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WINDOW FUNCTIONS OF COSMIC MICROWAVE BACKGROUND EXPERIMENTS

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ABSTRACT

We discuss the applicability and derivation of window functions for cosmic microwave background experiments on large and intermediate angular scales. These window functions describe the response of the experiment to power in a particular mode of the fluctuation spectrum. We give general formulae, illustrated with specific examples, for the most common observing strategies.

Subject headings: cosmic microwave background — cosmology: theory — methods: numerical

1. INTRODUCTION

It has become conventional in cosmic microwave background (CMB) studies to describe the sensitivity of experiments by "window" or "filter" functions. These functions describe the response of an experiment to the power in a particular mode of the underlying fluctuation spectrum. Plots of the window functions for various experiments are becoming common (see, e.g., Bond 1990; Crittenden et al. 1993; Gorski 1993; White, Scott, & Silk 1994) and are increasingly used to compare different experiments. We believe that it is useful to understand the derivation and meaning of these window functions from a generic point of view. Furthermore, there are experiments for which analysis by a window function is too complicated to be useful, and it is important to understand what features of an experiment lead to this situation.

We begin by assigning an "ideal" temperature $T(\boldsymbol{n})$ to every point on the sky; this is the temperature that would be measured by a perfect experiment with an infinitely thin beam. Here $\boldsymbol{n}$ is a unit vector which can be expressed in the usual way in terms of the polar and azimuthal angles:

$$\boldsymbol{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ . \tag{1}$$

We can expand the ideal temperature pattern in spherical harmonics:

$$T(\boldsymbol{n}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\boldsymbol{n})\ , \tag{2}$$

where we have removed the constant ($l = 0$) term. In *any* theory of fluctuations, the a_{lm} are treated as random variables; if we assume nothing more than rotational invariance, we must have

$$\langle a_{lm}^* a_{l'm'} \rangle_{\rm ens} \equiv C_l \delta_{l'l} \delta_{m'm}\ , \tag{3}$$

where the angle brackets denote an average over the statistical ensemble of temperature fluctuations. If the fluctuations are Gaussian, all higher point autocorrelation functions are given in terms of the two-point function, and the set of C_l's exhausts the content of the model. Thus, any Gaussian model is completely specified by its predictions for the values of the C_l's, which will depend on some underlying parameters. It is these parameters which we would like to measure with experimental data. The computation of the C_l's has been discussed by several authors, including Peebles & Yu (1970), Wilson & Silk (1981), Bond & Efstathiou (1984, 1987), Vittorio & Silk (1984, 1992), Holtzman (1989), Sugiyama & Gouda (1992), Dodelson & Jubas (1993), and Stompor (1994). In terms of the C_l's, the two-point autocorrelation function of the ideal temperatures $T(\boldsymbol{n})$ is given by

$$\langle T(\boldsymbol{n}_1)T(\boldsymbol{n}_2)\rangle_{\rm ens} = \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)C_l P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\ , \tag{4}$$

where $P_l(x)$ is a Legendre polynomial.

Of course no real experiment can measure the ideal temperatures. In fact, each experiment assigns a temperature, or a temperature "difference," to points on the sky in a way which is generally unique to that experiment. Call the temperature or temperature difference assigned to a point $\boldsymbol{n}$ by a particular experiment $\tilde{T}(\boldsymbol{n})$. In general, it will be linearly related to the ideal temperatures in some neighborhood of $\boldsymbol{n}$:

$$\tilde{T}(\boldsymbol{n}) = \int d\Omega_{\boldsymbol{n}'}\, M(\boldsymbol{n}, \boldsymbol{n}')T(\boldsymbol{n}')\ , \tag{5}$$

where the *mapping function* $M(\boldsymbol{n}, \boldsymbol{n}')$ depends on the detailed experimental strategy. The mapping function is usually too messy to compute in closed form, but it gives us a common way of thinking about experiments, and as we will see it is closely related to the more commonly used window function.

There are a number of different aspects of the experiment that go into the mapping function. One common to all experiments is the *beam profile function* $B(\boldsymbol{n}, \boldsymbol{n}')$, which accounts for the directional response of the antenna. For large-scale experiments such as *COBE* (Smoot et al. 1992) and Far-Infrared Survey (FIRS) (Ganga et al. 1993), which map the sky, this is the only effect, and the mapping function $M(\boldsymbol{n}, \boldsymbol{n}')$ is equal to the beam profile function $B(\boldsymbol{n}, \boldsymbol{n}')$.

However, experiments on smaller scales are usually more complicated. They typically use a chopping strategy, possibly coupled with a smooth scan. All cases are encompassed by the following treatment. We must specify the *beam position function* $\boldsymbol{n}(t)$, or equivalently $\theta(t)$ and $\phi(t)$, which tells us the position of the center of the beam at time t. We must also specify the *weighting* or *lock-in function* $L(t)$, which tells us how different portions of the beam trajectory are weighted in computing an experimental temperature, and the overall *normalization* N.

Now consider a particular time interval, labeled by i, which runs from $t = t_i - \frac{1}{2}\delta_i$ to $t = t_i + \frac{1}{2}\delta_i$. To this time interval we assign an average position $\bar{\boldsymbol{n}}_i$, given in terms of $\bar{\theta}_i$ and $\bar{\phi}_i$ by equation (1), where

$$\bar{\theta} = \frac{1}{\delta_i}\int_{t_i-\delta_i/2}^{t_i+\delta_i/2} dt\, \theta(t)\,, \quad \bar{\phi}_i = \frac{1}{\delta_i}\int_{t_i-\delta_i/2}^{t_i+\delta_i/2} dt\, \phi(t)\,. \tag{6}$$

To the average position $\bar{\boldsymbol{n}}_i$ we assign the temperature

$$\tilde{T}(\bar{\boldsymbol{n}}_i) \equiv \frac{N}{\delta_i}\int_{t_i-\delta_i/2}^{t_i+\delta_i/2} dt\, L(t)\int d\Omega_{\boldsymbol{n}'}\, B[\boldsymbol{n}(t), \boldsymbol{n}']T(\boldsymbol{n}')\,. \tag{7}$$

Equation (7) is completely general and applies to all experiments. It gives an implicit definition of the mapping function $M(\boldsymbol{n}, \boldsymbol{n}')$; compare equation (7) with equation (5). Below we will discuss a variety of possible choices for the beam position function $\boldsymbol{n}(t)$ and the lock-in function $L(t)$.

To summarize, an experiment is completely specified by giving the beam profile function $B(\boldsymbol{n}, \boldsymbol{n}')$, the beam position function $\boldsymbol{n}(t)$, the lock-in function $L(t)$, and the normalization N. All four must be given explicitly before the results of an experiment can be analyzed or understood. A measured value of "ΔT" without specification of all four of these experimental ingredients cannot be interpreted.

Given the mapping function $M(\boldsymbol{n}, \boldsymbol{n}')$, we can compute the *window function* $W_l(\boldsymbol{n}, \boldsymbol{n}')$ as follows. Let us begin with the two-point autocorrelation function of the *experimental* temperatures:

$$\begin{aligned}\langle \tilde{T}(\boldsymbol{n}_1)\tilde{T}(\boldsymbol{n}_2)\rangle_{\rm ens} &= \int d\Omega_{\boldsymbol{n}_1'}\int d\Omega_{\boldsymbol{n}_2'}\, M(\boldsymbol{n}_1, \boldsymbol{n}_1')M(\boldsymbol{n}_2, \boldsymbol{n}_2') \\ &\quad\times \langle T(\boldsymbol{n}_1')T(\boldsymbol{n}_2')\rangle_{\rm ens} \\ &= \int d\Omega_{\boldsymbol{n}_1'}\int d\Omega_{\boldsymbol{n}_2'}\, M(\boldsymbol{n}_1, \boldsymbol{n}_1')M(\boldsymbol{n}_2, \boldsymbol{n}_2') \\ &\quad\times \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)C_l P_l(\boldsymbol{n}_1'\cdot\boldsymbol{n}_2') \\ &= \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)C_l W_l(\boldsymbol{n}_1, \boldsymbol{n}_2)\,,\end{aligned} \tag{8}$$

where the last equation defines the window function:

$$W_l(\boldsymbol{n}_1, \boldsymbol{n}_2) \equiv \int d\Omega_{\boldsymbol{n}_1'}\int d\Omega_{\boldsymbol{n}_2'}\, M(\boldsymbol{n}_1, \boldsymbol{n}_1')M(\boldsymbol{n}_2, \boldsymbol{n}_2')P_l(\boldsymbol{n}_1'\cdot\boldsymbol{n}_2')\,. \tag{9}$$

Often the window function is plotted in the literature as a function of l only. This case corresponds to the window function at zero-lag, $W_l(\boldsymbol{n}, \boldsymbol{n})$. This is usually independent of the choice of $\boldsymbol{n}$ to a *very* good approximation. To specify slightly, we will assume that $W_l(\boldsymbol{n}, \boldsymbol{n})$ is indeed independent of $\boldsymbol{n}$ and use the shorthand notation $W_l \equiv W_l(\boldsymbol{n}, \boldsymbol{n})$. We will, however, also be interested in the complete window function $W_l(\boldsymbol{n}_1, \boldsymbol{n}_2)$.

2. SIMPLE WINDOW FUNCTIONS

2.1. *The Beam Profile*

As already noted, the simplest example of a window function is that which arises when the finite size of the beam is the only effect, as is the case for *COBE* (Smooth et al. 1992; Wright et al. 1994) and FIRS (Ganga et al. 1993). In this case, the mapping function $M(\boldsymbol{n}, \boldsymbol{n}')$ is equal to the beam width function $B(\boldsymbol{n}, \boldsymbol{n}')$. If the beam profile is isotropic, then $B(\boldsymbol{n}, \boldsymbol{n}')$ is a function of $\boldsymbol{n}\cdot\boldsymbol{n}'$ alone, and we specialize to this case from here on. We can expand $B(\boldsymbol{n}, \boldsymbol{n}')$ in Legendre polynomials:

$$B(\boldsymbol{n}, \boldsymbol{n}') = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)B_l P_l(\boldsymbol{n}\cdot\boldsymbol{n}')\,, \tag{10}$$

and rewrite the P_l, using the addition theorem for spherical harmonics, in equation (9) to find that the window function is simply

$$W_l(\boldsymbol{n}, \boldsymbol{n}') = B_l^2 P_l(\boldsymbol{n}\cdot\boldsymbol{n}')\,. \tag{11}$$

What we call B_l^2 is called G_l by Wright et al. (1994). For a Gaussian beam profile,

$$B(\boldsymbol{n}, \boldsymbol{n}') = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\theta^2}{2\sigma^2}\right), \tag{12}$$

where $\theta \equiv \cos^{-1}(\boldsymbol{n}\cdot\boldsymbol{n}')$, we have to a very good approximation (Silk & Wilson 1980; Bond & Efstathiou 1984; White 1992):

$$B_l(\sigma) = \exp\left[-\tfrac{1}{2}l(l+1)\sigma^2\right]. \tag{13}$$

In general, the effect of finite beam width is to provide a high-l cutoff at scales of the beam size $l \sim \sigma^{-1}$. We note in passing that uncertainties in the value of σ for a Gaussian beam profile, or more generally the shape of the beam profile, can result in significant uncertainties in comparing experiment with theory, especially if the high-l cutoff is in a range when C_l is changing rapidly with l.

2.2. *Constant Elevation Scans*

We now turn to small-scale experiments which use nontrivial beam position functions $\boldsymbol{n}(t)$ and lock-in functions $L(t)$. For these, it is possible to significantly simplify equations (7) and (9) only if the scan is performed at a constant θ. (Note that this needs to be the case only for one particular choice of coordinates, which need not be equivalent to any of the usual choices.) In this case the complete window function $W_l(\boldsymbol{n}, \boldsymbol{n}')$ is a function only of $|\phi - \phi'|$, an enormous simplification. We specialize to this case for now and will return to discuss the general case later.

We first consider a stepped (as opposed to smooth) scan. In this case the beam is centered at a particular point $\bar{\boldsymbol{n}} = (\theta_0, \phi_0)$ on the sky and then "chopped" back and forth in the ϕ-direction. The instantaneous beam position $\boldsymbol{n}(t)$ is given by

$$\begin{aligned}\theta(t) &= \theta_0\,,\\ \phi(t) &= \phi_0 + \alpha_0 \sin(\omega_c t)\,,\end{aligned} \tag{14}$$

where α_0 is half of the peak-to-peak chop angle, and $\omega_c/2\pi$ is the chop frequency. (In practice it is a few hertz, but the window function turns out to be independent of ω_c.) Note that the angular separation on the sky is measured by $\phi \sin\theta_0$.

In the case of a smooth scan, the beam is swept smoothly, at an angular velocity of ω_s, in addition to being chopped. The instantaneous beam position $\boldsymbol{n}(t)$ is now given by

$$\begin{aligned}\theta(t) &= \theta_0\,,\\ \phi(t) &= \phi_0 + \omega_s t + \alpha_0 \sin(\omega_c t)\,,\end{aligned} \tag{15}$$

and the data must be binned, as in equation (7), by integrating t over the duration time δ of a bin. It practice δ is always a multiple of the period of the chop, that is, $\omega_c\,\delta/2\pi$ is an integer.

We are now in a position to compute the window function, assuming either equation (14) or equation (15) for the instantaneous beam position. We further assume (again always the case in practice) that the lock-in function $L(t)$ has the same periodicity as the chopping function and that $\omega_s \ll \omega_c$. Again, making use of the addition theorem for spherical harmonics, we ultimately find from equations (7) and (9) that

$$W_l(\phi) = N^2 B_l^2(\sigma)\,\frac{4\pi}{2l+1}\sum_{m=-l}^{l} |Y_{lm}(\theta, 0)|^2 \times L_m^2(\alpha_0) S_m^2(\Delta\phi)\cos(m\phi)\,, \quad (16)$$

where, up to an irrelevant phase,

$$L_m(\alpha_0) \equiv \frac{\omega_c}{2\pi}\int_{-\pi/\omega_c}^{+\pi/\omega_c} dt\, L(t) e^{im\alpha_0 \sin(\omega_c t)}\,, \quad (17)$$

$$S_m(\Delta\phi) \equiv j_0\left(\frac{m\,\Delta\phi}{2}\right) = \frac{\sin(m\,\Delta\phi/2)}{m\,\Delta\phi/2}\,, \quad (18)$$

for a smooth scan. Here $\Delta\phi = \omega_s\,\delta$ is the size of the bins in ϕ. For a stepped scan, $S_m = 1$. For more details in the context of specific choices of $L(t)$, see Dodelson & Jubas (1993) and White, Krauss, & Silk (1993).

If we can neglect the curvature of the line and assume that it is an arc of a great circle (usually a very good approximation), then we can set $\theta = \pi/2$ and let ϕ be the angle on the sky; in this case

$$W_l(\phi) = N^2 B_l^2(\sigma)\sum_{r=0}^{l}\frac{(2l-2r)!(2r)!}{[2^l r!(l-r)!]^2} \times L_{l-2r}^2(\alpha_0) S_{l-2r}^2(\Delta\phi)\cos[(l-2r)\phi]\,, \quad (19)$$

which is easy to implement numerically. Note that now ϕ, $\Delta\phi$, and α_0 are all defined as angles *on the sky*.

2.3. *The Lock-in*

The simplest lock-in function is that for a "square wave chop" recently used by Tenerife (Hancock et al. 1994), MSAM (Cheng et al. 1994), OVRO (Myers, Readhead, & Lawrence 1993), and Python (Dragovan et al. 1994). In this strategy the temperature assigned to ϕ_0 is a weighted sum of temperatures along a line, which we assume to be of constant elevation. The telescope moves rapidly between the observed points, stopping and taking data at set positions on the line. The weights assigned to points on the sky for three different "switching strategies" are listed in Table 1. In our notation such a strategy is implemented by taking $L(t)$ to be a linear combination of Dirac delta functions. For a two-beam, three-beam, or four-beam switching strategy, we have

$$\frac{\omega_c}{2\pi}L(t) = \begin{cases} +\delta(t+t_c) - \delta(t-t_c) & \text{(2-beam)}\,,\\ -\tfrac{1}{2}\delta(t+t_c) + \delta(t) - \tfrac{1}{2}\delta(t-t_c) & \text{(3-beam)}\,,\\ +\tfrac{1}{4}\delta(t+t_c) - \tfrac{3}{4}\delta(t+\xi t_c) \\ \quad + \tfrac{3}{4}\delta(t-\xi t_c) - \tfrac{1}{4}\delta(t-t_c) & \text{(4-beam)}\,, \end{cases} \quad (20)$$

where $\xi = (2/\pi)\sin^{-1}(1/3)$ and $t_c = \pi/2\omega_c$ is the time to chop from $\phi = \phi_0$ to $\phi = \phi_0 + \alpha_0$. In this case the mapping function $M(\boldsymbol{n}, \boldsymbol{n}')$ defined through equation (7) reduces to a weighted sum of beam profile functions $B(\boldsymbol{n}, \boldsymbol{n}')$. From equation (17) we find immediately that, up to an irrelevant phase,

$$L_m(\alpha_0) = \begin{cases} 2\sin(m\alpha_0) & \text{(2-beam)}\,,\\ 2\sin^2(m\alpha_0/2) & \text{(3-beam)}\,,\\ \tfrac{1}{2}[\sin(m\alpha_0) - 3\sin(\tfrac{1}{3}m\alpha_0)] & \text{(4-beam)}\,. \end{cases} \quad (21)$$

Notice that $L_m(\alpha_0)$ scales as α_0^{n-1} for the n-beam switching strategy. In general, the window function for any kind of differencing experiment is suppressed at low l, since any long-wavelength perturbation is removed by the differencing. Since the low-l cutoff is controlled by α_0 while the high-l cutoff is specified by σ, one can increase both the height and width of W_l by separating these scales as much as possible.

To make contact with forms of W_l frequently quoted in the literature, we note that we can substitute equation (21) into equation (19) with $\phi = 0$ (zero lag) and use the addition theorem for spherical harmonics to obtain

$$W_l = B_l^2(\sigma)\begin{cases} 2[1 - P_l(\cos 2\alpha_0)] & \text{(2-beam)}\,,\\ \tfrac{1}{2}[3 - 4P_l(\cos\alpha_0) + P_l(\cos 2\alpha_0)] & \text{(3-beam)}\,,\\ \tfrac{1}{8}[10 - 15P_l(\cos\tfrac{2}{3}\alpha_0) \\ \quad + 6P_l(\cos\tfrac{4}{3}\alpha_0) - P_l(\cos 2\alpha_0)] & \text{(4-beam)}\,. \end{cases} \quad (22)$$

Three other illustrative choices of the lock-in function for differencing experiments are the "square-wave lock-in,"

$$L(t) = 2\,\mathrm{sgn}[\alpha_0\sin(\omega_c t)]\,, \quad L_m(\alpha_0) = 2H_0(m\alpha_0)\,, \quad (23)$$

used by SP91 (Gaier et al. 1992) and ARGO [de Bernardis et al. 1994; the ARGO team define ΔT as exactly half the SP91 definition shown here, so for ARGO $L_m = H_0(m\alpha_0)$], the "sine-wave lock-in,"

$$L(t) = \pi\sin(\omega_c t)\,, \quad L_m(\alpha_0) = \pi J_1(m\alpha_0)\,, \quad (24)$$

used by the Millimeter-wave Anisotropy Experiment (MAX) (Gundersen et al. 1993; Meinhold et al. 1993) and the double-angle "cosine lock-in,"

$$L(t) = \pi\cos(2\omega_c t)\,, \quad L_m(\alpha_0) = \pi J_2(m\alpha_0)\,, \quad (25)$$

used by Saskatoon (Wollack et al. 1993). Here $H_0(x)$ is the Struve function, and $J_n(x)$ are Bessel functions of the first kind. The numerical prefactors are chosen so that

$$\frac{\omega_c}{2\pi}\int_{-\pi/\omega_c}^{+\pi/\omega_c} dt\,|L(t)| = 2\,, \quad (26)$$

which is a common way of normalizing an experiment (more on this below). Sometimes the $L_m(\alpha_0)$'s of equations (23) and (24) are approximated by the $L_m(\alpha_0)$ for a "two-beam chop," as given in equation (20), but this can result in significant errors. For example, after taking into account the normalizations used by these experiments, we find that for SP91 the approximation is off 20% (see Dodelson & Stebbins 1994) while for MAX it

TABLE 1

WEIGHTS FOR SWITCHING STRATEGIES

	$(\phi - \phi_0)\alpha_0$				
SWITCHING STRATEGY	-1	$-\frac{1}{3}$	0	$+\frac{1}{3}$	$+1$
Two-beam	$+1$	0	0	0	-1
Three-beam	$-\frac{1}{2}$	0	$+1$	0	$-\frac{1}{2}$
Four-beam	$+\frac{1}{4}$	$-\frac{3}{4}$	0	$+\frac{3}{4}$	$-\frac{1}{4}$

differs by 10%. Also, the Saskatoon lock-in function of equation (25) is only roughly approximated by the three-beam result of equation (21).

2.4. *The Normalization*

Finally, we consider the normalization factor N. For SP91 and MAX this is chosen so that if there is a sharp boundary between two regions of constant temperatures T_1 and T_2, then aiming at a point on the boundary, and chopping perpendicular to the boundary, gives $\tilde{T} = T_2 - T_1$. For SP91, the normalization is computed assuming a perfect, pointlike beam, corresponding to $\sigma = 0$ in equation (13). For a perfect beam, the lock-in factor $L_m(\alpha_0)$ given in equations (21–25) is already normalized so that $N = 1$. However, if the normalization is done assuming the actual beam profile of the experiment (as is the case of MAX and Saskatoon), then there are corrections which must be computed. The result for MAX was presented in Srednicki et al. (1993), but here we give a more general treatment.

We first make the "flat sky approximation" near the $T_1 - T_2$ boundary, which we shall take to be the line of longitude $\phi = 0$. We treat $x \equiv \phi$ and $y \equiv (\pi/2) - \theta$ as Cartesian coordinates. For $\boldsymbol{n} = (0, 0)$ and $\boldsymbol{n}' = (x, y)$, equation (12) for the beam profile becomes

$$B(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} . \quad (27)$$

Equation (5) can now be written as

$$\tilde{T}(0, 0) = \int_{-\infty}^{+\infty} dx\, dy M(x, y) T(x, y) , \quad (28)$$

where the mapping function is given by

$$M(x, y) = \frac{N\omega_c}{(2\pi)^2\sigma^2} \int_{-\pi/\omega_c}^{+\pi/\omega_c} dt\, L(t) \times \exp\left\{-\frac{[x - \alpha_0 \sin(\omega_c t)]^2 + y^2}{2\sigma^2}\right\} . \quad (29)$$

Take the temperature profile to be $T(x, y) = T_0\, \theta(x)$, where $\theta(x)$ is the step function, and demand that $\tilde{T}(0, 0) = T_0$. For the MAX lock-in function given by equation (24), we get

$$1 = \tfrac{1}{4} N \int_{-\pi}^{+\pi} dr \sin r \operatorname{erf}(\gamma \sin r) = N(\tfrac{1}{2}\gamma\sqrt{\pi})\, {}_1F_1(\tfrac{1}{2}, 2; -\gamma^2) , \quad (30)$$

where $\gamma = \alpha_0/\sqrt{2}\sigma$, erf is the error function, and ${}_1F_1$ is the confluent hypergeometric function. For MAX, with $\alpha_0 = 0\overset{\circ}{.}65$ and $\sigma = 0.425 \times 0\overset{\circ}{.}5$, this gives $N^2 = 1.13$. (Note that it is N^2 which appears in the window function.)

For Saskatoon, the only change is that the normalizing temperature profile is T_0 where $M(x, y) \geq 0$, and zero elsewhere. The mapping function follows from substituting equation (25) into equation (29), which also implicitly defines the region where $T(x, y) \neq 0$ in equation (28). Setting $\tilde{T}(0, 0)$ to T_0 as before gives

$$1 = \frac{N}{2\sigma\sqrt{2\pi}} \int dx \int_{-\pi}^{\pi} dr \cos(2r) \exp\left[-\frac{(x - \alpha_0 \sin r)^2}{2\sigma^2}\right] , \quad (31)$$

where the integration is over the region of x where $M(x, y) \geq 0$. With $\alpha_0 = 2\overset{\circ}{.}45$ and $\sigma = 0.425 \times 1\overset{\circ}{.}44$, we find $N^2 = 1.74$, quite a large correction. Also note that the mapping $M(x, y)$ defined by equation (25) and equation (29) is not well approximated by a square wave chop pattern, which would be three Gaussian beam profiles at $x = -\alpha_0$, 0, and α_0 with weights $-\frac{1}{2}$, $+1$, and $-\frac{1}{2}$, respectively.

3. OTHER STRATEGIES

One other "differencing" strategy that has been proposed recently is used by the "White Dish" experiment (Tucker et al. 1993), which assigns to the point $\boldsymbol{n}$ a temperature $\tilde{T}(\boldsymbol{n})$ which is given by a particular weighted average of measured temperatures in a circle around that point. The actual strategy used is *very* difficult to model effectively. However, if we modify their "method II" analysis to neglect binning, then it is straightforward to compute $W_l = W_l(\boldsymbol{n}, \boldsymbol{n})$. While the off-diagonal elements cannot be simply constructed, a numerical procedure similar to that used in Srednicki et al. (1993) would be feasible.

To get W_l, we first rotate coordinates so that $\boldsymbol{n}$ is at $\theta = 0$ with the circle being in ϕ at fixed $\theta = \theta_0$. The analog of equation (7) is now to extract the nth harmonic of the temperature around the circle, which corresponds to a window function of the form (see eq. [16])

$$W_l = \frac{4\pi}{2l+1} N^2 B_l^2(\sigma) \frac{\pi^2}{2} | Y_{ln}(\theta_0, 0) |^2$$
$$\simeq \frac{\pi^2}{2} N^2 B_l^2(\sigma) \left[\frac{(l+n)!}{(l-n)!\, l^{2n}}\right] J_n^2(l\theta_0) , \quad (32)$$

where in the last line the limit $\theta_0 \ll 1$ has been used. The term in parentheses on the last line is very close to unity for $l \gg n$. Assuming that a temperature profile which is T_0 for both $0 \leq \phi < \pi/2$ and $\pi \leq \phi < 3\pi/2$ and zero elsewhere would be assigned a temperature difference of $\tilde{T} = T_0$, when $n = 2$, we get $N = 1$. A less artificial normalization would require more information than is specified in the paper (Tucker et al. 1993). The White Dish experiment has $\theta_0 = 14'$, $\sigma = 0.425 \times 12'$, and $n = 2$, with the resulting "temperatures" binned into four positions $T_1, \ldots, T_4$ in a square of side $\theta' = 23\overset{'}{.}6$. The four temperatures are assigned to consecutive corners clockwise around the square (Tucker et al. 1993).

Equation (32) can be compared with the window function for a "square wave chop" procedure where we simply sum the temperatures in a square: $\tilde{T} = \frac{1}{2}(T_1 - T_2 + T_3 - T_4)$, neglecting how they were measured, to obtain

$$W_l = B_l^2(\sigma)[1 - 2P_l(\cos\theta') + P_l(\cos\sqrt{2}\theta')] .$$

Assuming that $N = 1$, the two window functions peak in the same place ($l \sim 500$) but differ by 20% at the peak and have a different scaling with l off the peak. Additionally, neither of these methods accurately reflects the correlations induced by coarse binning of the data.

With an analysis procedure as difficult to model as White Dish, the window function approach is of limited utility and one should resort to Monte Carlo simulations of the observing strategy, for each theory being tested. Alternatively, the applicability of a window function should be kept in mind when the analysis procedure is designed.

4. SCANS AT VARYING ELEVATION

If data points are not taken at constant elevation, such as in the γ UMi scan of MAX (Gundersen et al. 1993), and chopping is used, computing the window function at nonzero lag ($\boldsymbol{n} \neq \boldsymbol{n}'$) is impossible analytically. The information which is needed to compare data with a theory is the autocorrelation function of the experimental temperatures, equation (8), computed with the C_l's of the theory in question. It is then usually easier to compute this directly, using numerical methods, than to compute the complete window function $W_l(\boldsymbol{n}, \boldsymbol{n}')$. However, even numerical methods become cumbersome if the observing strategy is complex. Accounting for chopping generally requires that a double integral be done numerically (Srednicki et al. 1993). If there is, in addition, a smooth (as opposed to stepped) scan, then a quadruple integral must be done numerically, and this is not feasible in general. If the data is binned finely enough, then the effects of the smooth scan are small, and this is not a problem. Here "finely enough" means $\Delta\phi \lesssim \alpha_0$, which is fortunately the case for MAX γ UMi.

5. CONCLUSIONS

Using the formulae presented in this paper it is possible to compute window functions for most of the current large- and intermediate-scale CMB anisotropy experiments. We stress, however, that there exist several generic cases in which the window function approach is not the optimal method of analysis. These are when the correlation matrix $\langle \tilde{T}(\boldsymbol{n}_1)\tilde{T}(\boldsymbol{n}_2)\rangle_{\rm ens}$ is anisotropic (such as in the case of the γ UMi scan of the MAX experiment or multiple scans of the SP91 experiment) or if the experimental procedure makes analytic calculation of the window function difficult. In these cases, it is generally much easier to compute equation (8) directly (by numerical methods) or to do a Monte Carlo analysis of the experiment than it is to try to calculate the window function $W_l(\boldsymbol{n}, \boldsymbol{n}')$.

Both diagonal and off-diagonal window functions are easy to construct for experiments in which the scanning and chopping directions are constant, regardless of the type of chopping (square wave, sine, cosine) and regardless of whether the scan is smooth or stepped. For these cases, the window function provides an efficient method for comparing theory with data. However, with the current refined state of CMB anisotropy measurements, it is important to use the *right* window function. A simple stepped-scan, square-wave chop approximation to all experiments is no longer accurate enough for the quality of data now available.

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