel'dovich (1972) treatment has been recognized. An accurate relativistic treatment of the scattering process for any value of τ was given by Wright (1979) and Taylor & Wright (1989). However, the specific calculations presented in these papers concerned Comptonization by very hot intergalactic gas, with gas parameters unlike those in clusters. Sunyaev (1980) partly generalized the Kompaneets-based treatment, but still in the context of a nonrelativistic calculation. Fabbri (1981) calculated relativistic corrections to the Sunyaev & Zel'dovich (1972) treatment and presented some analytic approximations. Relativistic Comptonization has been treated in a general context by Loeb, McKee & Lahav (1991). The work presented here is similar in part to their treatment, but is a more complete and explicit treatment of relativisitic CMB Comptonization for typical temperatures and densities in clusters. Our main goals here are to draw attention to the significance of relativistic corrections to the Kompaneets-based expressions in light of present experimental capabilities of high-frequency measurements of the thermal and kinematic S-Z effects, and to present the full calculation in a readily usable form. In particular, we stress the crucial effect

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of the shift to higher values of the cross-over frequency—at which the effect vanishes—on measurements of peculiar velocities of clusters.

2. COMPTON SCATTERING IN A HOT AND DILUTE GAS

An approximate description of the scattering of an isotropic Planckian radiation field by a nonrelativistic Maxwellian electron gas can be obtained by means of a solution to the Kompaneets (1957) equation. The change in the (spectral) intensity due to the scattering is (Zel'dovich & Sunyaev 1969)

$$\Delta I_{nr} = \frac{2(kT_0)^3}{(hc)^2}\, y g(x) , \quad (1)$$

where T_0 is the CMB temperature, $x = h\nu/kT_0$, and

$$g(x) = \frac{x^4 e^x}{(e^x - 1)^2}\left[\frac{x(e^x + 1)}{e^x - 1} - 4\right] . \quad (2)$$

The spectral form of this thermal S-Z effect is contained in the function $g(x)$, which is zero at the critical value $x_0 = 3.83$, or $\nu_0 = 217$ GHz for the *COBE*-determined value $T_0 = 2.726$ K (Mather et al. 1994). The Comptonization parameter, y, is

$$y = \int (kT_e/mc^2) n\sigma_T\, dl , \quad (3)$$

where n and T_e are the electron density and temperature, σ_T is the Thomson cross section, and the integral is over a line of sight through the cluster.

As stated in the previous section, the low value of τ in clusters (even in Coma—a rich cluster—$\tau < 0.01$; Rephaeli 1987) renders a diffusion approximation inadequate, and necessitates the calculation of the full re-distribution in frequency space, with explicit account taken of the low scattering probability. Moreover, with mean velocities in the range $v \simeq (0.1\text{–}0.3)c$, electrons in the IC space are near-relativistic, so their Maxwellian distribution has to be cast in a relativistic form. The exact photon redistribution can be calculated (Wright 1979) using Chandrasekhar's (1950) expression for the probability of scattering of a photon, originally moving in direction defined by $\mu = \cos(\theta)$ with respect to the electron velocity, into a direction $\mu' = \cos(\theta')$:

$$f(\mu, \mu', \beta) = \tfrac{3}{8}[1 + \mu^2\mu'^2 + \tfrac{1}{2}(1 - \mu^2)(1 - \mu'^2)] . \quad (4)$$

Both these directions are in the electron rest frame. The scattering off an electron moving at a velocity $v = \beta c$ changes the photon frequency from ν to ν'; the logarithmic frequency shift is

$$s = \ln\left(\frac{\nu'}{\nu}\right) = \ln\left(\frac{1 + \beta\mu'}{1 - \beta\mu}\right) . \quad (5)$$

Using the above probability distribution, the kinematics of the scattering (in the Thomson limit) then yields the following expression for the probability for a single scattering of a photon by an electron,

$$P(s, \beta) = \frac{1}{2\gamma^4\beta}\int \frac{(1 + \beta\mu')f(\mu, \mu', \beta)}{(1 - \beta\mu)^3}\, d\mu . \quad (6)$$

The overall distribution of frequency shifts in single scatterings, $P_1(s)$, is obtained by integration over the electron velocity distribution,

$$P_1(s) = \frac{\int \beta^2\gamma^5 e^{-\eta(\gamma-1)} P(s, \beta) d\beta}{\int \beta^2\gamma^5 e^{-\eta(\gamma-1)} d\beta} , \quad (7)$$

with $\eta \equiv mc^2/kT_e$.

The final frequency distribution, $P(s)$, is a weighted sum over the probabilities for any number of scatterings, with the probability for j scatterings multiplied by τ^j. Since in clusters $\tau \lesssim 0.01$, we can adequately approximate $P(s)$ by retaining only the first two terms in the sum. Doing so, we have

$$P(s) = (1 - \tau)\delta(s) + \tau P_1(s) , \quad (8)$$

where $\delta(s)$ is the delta function.

The incident radiation field has a Planckian frequency distribution which has to be convolved with $P(s)$ in order to obtain the final Comptonized spectrum. Integration over the incident spectrum yields the expression for the intensity change:

$$\Delta I = I_0(x)\tau[\Phi(x, \eta) - 1] , \quad (9)$$

where $I_0(x) \equiv [2(kT_0)^3/(hc)^2]x^3/(e^x - 1)$. The relative temperature change is

$$\frac{\Delta T}{T_0} = \frac{(e^x - 1)}{xe^x}\,\tau[\Phi(x, \eta) - 1] . \quad (10)$$

The essence of the calculation is contained in the function $\Phi(x, \eta)$ which is a three-dimensional integral over μ, β, and s. After careful determination of the appropriate integration intervals of μ and β, and with $\exp(-s) = t$, we obtain

$$\Phi(x, \eta) = A(\eta)[\phi_1(x, \eta) + \phi_2(x, \eta)] , \quad (11)$$

where

$$\phi_1(x, \eta) = \int_0^1 \frac{t(e^x - 1)dt}{e^{xt} - 1}\int_{\beta_m}^1 \gamma e^{-\eta(\gamma-1)} d\beta \int_{\mu_m}^1 q(t, \mu, \beta) d\mu , \quad (12)$$

$$\phi_2(x, \eta) = \int_0^1 \frac{(e^x - 1)dt}{t^3(e^{x/t} - 1)}\int_{\beta_m}^1 \gamma e^{-\eta(\gamma-1)} d\beta \int_{-1}^{\mu_M} q(t, \mu, \beta) d\mu , \quad (13)$$

$$q(t, \mu, \beta) = \frac{(1/\beta^2)(3\mu^2 - 1)[(1 - \beta\mu)/t - 1]^2 + (3 - \mu^2)}{(1 - \beta\mu)^2} , \quad (14)$$

and

$$A(\eta) = \frac{3}{32\int_0^1 \beta^2\gamma^5 e^{-\eta(\gamma-1)} d\beta} , \quad (15)$$

The minimal value of β for a given t is $\beta_m = (1 - t)/(1 + t)$; also, $\mu_m = t(1/t - 1 - \beta)/\beta$, and $\mu_M = (t - 1 + \beta)/t\beta$.

Equation (4) is an exact, general expression for the angular probability distribution in Compton scattering. If it is assumed that the scattering is isotropic in the electron rest frame, then a convenient approximation for the scattering probability can be obtained by averaging over the angle between the incident and scattered photon (Rybicki & Lightman 1980):

$$P_{\rm iso}(s, \beta) = \frac{e^s}{(2\gamma\beta)^2}\begin{cases}(1 + \beta)e^s - 1 + \beta , & \dfrac{1 - \beta}{1 + \beta} \le e^s \le 1 \\ 1 + \beta - (1 - \beta)e^s , & 1 < e^s \le \dfrac{1 + \beta}{1 - \beta}\end{cases} , \quad (16)$$

Use of this isotropic probability distribution yields a much more convenient expression for $\Phi(x, \eta)$:

$$\Phi_{\rm iso}(x, \eta) = B(\eta)\phi(x, \eta) , \quad (17)$$

where

$$\phi(x, \eta) = \int_0^1 (e^x - 1)\left[\frac{1}{e^{xt} - 1} + \frac{1}{t^3(e^{x/t} - 1)}\right]\zeta(x, \eta)dt , \quad (18)$$

and

$$\zeta(x, \eta) = \int_{\beta_m}^1 \gamma^3 e^{-\eta(\gamma - 1)}[(1 + \beta)t - 1 + \beta]d\beta . \quad (19)$$

The normalization of the velocity integral is now $B(\eta) = 8A(\eta)/3$.

Note that the intensity change depends explicitly on x, τ, and η, in contrast to the corresponding Sunyaev & Zel'dovich (1972) expression, equation (1), which depends only on x and $y = \tau/\eta$. Thus, in addition to x, in general both η and τ have to be separately specified in the calculation of the exact intensity change. This means that at a given frequency the intensity change may assume different values for a given value of the ratio $y = \tau/\eta$.

To avoid an appreciable loss of accuracy, we have not simplified the expression for $\Phi(x, \eta)$. ΔI was calculated for values of kT_e in the interval 1–50 keV (η = 511.00–10.22). The calculation was repeated with the isotropic form for the scattering probability, yielding practically identical results; thus, equations (17)–(20) can be used instead of equations (11)–(14).

In Figure 1 we compare $(hc)^2/[2\tau(kT_0)^3]\Delta I$, calculated using the more general expression for the scattering probability (eqs. [9], [11]–[15]) with the Kompaneets-based expression, equation (1), for kT_e = 5, 10, and 15 keV. As seen in the figure, the intensity change is appreciably different from what is computed using equation (1). The level of deviation, which obviously increases with T_e, is significant even at 5 keV. At this temperature, the relative intensity difference, $\xi \equiv (\Delta I - \Delta I_{nr})/\Delta I_{nr} \geq 20\%$ in the interval $3.7 < x < 4.2$, and at $x > 11.0$. For $kT_e = 10$ keV, $\xi > 20\%$ in the interval $3.6 \leq x \leq 5.1$, and at $x \geq 10.1$, while for $kT_e = 15$ keV, $\xi > 20\%$ for $3.6 \leq x \leq 6.2$, and at $x \geq 9.8$. The deviations are much larger at higher frequencies: Specifically, for the latter two values of kT_e, $\xi \geq 100\%$ not only very close to the crossover frequency, but also at $\xi \geq 13.4$ and 12.5, respectively. Thus, the higher the temperature, the wider is the frequency range where deviations are significant. Strictly speaking, the neglect of higher order terms in τ (eq. [8]) means that these numerical values are approximate. However, for $\tau \lesssim 0.01$—which is valid in clusters—a typical error is less than a few percent.

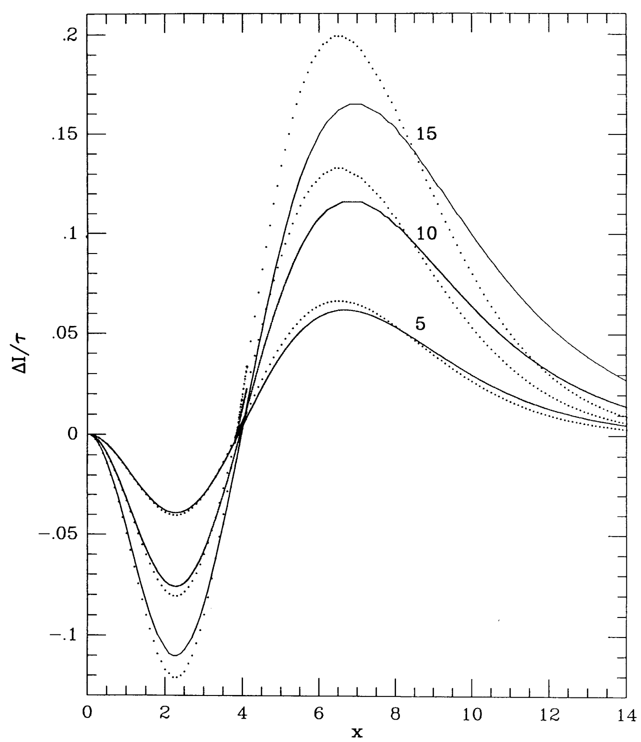

FIG. 1.—The spectral shape of the intensity change due to the Sunyaev-Zel'dovich effect. Solid lines show $\Delta I/\tau$ {in units of $(hc)^2/[2(kT_0)^3]$} for kT_e = 5, 10, and 15 keV. A dotted line next to each solid line shows the intensity change as calculated from the Kompaneets-based expression.

The main characteristic feature of the exact solution is generally a decrease of the intensity change for values of $x \leq$ 8.0–8.4 (in the temperature range kT_e = 1–15 keV), and an inrease at higher values of x, as compared with ΔI_{nr}. The higher the gas temperature, the higher is the Wien deviation from that predicted based on the Kompaneets equation. Thus, the inadequacy of the Kompaneets equation (in the description of Comptonization in *clusters*) largely stems from its nonrelativistic nature.

These features of the more accurate treatment are also manifested in the change in the value of the crossover frequency (where the thermal effect vanishes) which is pushed to highter values with increasing T_e, in contrast to the temperature-independent value $x_0 = 3.83$ of Sunyaev & Zel'dovich (1972). Deviations from the latter value are simply linear in kT_e/mc^2, and indeed a very good approximation in the interval kT_e = 1–50 keV is

$$x_0 \simeq 3.83(1 + 1/\eta) = 3.83(1 + kT_e/mc^2) , \quad (20)$$

For example, at $kT_e = 15$ keV, $x_0 = 3.94$, or $\nu_0 = 223$ GHz, which is higher by 6 GHz from the Sunyaev & Zel'dovich (1972) value. Note that Fabbri's (1981) similar expression somewhat overestimates x_0 at high temperatues.

3. DISCUSSION

Our main conclusion is that the Kompaneets-based expression for the CMB intensity change caused by Compton scattering in clusters of galaxies is not sufficiently accurate for use of the effect as a cosmological and cluster probe. The combined effects of the near-relativistic electron velocities, and low optical thickness to Compton scattering in clusters, account for the significantly different degree of Comptonization on the Wien side than that calculated by Sunyaev & Zel'dovich (1972).

The adequacy of the simple—and convenient—Sunyaev & Zel'dovich (1972) expression for ΔI (eq. [1]) depends on the gas temperature and the frequency at which the effect is determined. In the observed range of IC gas temperatures, $kT_e \leq 15$ keV, eq. (1) is accurate to within 10% for $x \leq 3.5$. Closer to x_0, ΔI differs substantially from ΔI_{nr}, even for $kT_e \sim 5$ keV. Since most previous measurements of the S-Z effect were made at frequencies $x < 1$, equation (1)—and the related expression for the temperature change, $\Delta T/T \simeq -2y$—were generally adequate approximations. The currently operational balloon-borne experiments, such as MAX and MSAM, operate at frequencies in the range $x \sim$ 1.6–11.9. Initial attempts to observe the S-Z effect with these experiments have already been made. The exact expression for ΔI will have to be used in analyses of these measurements.

The exact value of the crossover frequency (x_0) has to be known in order to optimize measurements aimed at determining peculiar velocities of clusters (as is possible with the SUZIE experiment; Wilbanks et al. 1994) from measurements of the kinematic S-Z effect. The additional intensity change due to motion of the cluster at a velocity V along the line of sight is (Sunyaev & Zel'dovich 1980b)

$$\Delta I_k = -\frac{2(kT_0)^3}{(hc)^2}\frac{x^4 e^x}{(e^x - 1)^2}\frac{V}{c}\tau . \tag{21}$$

(V is positive [negative] for a receding [approaching] cluster.) In order for the thermal S-Z effect not to dominate over the kinematic effect, measurements at exactly this frequency are *essential* for this purpose (Rephaeli & Lahav 1991). In the relevant range of gas temperatures in clusters, x_0 can be as high as 3.94, i.e., a value which differs from that of Sunyaev & Zel'dovich (1972) by $\simeq 3\%$. Because of the very steep profile of ΔI near x_0, even such a small deviation is an important consideration in the precise tuning of the detector passband to the correct value of the crossover frequency.

As an illustration of the significance of this relativistic shift in the value of x_0 in determination of peculiar velocities of clusters, consider the specific frequency response of the SUZIE experiment (kindly given to us by T. Wilbanks) near x_0, used in measurement of a cluster whose gas temperature is $kT = 13.8$ keV (such as has been determined for A2163; Arnaud et al. 1992). Integrating ΔI, ΔI_{nv}, and ΔI_k over this spectral response, we find that ignoring the relativistic shift in the value of x_0 introduces a systematic error of ~ 650 km s^{-1} in the deduced value of V. For example, a cluster moving at 500 (-500) km s^{-1} will be incorrectly determined to have a velocity of 1150 (150) km s^{-1}. We conclude that ignoring the (relativistic) shift in the value of the crossover frequency can lead to a grossly inaccurate value of the deduced peculiar velocity of a high-temperature cluster.

Sunyaev (1980) concluded that the original Kompaneets expression is sufficiently accurate for $x < 10$. In contrast, we find that an exact calculation yields significantly different values for the intensity change at lower values of x. The calculation presented here is more accurate: While Sunyaev's calculation adequately accounts for the low τ in clusters, it is still a nonrelativistic calculation which is not as accurate as implied in his paper. As was mentioned in the Introduction, relativistic Comptonization had been treated in a more general context by Loeb, McKee, & Lahav (1991); the work presented here is similar to theirs. However, we use the exact probability distribution, and focus on the parameter regime relevant to clusters. Also, we give full details of the calculation so it can be readily used to compute the intensity and temperature changes at any frequency and gas temperature.

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