THE ASTROPHYSICAL JOURNAL, 429:797–807, 1994 July 10

MAGNETOCENTRIFUGALLY DRIVEN FLOWS FROM YOUNG STARS AND DISKS. II. FORMULATION OF THE DYNAMICAL PROBLEM

FRANK H. SHU AND JOAN NAJITA
Astronomy Department, University of California, Berkeley, CA 94720

STEVEN P. RUDEN
Physics Department, University of California, Irvine, CA 92717

AND

SUSANA LIZANO
Instituto de Astronomia, Universidad Nacional Autonoma de Mexico, Apdo. Postal 70-264, DF 04510 Mexico City, Mexico
Received 1993 September 9; accepted 1994 January 11

ABSTRACT

We formulate the dynamical problem of a cool wind centrifugally driven from the magnetic interface of a young star and an adjoining Keplerian disk. We examine the situation for mildly accreting T Tauri stars that rotate slowly as well as rapidly accreting protostars that rotate near break-up. In both cases a wind can be driven from a small X-region just outside the stellar magnetopause, where the field lines assume an open geometry and are rooted to material that rotates at an angular speed equal both to the local Keplerian value and to the stellar angular speed. Assuming axial symmetry for the ideal magnetohydrodynamic flow, which requires us to postpone asking how the (lightly ionized) gas is loaded onto field lines, we can formally integrate all the governing equations analytically except for a partial differential equation that describes how streamlines spread in the meridional plane. Apart from the difficulty of dealing with PDEs of mixed type, finding the functional forms of the conserved quantities along streamlines—the ratio β of magnetic field to mass flux, the specific energy H of the fluid in the rotating frame, and the total specific angular momentum J carried in the matter and the field—constitutes a standard difficulty in this kind of (Grad-Shafranov) formalism. Fortunately, because the ratio of the thermal speed of the mass-loss regions to the Keplerian speed of rotation of the interface constitutes a small parameter ϵ, we can attack the overall problem by the method of matched asymptotic expansions. This procedure leads to a natural and systematic technique for obtaining the relevant functional dependences of β, H, and J. Moreover, we are able to solve analytically for the properties of the flow emergent from the small transsonic region driven by gas pressure without having to specify the detailed form of any of the conserved functions, β, H, and J. This analytical solution provides inner boundary conditions for the numerical computation in a companion paper by Najita & Shu of the larger region where the main acceleration to terminal speeds occurs.

Subject headings: accretion, accretion disks — MHD — stars: formation — stars: magnetic fields — stars: mass-loss — stars: pre–main sequence — stars: rotation

1. INTRODUCTION

1.1. *Summary of Paper I*

A magnetocentrifugal mechanism for driving winds from strongly magnetized, rapidly rotating, astrophysical sources constitutes perhaps the leading contender for explaining collimated outflows from a wide range of objects, ranging from radio jets in active galactic nuclei (AGNs) to optical jets and bipolar flows in young stellar objects (YSOs) (see, e.g., Blandford & Payne 1982; Pudritz & Norman 1983, 1986; Königl 1989; Wardle & Königl 1993). In the first of this series (Shu et al. 1994, hereafter Paper I), we demonstrated that the specific mechanism proposed by Shu et al. (1988, hereafter SLRN)—an X-celerator wind driven magnetocentrifugally along open field lines from the equator of a protostar spun up to breakup speed by rapid accretion from an adjoining Keplerian disk—could be generalized to accommodate the case of slow rotators such as classical T Tauri stars (CTTSs) which also possess disks but rotate typically at only a tenth of breakup (see, e.g., Edwards et al. 1993). This development is fortunate because in the interim we have developed a mathematical formalism for calculating the detailed properties of the flow in the former case under the simplifying assumptions of time independence and axial symmetry. To make the generalization to the slow rotator case, we only need to suitably scale the nondimensional results that we have already obtained for the rapid rotator case.

1.2. *Aims of Present Paper*

Addressing the problem of massive winds from protostars, SLRN outlined the form of the so-called Grad-Shafranov equation (hereafter GSE, Grad & Rubin 1958; Shafranov 1966)[1] that one could use, in parallel with the Bernoulli equation (BE), to analyze the steady axisymmetric X-celerator flow under the approximation of ideal magnetohydrodynamics (MHD). (See also Lovelace, Berk, & Contopoulos 1991.) The key to simplifying the calculations lies in taking advantage of the existence of a small parameter in the problem. This parameter, the ratio ϵ of the isothermal sound speed a_x to the orbital

[1] We follow the convention of the plasma physics community in this nomenclature (GSE), but we note that the use of scalar magnetic potentials to describe two-dimensional magnetostatic configurations in solar physics was pioneered by Dungey (1953), while the use of scalar streamfunctions to describe steady two-dimensional fluid flows have been extant since their introduction by Stokes (1842).

speed $R_x \Omega_x$ at the X-point of the effective potential,

$$\epsilon \equiv \frac{a_x}{R_x \Omega_x}, \tag{1.1}$$

was introduced in Paper I to obtain the important scalings from a physical point of view. In the present paper, we use the method of matched asymptotic expansions to formalize the procedure and to obtain more detailed results.

In particular, a solution of the GSE by our techniques (see below and Najita & Shu 1994, hereafter Paper III) yields an a priori calculation of the divergence of the streamlines and field lines in the meridional plane (see, e.g., Heinemann & Olbert 1978; Sakurai 1985) that is usually avoided in other treatments of the problem. The present paper presents a full derivation of the SLRN formalism; we also indicate a practical method for the numerical solution of the resulting equations in the crucial part of the flow between the sonic and Alfvénic transitions, where the main acceleration takes place.[2]

2. BASIC EQUATIONS

2.1. *Natural Scalings and Numerical Examples*

To begin, we introduce the following fiducial units of length, velocity, density, and magnetic field (cf. eqs. [2.5b] and [2.5c] of Paper I, hereafter eqs. [I.2.5b] and [I.2.5c]):

$$R_x\,, \quad \Omega_x R_x\,, \quad \dot{M}_w/4\pi R_x^3 \Omega_x\,, \quad (\Omega_x \dot{M}_w/R_x)^{1/2}\,. \tag{2.1a}$$

In the above, Ω_x equals the Keplerian angular velocity at a distance R_x from a star of mass M_*:

$$\Omega_x = \left(\frac{GM_*}{R_x^3}\right)^{1/2}, \tag{2.1b}$$

and $\dot{M}_w$ is the mass-loss rate in the X-wind, which we may obtain either observationally from direct empirical measurements, or theoretically by using equations (I.1.1.) and (I.4.7a) if we know the disk accretion rate $\dot{M}_D$. Implicit in our model is the assumption that the stellar magnetic fields (and any that they induce in the disk) rotate at an angular velocity $\Omega_* = \Omega_x$.

2.2. *Nondimensional Equations for Isothermal Ideal MHD Flow*

If we confine our attention to the region of the sonic transition (which determines the inner boundary conditions for the rest of the flow), the assumption of isothermality represents a good approximate replacement for a full heating and cooling calculation (cf. Ruden, Glassgold, & Shu 1990, hereafter RGS). Elsewhere in the flow, we may ignore the role of thermal pressure altogether, i.e., we may take the limit of a cold flow, $\epsilon \to 0$.

The calculations of RGS and Paper I also indicate that slip speeds in the X-wind due to ambipolar diffusion amount to a fraction of a km s^{-1}; as a consequence, we may adopt to good approximation the assumptions of ideal MHD. With the nondimensionalization (2.1a), we can write the steady equations in a frame of reference that rotates at the angular velocity Ω_x as

$$\nabla \cdot (\rho \boldsymbol{u}) = 0\,, \tag{2.2a}$$

$$\nabla\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + (2\boldsymbol{e}_z + \nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\frac{\epsilon^2}{\rho}\nabla\rho - \nabla V_{\text{eff}} + \frac{1}{\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\,, \tag{2.2b}$$

$$\boldsymbol{B} \times \boldsymbol{u} = 0\,, \tag{2.2c}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{2.2d}$$

where V_{eff} is the effective potential associated with a rotating frame of reference and the gravitational field of a centrally concentrated star:

$$V_{\text{eff}} = -\frac{1}{r} - \frac{1}{2}\varpi^2 + \frac{3}{2}\,. \tag{2.3}$$

In equation (2.2b) the term $2\boldsymbol{e}_z \times \boldsymbol{u}$ represents the Coriolis acceleration associated with being in a dimensionless reference frame that rotates at unit angular speed about the z-axis. In equation (2.3) $r \equiv (\varpi^2 + z^2)^{1/2}$ and ϖ are, respectively, the radii in spherical (r, θ, φ) and cylindrical (ϖ, φ, z) coordinate systems with origin at the center of the star. We have also defined the arbitrary constant in V_{eff} so as to make its numerical value conveniently equal to zero at the X-point, $r = \varpi = 1$, where $\nabla V_{\text{eff}} = 0$.

Equation (2.2c) does not yield the most general solution of the field-freezing equation. For steady axisymmetric flow, however, it is possible to show that $\boldsymbol{u}$ must generally be proportional to $\rho^{-1}\boldsymbol{B}$, except for an additive term that corresponds to fluid rotation about the z-axis at an arbitrary uniform angular speed (Mestel 1968). If we choose to work in the frame that rotates with the star and the X-region, this arbitrary term vanishes because we impose the boundary condition that $\boldsymbol{u}$ and $\rho^{-1}\boldsymbol{B}$ must both approach zero as we go deep into the star (for the protostellar case) or the X-region of the disk (for the T Tauri star case). We may then write the condition $\boldsymbol{u} \propto \rho^{-1}\boldsymbol{B}$ as

$$\boldsymbol{B} = \beta\rho\boldsymbol{u}\,, \tag{2.4}$$

where equations (2.2a) and (2.2d) require that $\boldsymbol{u} \cdot \nabla\beta = 0$, i.e., that β is constant on a streamline.

The quantity β governs how matter is loaded onto field lines and has a peculiar property near the midplane $z = 0$ that we should elucidate at the outset. For definiteness, focus on the T Tauri star case depicted in Figure 2*b* of Paper I. An outflow occurs from above and below the disk, so the magnetic field $\boldsymbol{B}$ threading *through* the disk is, say, parallel to $\boldsymbol{u}$ in the wind above the disk, and antiparallel below. In other words, β must have opposite signs above and below $z = 0$. On the other hand, β has a nonzero value as $z \to 0$ from either above or below, if a field strong enough to launch a magnetocentrifugally driven wind exists in the first place. The dual requirement seemingly involves a contradiction: if β suffers a finite jump, from a positive value to its opposite negative value, as we cross the midplane, we can have $\nabla \cdot \boldsymbol{B} = 0$ everywhere including $z = 0$, but then $\nabla \cdot (\rho\boldsymbol{u}) \propto \delta(z)$ if $\boldsymbol{B} = \beta\rho\boldsymbol{u}$. The problem lies in our assumption of field freezing. If strict freezing held even inside the disk, there would be *no* way to load matter onto field lines for a wind other than to suppose a source of mass on the midplane, i.e., to accept a δ function $\delta(z)$. In actuality, field freezing must break down inside the disk, e.g., because of ambipolar diffusion (see Paper I). Then we can have both $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot (\rho\boldsymbol{u}) = 0$ everywhere and still load matter onto field lines; i.e., the relation (2.4) must not hold in the deep interior of

[2] Apart from the issue of an Alfvénic transition, we can use the same formulation to solve the problem of magnetic accretion from a disk onto the central star by funnel flow (see Paper I). We leave this task for a future endeavor.

the disk. The formulation given in this paper can represent, at best, only the *outer limit* of some inner solution (in the sense of matched asymptotic expansions), where field freezing is an invalid approximation near the midplane $z = 0$ of the X-region, but becomes increasingly valid as we leave it on an ϵ scale. Without examining the inner problem, where we need explicitly to account for finite magnetic diffusivity, we cannot hope to have a priori knowledge of how matter is physically loaded onto field lines. In this paper and the next (III), we absorb this information in the adoption of the arbitrary free function β, which we henceforth assume to be positive above the midplane, where we perform all our calculations.

2.3. *Stream Function and Conserved Quantities*

For axisymmetric flow with rotation, it is further possible to introduce a stream function ψ for the poloidal part of the flow via

$$\rho u_\varpi \equiv \frac{1}{\varpi}\frac{\partial \psi}{\partial z}, \quad \rho u_z \equiv -\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}, \tag{2.5}$$

so that the equation of continuity is satisfied identically:

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi \rho u_\varpi) + \frac{\partial}{\partial z}(\rho u_z) = 0 . \tag{2.6}$$

Equation (2.5) implies that $\boldsymbol{u} \cdot \nabla \psi = 0$; i.e., that the lines of constant ψ define streamlines in the meridional plane (the poloidal part of the flow). Since $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel to one another, we see that lines of constant ψ also define magnetic field lines in the meridional plane (the poloidal components of the field). In any case, the condition $\boldsymbol{u} \cdot \nabla \beta = 0$ can now be expressed as the requirement,

$$\beta = \beta(\psi) . \tag{2.7}$$

We require that the integral of the mass flux gives the assumed mass-loss rate, i.e., that for all ϖ,

$$1 = \int_0^{z_1} \varpi \rho u_\varpi \, dz = \int_0^{z_1} \frac{\partial \psi}{\partial z} dz = \psi(\varpi, z_1) , \tag{2.8}$$

since reflection symmetry allows ψ to be an odd function of z so that $\psi(\varpi, z) = 0$ for $z = 0$. Thus, the value of ψ from 0 to 1 labels the fraction of the total mass flux carried by all streamlines from the midplane to that value of ψ, with $\psi = 1$ on $z = z_1(\varpi)$.

If we dot equation (2.2b) with $\boldsymbol{u}$, we obtain Bernoulli's theorem:

$$\boldsymbol{u} \cdot \nabla\left(\frac{1}{2}|\boldsymbol{u}|^2 + \epsilon^2 \ln \rho + V_{\text{eff}}\right) = 0 , \tag{2.9a}$$

which implies

$$\tfrac{1}{2}|\boldsymbol{u}|^2 + \epsilon^2 \ln \rho + V_{\text{eff}} = H(\psi) . \tag{2.9b}$$

Except for the "heat function" term $\epsilon^2 \ln \rho$ in equation (2.9b), H represents Jacobi's constant, $E_g - J_g$, with E_g and J_g being, per unit mass, the nondimensional energy and z-component of the angular momentum of the gas in an inertial frame. The conservation of $H(\psi)$ along a streamline, independent of the magnetic field $\boldsymbol{B}$, arises because $\boldsymbol{u}$ is parallel to $\boldsymbol{B}$, so the Lorentz force $\propto (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ can do no work on the fluid in the rotating frame of reference. The specific energy of the gas E_g in an inertial frame $\approx H + J_g$ can change, however, because the Lorentz force does exert torque, and this can change the specific angular momentum J_g of the gas along a streamline.

Equation (2.9a) can be combined with equation (2.2a) to give (cf. Lubow & Shu 1975)

$$(|\boldsymbol{u}|^2 - \epsilon^2)(\nabla \cdot \boldsymbol{u}) = |\boldsymbol{u}|^3 \nabla \cdot \left(\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\right) - \boldsymbol{u} \cdot \nabla V_{\text{eff}} . \tag{2.10}$$

Independent of the assumption of axial symmetry that we adopt elsewhere for the sake of mathematical tractability, equation (2.10) implies that the sonic transition must be made within an ϵ neighborhood of the X-point, a conclusion already reached by physical argument in Paper I. To derive the same conclusion here, note that the fluid reaches acoustic speed when $|\boldsymbol{u}| = \epsilon$. If this occurs by smooth acceleration from subsonic to supersonic values, $\nabla \cdot \boldsymbol{u}$ remains finite, and the left-hand side of equation (2.10) vanishes. The two terms on the right-hand side must therefore cancel on the sonic surface. The first term $|\boldsymbol{u}|^3 \nabla \cdot (\boldsymbol{u}/|\boldsymbol{u}|) \sim \epsilon^3/l$ if the unit vector $\boldsymbol{u}/|\boldsymbol{u}|$ changes direction on the scale of l in the X-region. On the other hand, the gradient of the effective potential vanishes by definition at the X-point, and $\nabla V_{\text{eff}} \sim l$ for small distances l from the X-point. Thus, the term $\boldsymbol{u} \cdot \nabla V_{\text{eff}} \sim \epsilon l$ in the X-region. For the two terms on the right-hand side of equation (2.10) to balance, $\epsilon^3/l \sim \epsilon l$, i.e., $l \sim \epsilon$. (Q.E.D.)[3]

Although the constraint for passing through a sonic point imposed by equation (2.10) seems only to involve thermal pressure (via the parameter ϵ), magnetic forces do enter in a subtle fashion. Bernoulli's theorem (2.9b) states that increases in the specific kinetic energy $|\boldsymbol{u}|^2/2$ come only via decreases in the heat function $\epsilon^2 \ln \rho$ (as ρ decreases in the expansion to supersonic speeds) or in the effective gravitational potential V_{eff} (if the gas flows toward the "downhill side" of the saddle at the X-point). For launching a wind, however, increases in $|\boldsymbol{u}|^2/2$ do no good if they represent only a lag behind the frame rotation speed that occurs because each outwardly moving fluid element tends to preserve its original specific angular momentum in the absence of magnetic torques. To put the increase mostly into the poloidal part $(u_\varpi^2 + u_z^2)/2$ rather than the (lagging) toroidal part $u_\varphi^2/2$, the gas needs to be kept nearly corotating throughout the X-region. As we shall see in § 3, the X-celerator mechanism accomplishes precisely this result: while a gradient in the gas pressure accelerates u_ϖ and u_z to speeds of order ϵ in the X-region, strong magnetic fields maintain u_φ within order ϵ^2 of frame corotation (see also Paper I).

Other workers (e.g., Weber & Davis 1967; Goldreich & Julian 1970; Belcher & MacGregor 1976; Hartmann & MacGregor 1982, hereafter HM) obtain as the critical speed the propagation of slow MHD waves rather than acoustic waves because they consider a particular projection of the velocity (the radial component) that makes an angle with respect to the magnetic field (which has an azimuthal or toroidal component in the equatorial plane). When one considers the *total* velocity in the corotating frame, as we do here, the relevant signal speed is that for sound waves (and Alfvén waves, as we shall see below).

[3] Without coming to a fluid boundary (e.g., the interface with a dead zone), the sonic surface cannot just end, but must extend to distances larger than order ϵ from the X-point. For ∇V_{eff} of order unity, eq. (2.10) permits the sonic transition to happen only if the direction of $\boldsymbol{u}$, when its magnitude is ϵ, lies within an angle of order ϵ^2 parallel to the locus of the critical surface $V_{\text{eff}} = 0$. However, streamlines that make sonic crossings in this manner at an order unity distance from the X-point probably carry very little matter on them (cf. Fig. 3 in Paper I).

Since the fluid speed and density are of order ϵ and ϵ^{-2}, respectively, in the X-region, and since $V_{\rm eff}$ is itself of order ϵ^2 in the same neighborhood, we see from equation (2.9b) that $H(\psi)$ is of order $\epsilon^2 \ln \epsilon^{-2}$ on all matter-carrying streamlines in the wind flow. Thus, we may write for later convenience,

$$H \equiv -\epsilon^2 \ln(\epsilon^2 h) , \tag{2.11}$$

where $h(\psi)$ is an order unity function.

To reduce the equations of motion further, we use equation (2.9b) to rewrite equation (2.2b) as

$$H'\nabla\psi + [(2\boldsymbol{e}_z + \nabla \times \boldsymbol{u} - \beta\nabla \times \beta\rho\boldsymbol{u}) \times \boldsymbol{u}] = 0 . \tag{2.12}$$

In equation (2.12) we have denoted $dH/d\psi$ by H' and substituted equation (2.4). In cylindrical coordinates (ϖ, φ, z), define the quantities

$$J \equiv \varpi^2 + \varpi(1 - \beta^2\rho)u_\varphi , \tag{2.13a}$$

$$\omega \equiv \frac{\partial}{\partial z}[(1 - \beta^2\rho)u_\varpi] - \frac{\partial}{\partial \varpi}[(1 - \beta^2\rho)u_z] . \tag{2.13b}$$

We note that $\beta^2\rho$ represents the inverse square of the Alfvén Mach number, $(|\boldsymbol{u}|/v_{\rm A})^{-2}$ (cf. Paper I); thus, the quantity $(1 - \beta^2\rho)$ is an Alfvén discriminant. When it is negative, the flow is sub-Alfvénic; when it is positive, the flow is super-Alfvénic. Equation (2.13a) implies that an Alfvén crossing is made at the axial distance.

$$\varpi_{\rm A} = J^{1/2} . \tag{2.13c}$$

The quantity J in equation (2.13a) is the sum of the z-component of the specific angular momentum carried in an inertial frame by the gas, $J_g \equiv \varpi(\varpi + u_\varphi)$, and that by the field (in a torsional Alfvén wave) in the flow direction $\hat{\boldsymbol{n}}$, $J_B \equiv -\varpi B_\varphi \boldsymbol{B} \cdot \hat{\boldsymbol{n}}/\rho u \cdot \hat{\boldsymbol{n}} = -\varpi\beta^2\rho u_\varphi$. Although the sum $J_g + J_B = J$ is conserved along a streamline (see eq. [2.15b] below), the amount contained in the gas J_g can increase at the expense of that carried by the field J_B in the form of a Maxwell torque per unit mass flux, $-\varpi B_\varphi \boldsymbol{B} \cdot \hat{\boldsymbol{n}}/\rho u \cdot \hat{\boldsymbol{n}}$, that reacts back to spin down the footpoint of the field in the X-region.

The quantity ω is the sum of the φ-component of the vorticity in the gas and an analogous contribution from the field. The fluid part of the φ-vorticity ω gives a measure of how strongly an initially equatorial flow is deflected toward the rotational poles. In terms of J and ω, the term inside the brackets of equation (2.12) has the toroidal part,

$$\boldsymbol{e}_\varphi\left(\frac{u_\varpi}{\varpi}\frac{\partial J}{\partial \varpi} + \frac{u_z}{\varpi}\frac{\partial J}{\partial z}\right) , \tag{2.14a}$$

and the poloidal part,

$$\boldsymbol{e}_\varpi\left(\omega u_z - \frac{u_\varphi}{\varpi}\frac{\partial J}{\partial \varpi} + \varpi\beta\beta'\rho^2|\boldsymbol{u}|^2 u_z\right) - \boldsymbol{e}_z\left(\omega u_\varpi + \frac{u_\varphi}{\varpi}\frac{\partial J}{\partial z} + \varpi\beta\beta'\rho^2|\boldsymbol{u}|^2 u_\varpi\right) . \tag{2.14b}$$

In expression (2.14b), we have made use of equation (2.5) to eliminate $\partial\psi/\partial z$ and $\partial\psi/\partial\varpi$. Since $\nabla\psi$ has no toroidal part, the φ component of equation (2.12) requires expression (2.14a) to be zero, i.e.,

$$\boldsymbol{u} \cdot \nabla J = 0 , \tag{2.15a}$$

or

$$J = J(\psi) . \tag{2.15b}$$

With $J = J(\psi)$ and with equation (2.5) used to eliminate the first derivatives of ψ, the poloidal part of equation (2.12) reads

$$(u_z \boldsymbol{e}_\varpi - u_\varpi \boldsymbol{e}_z)(\omega + J'\rho u_\varphi + \varpi\beta\beta'\rho^2|\boldsymbol{u}|^2 - H'\varpi\rho) = 0 . \tag{2.16}$$

For nontrivial solutions of equation (2.16) to exist, we require

$$\frac{\omega}{\varpi\rho} + J'\frac{u_\varphi}{\varpi} + \beta\beta'\rho|\boldsymbol{u}|^2 - H' = 0 . \tag{2.17}$$

With the definitions (2.5) and (2.13b), we can reexpress equation (2.17) as a partial differential equation (PDE) for ψ:

$$\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi\mathscr{A}\frac{\partial\psi}{\partial\varpi}\right) + \frac{\partial}{\partial z}\left(\mathscr{A}\frac{\partial\psi}{\partial z}\right) = \mathscr{Q} , \tag{2.18}$$

where

$$\mathscr{A} \equiv \frac{\beta^2\rho - 1}{\varpi^2\rho} , \tag{2.19a}$$

$$\mathscr{Q} \equiv \rho\left[J'\frac{u_\varphi}{\varpi} + \beta\beta'\rho|\boldsymbol{u}|^2 - H'\right] . \tag{2.19b}$$

Notice that $\mathscr{A}$ is proportional to the negative of the Alfvén discriminant and that $\mathscr{Q} = 0$ for $\psi = 0$ because β^2, J, and H are all even functions of ψ by reflection symmetry. (We remind the reader, however, that the physical equations break down near the equatorial plane, where we should not ignore the effects of magnetic diffusivity.)

2.4. *Character of Governing Partial Differential Equation*

We find it computationally convenient to use equations (2.19a) and (2.13a) to eliminate ρ and u_φ via

$$\rho = \frac{1}{\beta^2 - \varpi^2\mathscr{A}} , \tag{2.20a}$$

$$u_\varphi = -\varpi^{-1}\mathscr{L}(\beta^2 - \varpi^2\mathscr{A}) , \tag{2.20b}$$

where

$$\mathscr{L} \equiv \frac{1}{\mathscr{A}}\left(\frac{J}{\varpi^2} - 1\right) . \tag{2.20c}$$

Making the substitutions (2.11a), (2.20a), and (2.20b), we may rewrite equations (2.18) and (2.9b) as

$$\nabla \cdot (\mathscr{A}\nabla\psi) = \mathscr{Q} , \tag{2.21a}$$

$$(\beta^2 - \varpi^2\mathscr{A})^2[|\nabla\psi|^2 + \mathscr{L}^2] + 2\varpi^2\left[V_{\rm eff} + \epsilon^2\ln\left(\frac{\epsilon^2 h}{\beta^2 - \varpi^2\mathscr{A}}\right)\right] = 0 , \tag{2.21b}$$

where

$$\mathscr{Q} = -\mathscr{L}\frac{J'}{\varpi^2} + (|\nabla\psi|^2 + \mathscr{L}^2)\frac{\beta\beta'}{\varpi^2} + \frac{\epsilon^2 h'/h}{\beta^2 - \varpi^2\mathscr{A}} . \tag{2.22}$$

The substitution of $\mathscr{A}$ for ρ proves a computationally sound strategy because $\mathscr{A}$ remains of order unity in the important part of the flow, whereas ρ can range over several orders of magnitude.

Notice that the direction of the magnetic field does not matter for the macroscopic dynamics; i.e., equations (2.20a)–(2.22) do not depend on the sign of β. Although we shall formally assume that β is positive in the upper hemisphere and negative in the lower; in the real problem, as in the case of the solar wind, any given hemisphere may have several "magnetic sectors" across whose boundaries the magnetic field reverses direction. Introduction of such sectors would completely change the global distribution of electric currents implied by the axisymmetric models without modifying to any significant extent the large-scale flow dynamics.

Notice finally that the meridional spreading of streamlines governed by equation (2.21a) resembles steady state "heat conduction," with $\mathscr{Q}$ being a net "source" term and $\mathscr{A}$ being a "conductivity coefficient" that is positive for sub-Alfvénic flow and negative for super-Alfvénic flow. The "conductivity" going negative in the current problem does not carry the implication of "negative diffusivity" (i.e., an automatic bunching of the streamlines) because the character of the governing PDE (elliptic or hyperbolic) is determined by the second-order derivatives of ψ in equation (2.21a) only after we substitute in the implicit dependence of $\mathscr{A}$ on the first derivatives of ψ implied by the Bernoulli equation (2.21b). In the general case when ϵ has an arbitrary value (see, e.g., Heinemann & Olbert 1978; Sakurai 1985), there are three "critical surfaces" (corresponding to slow MHD, Alfvén, and fast MHD crossings) about which one needs to worry, where the governing second-order PDE might or might not change character. A great simplification occurs if $\epsilon \ll 1$. The overall problem then divides by the method of matched asymptotic expansions into two parts: an *inner problem* near the X-point of the effective equipotential with size of order ϵ where the flow in the rotating frame of reference accelerates from rest on the star to supersonic speeds, and an *outer problem* beyond the X-point with size of order unity where we may take the formal limit $\epsilon \to 0$ so that the issues of Alfvén and fast MHD crossings telescope down to a single surface at $\mathscr{A} = 0$. In the same limit, the function $H(\psi)$ in equation (2.11) may be taken to equal zero independent of the functional form of $h(\psi)$, and the sonic transition appears to occur at a single point in the meridional plane—the X-point of the effective potential of the protostar. For the outer problem, the resultant PDE is elliptic in the sub-Alfvénic region ($\mathscr{A} > 0$) and hyperbolic in the super-Alfvénic region ($\mathscr{A} < 0$).

To summarize, if we are given the conserved functions $\beta(\psi)$, $H(\psi)$, and $J(\psi)$ that represent the distribution of magnetic flux to mass flux, specific energy in a rotating frame of reference, and total specific angular momentum, we can reduce the steady MHD flow problem to solving the second-order PDE (2.21a) and the transcendental equation (2.21b) for the stream function ψ and Alfvén discriminant $\mathscr{A}$ (with $\mathscr{L}$ and $\mathscr{Q}$ being given by eqs. [2.20c] and [2.22]). The density, azimuthal velocity, poloidal velocity, and vector magnetic field, can then be obtained from equations (2.20a), (2.20b), (2.5), and (2.4). Our formulation differs from the standard Grad-Shafranov equations (e.g., Hameiri 1983 or Lovelace et al. 1986) in that we choose to replace the magnetic flux function by the fluid stream function for the sake of convenience in a problem where the mass-loss rate is known but the magnetic field is not. Furthermore, by adopting the technique of matched asymptotic expansions, we can concentrate on crossing the Alfvén surface in the outer problem (where the governing PDE [2.21a] changes character from elliptic to hyperbolic) after we have successfully crossed the sonic surface in the inner problem. Finally, our method (see below) allows a tractable scheme for dealing with the practical difficulty that the unknown functions $\beta(\psi)$, $H(\psi)$, and $J(\psi)$ are *not* given to us a priori, but are to be found as part of the overall solution.

2.5. *Boundary Conditions for the Sub-Alfvénic Region of the Outer Problem*

We begin by examining the constraints set on the sub-Alfvénic flow by the boundary conditions. In the limit $\epsilon \to 0$, when we can approximate the adjoining disk as being infinitesimally thin, the boundary conditions to be imposed on the PDE (2.21a) are that the first streamline above the disk just skims the surface of the disk:

$$\psi = 0 \quad \text{for} \quad z = 0 \; ; \tag{2.23a}$$

the passage through the Alfvén surface is smooth (which defines the normal derivative of ψ):

$$\nabla \mathscr{A} \cdot \nabla \psi = \mathscr{Q} \quad \text{on} \quad \mathscr{A} = 0 \; ; \tag{2.23b}$$

and the limiting streamline corresponds to a free boundary across which the flow is in pressure balance with an O-wind or a dead zone (cf. Figs. 1, 2*b*, 2*c* and 4 of Paper I):

$$\tfrac{1}{2}(\beta\rho\,|\boldsymbol{u}|)^2 = p_{\rm ext} + \tfrac{1}{2}\,|\boldsymbol{B}_{\rm ext}|^2 \quad \text{on} \quad \psi = 1 \; . \tag{2.23c}$$

The term on the left-hand side of equation (2.23c) represents the magnetic pressure of the X-wind. We have ignored the thermal pressure (on the left-hand but not the right-hand side) on the grounds that it is negligible for the X-wind in the outer region of the flow. The ram pressures of the X- and O-winds do not enter on either side of equation (2.23c) because by definition all velocities are directed parallel to the interface between the two flows.

The boundary conditions (2.23a)–(2.23c), if extended to include the X-point, form a closed curve in the meridional plane that serves to fix the solution in the sub-Alfvénic region, $\mathscr{A} > 0$, where the governing PDE is elliptic. In practice, in the limit $\epsilon \to 0$, ψ takes on all values from 0 to 1 at the X-point, which constitutes a singularity for the formal outer problem (because all the streamlines seemingly emanate from a single point). For numerical work, therefore, we need to remove from the outer problem the small X-region and "patch" onto it the values that apply to the solution of the inner problem. In the current paper, we supply the mathematical justification for the patching procedure by demonstrating *analytically* that there exists a region of overlap where the solutions asymptotically match (see §§ 3 and 4). The analytical outer limit of our inner solution then provides inner boundary conditions for an outer problem that we solve numerically (see Paper III).

Once the sub-Alfvénic part of the outer problem has been solved (say, by a relaxation technique), both ψ *and* its normal derivative are fixed on the surface $\mathscr{A} = 0$; thus *no* outer surface boundary condition (say, at infinity) can be applied to the super-Alfvénic part of the problem. Instead, because the governing PDE is now hyperolic, it should be possible, in principle, to *march outward* from the Alfvén surface (with the help of the method of characteristics and the two boundary conditions, eqs. [2.23a] and [2.23c]) to complete the solution in the super-Alfvénic region. This relatively straightforward marching problem we leave for future work.

One obstacle to carrying out this overall plan is that the *spatial location* of the Alfvén surface $\mathscr{A} = 0$ in equation (2.23b) is not specified in advance. This apparent complication turns out, however, actually to be a boon, because it implies that one of the arbitrary functions—say, $\beta(\psi)$ for the sake of definiteness—is free for us to specify. Alternatively, we may fix the spatial location of the Alfvén surface in advance, and compute the ratio of magnetic field to mass flux $\beta(\psi)$ needed to accomplish this feat. If the calculated $\beta(\psi)$ does not satisfy a future calculation of how matter is actually loaded onto field lines (e.g., by ambipolar diffusion of gas relative to field deep inside the X-region), we will need to modify the location of the Alfvén surface (which will change the distribution of terminal velocities reached in the X-wind).

Having the freedom to choose the location of the Alfvén surface produces another immediate benefit, because equation (2.13c) fixes the unknown function J as a function of position on the Alfvén surface, which determines $J(\psi)$ if we know the distribution of ψ on this surface,

$$J(\psi) = \varpi^2 \quad \text{on} \quad \varpi = \varpi_A(z) , \tag{2.24a}$$

where $\varpi = \varpi_A(z)$ describes the locus of the curve $\mathscr{A} = 0$. Equations (2.20c) and (2.24a) imply that $\mathscr{L}$ on the Alfvén surface must be calculated by l'Hôpital's rule:

$$\mathscr{L} = \lim_{\mathscr{A}\to 0} \frac{1}{\mathscr{A}}\left(\frac{J}{\varpi^2} - 1\right) = \frac{\hat{\boldsymbol{n}} \cdot \nabla(J/\varpi^2)}{\hat{\boldsymbol{n}} \cdot \nabla\mathscr{A}} \quad \text{on} \quad \varpi = \varpi_A(z) , \tag{2.24b}$$

where $\hat{\boldsymbol{n}}$ is now a unit normal to the Alfvén surface $\mathscr{A} = 0$.

Imagine that we have the spatial dependence of ψ everywhere inside the Alfvén surface, via a tentative solution of the PDE (2.21a) if we know all other relevant information. An iterative method then immediately suggests itself if we can use all of the equations independent of equation (2.21a) to discover this other relevant information. In particular, the distribution of ψ on the Alfvén surface can itself be fixed by the constraint that Bernoulli's equation (2.21b) must yield a solution for $\mathscr{A}$ on the Alfvén surface that satisfies

$$\mathscr{A} = 0 \quad \text{on} \quad \varpi = \varpi_A(z) . \tag{2.25}$$

Equation (2.25) is equivalent to the requirement that $\mathscr{L}$ computed by l'Hôpital's rule must equal the ratio of $(J/\varpi^2 - 1)$ to $\mathscr{A}$ extrapolated to the Alfvén surface (cf. eq. [2.24b]).

On the other hand, if $J(\psi)$, $H(\psi)$ (taken to be zero in the outer problem), and $\psi(\varpi, z)$ are all known (so that $\mathscr{A}$ can everywhere be found from Bernoulli's eq. [2.21b]), equation (2.23b), with $\mathscr{Q}$ given by equation (2.22) in the limit $\epsilon \to 0$, can be regarded as an ordinary differential equation (ODE) for β:

$$\beta\beta' = \frac{\varpi^2 \nabla\mathscr{A} \cdot \nabla\psi + \mathscr{L}J'}{|\nabla\psi|^2 + \mathscr{L}^2} \quad \text{on} \quad \mathscr{A} = 0 . \tag{2.26}$$

Equation (2.26) allows us to integrate for $\beta(\psi)$ from $\psi = 0$ to $\psi = 1$ along the Alfvén surface when ψ (and therefore its tangential derivative) and the normal derivative of ψ (proportional to $\nabla\mathscr{A} \cdot \nabla\psi$) are given along it. The starting value of β at $\psi = 0$ ($z = 0$) in the integration of this ODE is to be determined, in principle, so that the boundary condition (2.23c) is satisfied at $\psi = 1$.

Unfortunately, equation (2.23c) does not form a practical boundary condition since no one possesses either a solution for the dead zone or a well-developed theory for the O-wind (and therefore definitive values for $p_{\text{ext}} + |\boldsymbol{B}_{\text{ext}}|^2/2$). Hence, we choose in practice to replace equation (2.23c) by the simpler expedient of a fixed upper boundary:

$$\psi = 1 \quad \text{on} \quad z = z_1(\varpi) , \tag{2.27}$$

where $z_1(\varpi)$ has a freely specifiable functional form. (See the Appendix for the introduction of a curvilinear set of coordinates that takes advantage of this freedom.) Again, if the left-hand side of equation (2.23c) computes after the fact to some value that does not conform with pressure balance in some future development for the dead zone or the O-wind, we will need to choose some other function for $z_1(\varpi)$ in equation (2.27).

The adoption of the conditions (2.23a) and (2.27) implies that we treat the physical interfaces between the X-wind and the disk or the X-wind and the O-wind as laminar slip surfaces. In practice, such interfaces are likely to become turbulent boundary layers. The astronomical difference may be considerable because observational selection effects (e.g., radiation by shocks in highly collimated optical jets [see, e.g., Mundt, Ray, & Raga 1991] or by masing action in wind-disk interactions [see, e.g., Plambeck, Wright, & Carlstrom 1988]) may especially highlight the regions of turbulent mixing. Nevertheless, because astronomers' knowledge of the properties of supersonic entrainment and hydromagnetic mixing is so poorly developed, we remain content at this stage to construct a zeroth-order theory based on the assumption of laminar slip surfaces. This theory may then serve as a starting point for more sophisticated calculations of the mixing to be expected for turbulent boundary layers (see, e.g., Cantó & Raga 1991).

2.6. *Scaling for Inner Problem*

Since J has to be larger than 1 by an order unity amount for the centrifugal mechanism to work at all, $\mathscr{L}$ and β must be of order unity if equations (2.20c) and (2.26) are to be obeyed. On the other hand, in the X-region where the sonic transition is made, V_{eff} is of order ϵ^2, and $\varpi^2\mathscr{A}$ must equal β^2 to this order if equation (2.21b) is to be satisfied. Equation (2.20a) then implies (as we have already argued physically in Paper I) that ρ is of order ϵ^{-2} (to carry unit mass loss in accordance with eq. [2.8] when the poloidal flow speed is $\sim\epsilon$ in an equatorial belt of height $\sim\epsilon$), and the departure from corotation speeds u_φ is of order ϵ^2 (to not carry more than order unity J in eq. [2.13a]). Equation (2.4) now shows that the dimensionless toroidal magnetic field is of order unity at the sonic point while the dimensionless poloidal magnetic field is of order ϵ^{-1} times larger (in units of the fiducial value given by the last expression in eq. [2.1a]). In other words, the magnetic field in the X-region is almost all poloidal, which makes good physical sense.

Since ψ from equation (2.8) is by definition an order unity function, the derivatives of β and J with respect to ψ (i.e., β' and J') are order unity functions (or zero). The largest term in equation (2.21c) for $\mathscr{Q}$ is then of order ϵ^{-2} (because $|\nabla\psi| \sim \epsilon^{-1}$) and reads $\mathscr{Q} \approx |\nabla\psi|^2\beta\beta'/\varpi^2$. With this approximation and $\mathscr{A} \approx \beta^2/\varpi^2$, equation (2.18) in the X-region (where $\varpi \approx 1$) takes the simplified form:

$$\frac{\partial}{\partial x}\left(\beta \frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial z}\left(\beta \frac{\partial\psi}{\partial z}\right) \approx 0 . \tag{2.28a}$$

In equation (2.28a), we have defined the pseudo-rectilinear coordinate $x \equiv \varpi - 1$, and it is important to remember that x and z are to be regarded as order ϵ variables (all this will be formalized better in § 3).

Equation (2.28a) has a subtle physical interpretation. The left-hand side is equal to the φ component of $\nabla \times B$ in the X-region. Thus, equation (2.24a) expresses the approximate current-free condition (in the important direction),

$$(\nabla \times \boldsymbol{B})_\varphi \approx 0 , \tag{2.28b}$$

that must hold for very strong fields anchored deep inside the X-region when they emerge into the surface layers. Current-free fields are, of course, a special example of force-free fields. The difference between the fluid involved dynamically in the wind and the dead zone becomes blurred in the X-region (see Fig. 3 in Paper I). Closed field lines are strong and nearly force-free as they protrude into the surface layers of the X-region. But when gas emerges dynamically onto open field lines in the wind (or onto closed ones in the funnel flow), the expansion and acceleration will reduce the importance of magnetic forces to fluid inertia, with the Alfvén speed remaining roughly constant, but with the fluid speed increasing. The force-free approximation will then break down. This change in the dynamical role of the field ultimately causes the field to whip the gas magnetocentrifugally through an Alfvén point.[4] The modification in physical behavior (from near-corotation to significant lags and outward magnetocentrifugal acceleration) occurs mathematically when equation (2.28a) has a region of common validity with equation (4.1) below. The rapid nature of the transition in the limit $\epsilon \to 0$ makes the problem an ideal one for attack by singular perturbation theory.

3. THE OUTER LIMIT OF THE INNER PROBLEM

3.1. *Basic Equations*

Motivated by the comments of the previous section, let us introduce, for the flow problem in the X-region, the scaled coordinate variables:

$$\varpi - 1 = \epsilon\xi\,, \quad z = \epsilon\zeta\,, \tag{3.1a}$$

and the scaled dependent variables:

$$\rho = \epsilon^{-2}p\,, \quad H = -\epsilon^2 \ln(\epsilon^2 h)\,. \tag{3.1b}$$

To lowest nonvanishing order for small ϵ, equation (2.13a) yields

$$u_\varphi = \epsilon^2\left[\frac{1 - J(\psi)}{\beta^2 p}\right], \tag{3.2}$$

which can be calculated after the solution for ψ has been obtained from (cf. eq. [2.28]):

$$\frac{\partial}{\partial\xi}\left(\beta\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\beta\frac{\partial\psi}{\partial\zeta}\right) = 0\,. \tag{3.3}$$

A Taylor expansion of $V_{\rm eff}$ about $\xi = 0 = \zeta$, where $\partial V_{\rm eff}/\partial\xi = \partial V_{\rm eff}/\partial\zeta = 0 = V_{\rm eff}$, yields to lowest nonvanishing order,

$$\epsilon^{-2}V_{\rm eff} = -\tfrac{3}{2}\xi^2 + \tfrac{1}{2}\zeta^2\,. \tag{3.4a}$$

Once the function $h(\psi)$ has been specified, the fluid pressure (and scaled density) p is given as a function of $\partial\psi/\partial\xi$ and $\partial\psi/\partial\zeta$ from the transcendental equation represented by the dominant terms in Bernoulli's equation:

$$\frac{1}{2p^2}\left[\left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\zeta}\right)^2\right] - \frac{3}{2}\xi^2 + \frac{1}{2}\zeta^2 + \ln(hp) = 0\,. \tag{3.4b}$$

Notice that once $\beta(\psi)$ has also been specified (i.e., a specification of how matter is loaded onto field lines), equation (3.3) for the stream function ψ is completely decoupled from the equation (3.4b) for the pressure (and density) p. Thus, if we take care to state the boundary conditions in terms of ψ and its derivatives alone, we can solve for ψ from equation (3.3) *independently* of any assumption for $h(\psi)$ or the resulting density field. For the same ψ, different solutions for p (and therefore $\boldsymbol{u}$) can be obtained, after the fact, as the root of the transcendental equation (3.4b) for different $h(\psi)$.

As we have noted earlier and in Paper I, however, the problem of how matter is loaded (subsonically) onto field lines cannot be addressed properly within the context of the field-freezing approximation. Hence, we can use equations (3.4a)–(3.4b) only in the limit $(\xi^2 + \zeta^2)^{1/2} \to \infty$ within the rightmost sector of Figure 3 in Paper I, i.e., the "downhill" sector into which the X-wind blows in an outward direction. Because ψ has a limited range of values (from 0 to 1), its derivative with respect to distance $s \equiv \epsilon\sigma$ (on an ϵ scale) from the X-point must go to zero as we take the limit $\sigma \to \infty$. (Do not confuse the stretched coordinate σ here with the surface density of the disk σ introduced in Paper I.) This property of the physical solution motivates the development of the next subsection.

3.2. *Pseudopolar Coordinates*

Introduce the scaled pseudopolar coordinates (σ, φ, ϑ), where

$$\xi = \sigma\cos\vartheta\,, \quad \zeta = \sigma\sin\vartheta\,. \tag{3.5}$$

Equations (3.3) and (3.4b) now become

$$\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\beta\frac{\partial\psi}{\partial\sigma}\right) + \frac{1}{\sigma^2}\frac{\partial}{\partial\vartheta}\left(\beta\frac{\partial\psi}{\partial\vartheta}\right) = 0\,, \tag{3.6a}$$

$$\frac{1}{2p^2}\left[\left(\frac{\partial\psi}{\partial\sigma}\right)^2 + \frac{1}{\sigma^2}\left(\frac{\partial\psi}{\partial\vartheta}\right)^2\right] - \left(2\cos^2\vartheta - \frac{1}{2}\right)\sigma^2 + \ln(hp) = 0\,, \tag{3.7}$$

In the limit $\sigma \to \infty$, the lowermost streamline (in the upper hemisphere) skims over a thin disk:

$$\psi = 0 \quad \text{for} \quad \vartheta = 0\,, \tag{3.8a}$$

whereas the uppermost streamline leaves the X-region at an asymptotic latitudinal angle $\vartheta_x(0) \le 60°$:

$$\psi = 1 \quad \text{for} \quad \vartheta = \vartheta_x(0)\,. \tag{3.8b}$$

In the same limit, physically meaningful solutions of ψ must depend asymptotically only on ϑ and not on σ. Equation (3.6) then becomes

$$\frac{\partial}{\partial\vartheta}\left(\frac{\beta\,\partial\psi}{\partial\vartheta}\right) \to 0 \quad \text{for} \quad 0 \le \vartheta \le \vartheta_x(0) \quad \text{as} \quad \sigma \to \infty\,. \tag{3.9}$$

With the boundary conditions (3.8a) and (3.8b), equation (3.9) may be integrated to yield

$$\frac{1}{\bar\beta}\int_0^\psi \beta(\psi_0)d\psi_0 \to \frac{\vartheta}{\vartheta_x(0)} \quad \text{for} \quad 0 \le \vartheta \le \vartheta_x(0) \quad \text{as} \quad \sigma \to \infty\,, \tag{3.10a}$$

where ψ_0 is a dummy variable. In equation (3.10a) we have defined the constant $\bar\beta$ as the streamline-averaged value of ψ:

$$\bar\beta \equiv \int_0^1 \beta(\psi_0)d\psi_0\,. \tag{3.10b}$$

[4] The funnel flow, which remains sub-Alfvénic along its entire length, differs from the wind flow on this fundamental point. Thus the GSE is elliptic in the funnel domain, requiring the application of appropriate boundary conditions at the stellar surface (e.g., the requirement that the poloidal fields become dipole in character).

In the same limit equation (3.7) becomes

$$\frac{1}{2p^2}\left(\frac{\partial\psi}{\partial\vartheta}\right)^2 \to \left(2\cos^2\vartheta - \frac{1}{2}\right)\sigma^4$$

$$\text{for} \quad 0 \le \vartheta \le \vartheta_x(0) \quad \text{as} \quad \sigma \to \infty\ . \quad (3.10c)$$

Independent of $h(\psi)$, therefore, equation (3.10c) yields the asymptotic solution for the pressure $p = \epsilon^2 \rho$ as

$$p \to \left[\frac{\bar{\beta}}{\vartheta_x(0)\beta}\right]\sigma^{-2}(4\cos^2\vartheta - 1)^{-1/2}$$

$$\text{for} \quad 0 \le \vartheta \le \vartheta_x(0) \quad \text{as} \quad \sigma \to \infty\ . \quad (3.10d)$$

The corresponding solution for the dominant component of the (scaled) velocity field reads

$$\epsilon^{-1}u_\sigma = \frac{1}{p\sigma}\left(\frac{\partial\psi}{\partial\vartheta}\right) \to \sigma(4\cos^2\vartheta - 1)^{1/2}$$

$$\text{for} \quad 0 \le \vartheta \le \vartheta_x(0) \quad \text{as} \quad \sigma \to \infty\ . \quad (3.10e)$$

Equations (3.10a)–(3.10e) have interesting physical interpretations. Equation (3.10a) (or, equivalently, eq. [3.9]) implies that the σ-component of the magnetic field, proportional to $\beta(\psi)\partial\psi/\partial\vartheta$, has the same asymptotic strength on every streamline that leaves the X-region. On the other hand, equation (3.10e) implies that the fluid velocity asymptotically emerges along the σ-direction, increasing linearly with σ for every direction $0 \le \vartheta \le \vartheta_x(0) < 60°$, where the last angle (the complement of 30°) gives the locus of the upper-right branch of the critical equipotential in the X-region. Since $\boldsymbol{B} \propto \boldsymbol{u}$, this implies that the σ-component of $\boldsymbol{B}$ represents essentially the entire poloidal field (dominant over the toroidal field in the X-region). These conclusions hold *independently* of any specific choices for $\beta(\psi)$ or $h(\psi)$ and constitutes one of the principal findings of this section: magnetic fields strong enough to provide the requisite magnetocentrifugal acceleration in the outer region exert so much stress in the inner region that a uniform distribution of field strengths emerges for the mass-carrying streamlines. In turn, such a field configuration leads to the unique asymptotic distribution of pressure and gas density given by equation (3.10d).

Finally, we notice that if ϑ_x should equal 60° rather than be less than it (i.e., if the uppermost streamline asymptotically approaches the critical equipotential), then equation (3.10d) implies that the gas pressure goes to infinity on this streamline *unless*, for $\psi \to 1$, $\beta(\psi) \to \infty$. Only for infinite $\beta \propto (4\cos^2\vartheta - 1)^{-1/2}$ as $\vartheta \to 60°$ can the density $\rho = \epsilon^{-2}p$ and the magnetic field $\boldsymbol{B} = \beta\rho\boldsymbol{u}$ remain finite (to balance the pressure of the dead zone, say) when the fluid velocity $\boldsymbol{u}$ vanishes (see eqs. [3.10d]–[3.10e]). We regard this property of the outer limit of the inner solution the strongest hint that the outer branch of the critical equipotential might correspond asymptotically to the separatrix between gas that can flow dynamically in the wind and (exponentially small amounts of) matter that remains static in the dead zone. Unlike the case for the wind proper beyond the X-region, the magnetic field in the dead zone will generally have little or no toroidal component B_φ; consequently, we anticipate a fairly rapid change in the direction of the field (if not in its strength) as we cross the separatrix, so the separatrix should also correspond to a current sheath that mediates the transition between open and closed field lines (see § 2 of Paper I).

Nevertheless, we cannot prove that the locus of the upper streamline $\vartheta = \vartheta_x(s)$ must follow the locus of the critical equipotential, $V_{\rm eff}(s, \vartheta) = 0$. For example, the ram pressure of an O-wind (before it is turned into gas pressure by a shock) would undoubtedly push the $\psi = 1$ streamline somewhat away from the zero-velocity surface represented by the critical potential. Even with a dead zone, without a complete solution that includes the structure of the (near-vacuum) field, we cannot rule out the possibility that the $\psi = 1$ surface lies a finite distance within the "downhill" sector of the equipotential, rather than right on it.

In what follows, we allow for the more general possibility concerning the location of the surface $\vartheta = \vartheta_x(s)$. The Alfvén surface may then occur at a finite distance s from the X-point on the last streamline (which has a finite value of β associated with it), a considerable convenience for a numerical approach to the solution of the governing equations (see Paper III). Paper III also considers Alfvén surfaces that are concave *toward* the central star, rather than concave *away* from it (like the loci for the equipotentials within the outer downhill sector). The former geometry has smaller values of β than the latter geometry for ψ close to 1. The models of Paper III therefore represent *conservative* estimates for the amount of collimation toward the z-axis that we might realistically expect for the last streamline of the X-wind.

These comments conclude our discussion of the outer limit of the inner solution. We now proceed to consider the outer problem, in particular, the problem of asymptotic matching with the inner region.

4. THE OUTER PROBLEM

4.1. *Governing Equations*

To lowest order in ϵ, the governing equations in the outer region read

$$\nabla \cdot (\mathscr{A}\nabla\psi) = \mathscr{Q}\ , \quad (4.1)$$

$$(\beta^2 - \varpi^2\mathscr{A})^2\left[|\nabla\psi|^2 + \frac{1}{\mathscr{A}^2}\left(\frac{J}{\varpi^2} - 1\right)^2\right] + 2\varpi^2 V_{\rm eff} = 0\ , \quad (4.2)$$

with $\mathscr{Q}$ given by

$$\mathscr{Q} = -\frac{1}{\mathscr{A}}\left(\frac{J}{\varpi^2} - 1\right)\frac{J'}{\varpi^2} - \frac{2V_{\rm eff}}{(\beta^2 - \varpi^2\mathscr{A})^2}\beta\beta'\ . \quad (4.3)$$

The boundary conditions to be imposed on the PDE (4.1) are as stated in § 2.3.

In what follows (see § 3.2 and the Appendix, as well as Paper III), we find it convenient to introduce pseudopolar coordinates (s, ϑ) with the origin of s lying—not at the center of the star—but at the X-point, and with the angle ϑ measured in the meridional plane starting from zero at the equator:

$$x \equiv \varpi - 1 \equiv s\cos\vartheta\ , \quad (4.4a)$$

$$z \equiv s\sin\vartheta\ . \quad (4.4b)$$

The variables x, z, and s are identical to their counterparts ξ, ζ, and σ introduced in equations (3.1a) and (3.6) except that they naturally represent spatial scales on the order of unity rather than of ϵ. In terms of the (right-handed) coordinate system (s, φ, ϑ), the metric coefficients are $h_s = 1$, $h_\varphi = 1 + s\cos\vartheta$, $h_\vartheta = s$.

4.2. *The Inner Limit of the Outer Solution*

We now wish to find the form taken by the outer solution in the inner limit $s \to 0$ to see if it matches the form taken by the inner solution in the outer limit $\sigma \to \infty$. For small s, $V_{\rm eff}$ can be expanded in a Taylor series as (cf. eq. [3.4a]):

$$V_{\rm eff} = -\tfrac{3}{2}x^2 + \tfrac{1}{2}z^2 + \cdots = -s^2(2\cos^2\vartheta - \tfrac{1}{2}) + \cdots . \quad (4.5a)$$

We also try the series expansions,

$$\psi = \psi_0(\vartheta) + s\psi_1(\vartheta) + s^2\psi_2(\vartheta) + \cdots , \quad (4.5b)$$

$$\mathscr{A} = \beta^2(\psi_0) + s\mathscr{A}_1(\vartheta) + s^2\mathscr{A}_2(\vartheta) + \cdots , \quad (4.5c)$$

where the terms in s^0 and s^1 for $\mathscr{A}$ are determined so that $\beta^2 - \varpi^2\mathscr{A}$ is zero to these orders, i.e.,

$$\mathscr{A}_1 = 2\beta\beta'\psi_1 - 2\beta^2\cos\vartheta . \quad (4.5d)$$

In equation (4.5d), and for the rest of this section, we mean $\beta(\psi_0)$ and $\beta'(\psi_0)$ when we write β and β'. With the choices (4.5c) and (4.5d), the expansion for the specific volume becomes

$$\rho^{-1} \equiv \beta^2 - \varpi^2\mathscr{A} = s^2 v_2 + s^3 v_3 + \cdots , \quad (4.5e)$$

where

$$v_2 = 2\beta\beta'\psi_2 - \mathscr{A}_2 + (\beta'^2 + \beta\beta'')\psi_1^2 + 3\beta^2\cos^2\vartheta - 4\beta\beta'\psi_1\cos\vartheta . \quad (4.5f)$$

The behaviors that $\rho \propto s^{-2}$ and $u_s \propto s$ as $s \to 0$ make an asymptotic match possible (cf. eqs. [3.10d] and [3.10e]). The substitution of equations (4.5a), (4.5b), (4.5c), and (4.5e) into equation (4.2) yields the identification

$$v_2\left(\frac{d\psi_0}{d\vartheta}\right) = (4\cos^2\vartheta - 1)^{1/2} . \quad (4.5g)$$

Equation (4.5g), coupled with equation (4.5f), gives $\mathscr{A}_2$ if ψ_0, ψ_1, and ψ_2 are known functions of ϑ.

To determine ψ_0 as a function of ϑ, consider the lowest order (s^{-2}) requirement of equation (4.1). The right-hand side (cf. eq. [4.3]) has the expansion

$$\mathscr{Q} = s^{-2}\mathscr{Q}_{-2} + s^{-1}\mathscr{Q}_{-1} + \cdots , \quad (4.6a)$$

where

$$\mathscr{Q}_{-2} = \beta\beta'\left(\frac{d\psi_0}{d\vartheta}\right)^2 , \quad (4.6b)$$

$$\mathscr{Q}_{-1} = 2\beta\beta'\left(\frac{d\psi_0}{d\vartheta}\right)\left[\left(\frac{d\psi_1}{d\vartheta}\right) - \left(\frac{d\psi_0}{d\vartheta}\right)\cos\vartheta\right] + (\beta'^2 + \beta\beta'')\left(\frac{d\psi_0}{d\vartheta}\right)^2\psi_1 . \quad (4.6c)$$

To lowest order in small s, equation (4.1) requires

$$\frac{d}{d\vartheta}\left(\beta\frac{d\psi_0}{d\vartheta}\right) = 0 , \quad (4.6d)$$

which has the solution

$$\int_0^{\psi}\beta(\psi_0)d\psi_0 = K\vartheta , \quad (4.6e)$$

where

$$K \equiv \frac{\bar\beta}{\vartheta_x(0)} , \quad (4.6f)$$

with (cf. eq. [3.10b])

$$\bar\beta \equiv \int_0^1 \beta(\psi_0)d\psi_0 . \quad (4.6g)$$

In equation (4.6f) we have assumed that the shape of the upper boundary, $z = z_1(\varpi)$ where $\psi = 1$, can be specified by an equation $\vartheta = \vartheta_x(s)$. In equation (4.6d), we have required $\psi = 0$ for $\vartheta = 0$ and $\psi = 1$ for $\vartheta = \vartheta_x(0)$ in the limit as $s \to 0$, with $\vartheta_x(0)$ being the initial opening angle (see § 3.2). Equation (4.5g) now yields the following formula for fixing the coefficient v_2:

$$v_2 = \vartheta_x(0)\frac{\beta}{\bar\beta}(4\cos^2\vartheta - 1)^{1/2} , \quad (4.6h)$$

where we must take the positive square root to make ρ positive as given by equation (4.5e). The equations (4.6e)–(4.6h), etc., demonstrate that the inner limit of the outer solution does indeed provide an asymptotic match to the outer limit of the inner solution (3.10a)–(3.10e). Thus, the X-celerator mechanism can effect a smooth crossover from the region of trans-sonic flow to the region of trans-Alfvénic flow.

For higher accuracy in the numerical work (which involves "patching"), we obtain also the next order (s^{-1}) term in the expansion for ψ. A little algebra demonstrates that equation (4.1) yields a second-order ODE in ϑ for ψ_1:

$$\frac{d^2}{d\vartheta^2}(\beta\psi_1) + \beta\psi_1 = -K\sin\vartheta , \quad (4.7a)$$

where we have used equation (4.6e) to write $\beta\psi_0' = K$. The solution of the homogeneous counterpart of equation (4.7a) reads $\beta\psi_1 = A\sin\vartheta + B\cos\vartheta$, where A and B are integration constants. On the other hand, a particular solution of the actual equation (4.7a) is $\beta\psi_1 = (K/2)\vartheta\cos\vartheta$; thus the most general solution reads

$$\beta\psi_1 = \frac{\bar\beta}{\vartheta_x(0)}\left(\frac{1}{2}\vartheta\cos\vartheta + a\sin\vartheta + b\cos\vartheta\right) . \quad (4.7b)$$

The lower boundary condition,

$$\psi_1 = 0 \quad \text{at } \vartheta = 0 , \quad (4.7c)$$

implies $b = 0$. A Taylor series expansion about $s = 0$ of the upper boundary condition, $\psi = 1$ at $\vartheta = \vartheta_x(s)$, yields for the coefficient proportional to s:

$$\frac{d\psi_0}{d\vartheta}\frac{d\vartheta_x}{ds}(0) + \psi_1 = 0 \quad \text{at} \quad \vartheta = \vartheta_x(0) . \quad (4.7d)$$

With $\beta\, d\psi_0/d\vartheta = \bar\beta/\vartheta_x(0)$, equation (4.7d) implies that a satisfies

$$a\sin\vartheta_x(0) = -\frac{d\vartheta_x}{ds}(0) - \frac{1}{2}\vartheta_x(0)\cos\vartheta_x(0) . \quad (4.7e)$$

This identification completes our solution for $\psi_1(\vartheta)$ in equation (4.7b).

5. SUMMARY AND CONCLUSIONS

In this paper we have formulated the governing equations for a steady, axisymmetric, X-celerator wind in terms of the mathematics of matched asymptotic expansions. By a judicious choice of reference frame, variables, and expression of boundary conditions, we can avoid many of the standard difficulties associated with a Grad-Shafranov formulation: the indeterminacy of unknown functions, the existence of a PDE of multiple mixed type, etc. A particularly useful feature of our approach is the analytical derivation of the properties of the emergent flow from the narrow interface (X-region), resulting from the interaction of a strongly magnetized star and an electrically conducting accretion disk, where the gas accelerates from rest to supersonic speeds.

The transsonic acceleration occurs by means of a combination of gas pressure and magnetic forces. Basically, gas under high pressure at deeper layers of the X-region—surrounding the X-point of the critical equipotential, where the effective gravity vanishes—finds itself facing regions above and below the disk of relatively low gas pressure. The high-pressure gas, finding little resistance from the effective gravity, flows outward in a X-wind frozen to strong poloidal fields that asymptotically fan radially away from the X-region and that have almost the same strength on every mass-carrying streamline. The continuation of this outflow into the main acceleration zone of the X-celerator model constitutes the subject of study of Paper III.

During the years between the gestation (1986–1987) of the ideas presented in this paper and their final fruition (1993), we have benefited from discussions with numerous friends and colleagues—too many to thank individually, although Al Glassgold and Charlie Lada come to mind for special mention. This work was funded in part by NSF grant AST 90-24260 and in part under the auspices of a special NASA Astrophysics Theory Program that supports a joint Center for Star Formation Studies at NASA/Ames Research Center, the University of California at Berkeley, and the University of California at Santa Cruz.

APPENDIX

SPECIAL CURVILINEAR COORDINATE SYSTEM

The pseudopolar coordinate system (s, ϑ) introduced in § 4 (see also § 3 for its scaled counterpart) yields a convenient grid for meridional plane calculations (1) in the X-region or (2) throughout the flow if the uppermost streamline is constrained to remain straight, i.e., if $\vartheta_x(s)$ retains a constant value $\vartheta_x(0)$ for all s. If, however, $\vartheta_x(s)$ varies with s—e.g., if the uppermost streamline follows a curve that resembles the upturning locus of the critical equipotential—then we find it computationally more convenient to introduce a curvilinear coordinate system (q, t) such that the upper streamline is defined by a constant value of t, chosen for definiteness to be $t = 1$ (see Fig. 1 of Paper III). To discover the transformation properties from (s, ϑ) to (q, t), we begin by imposing the orthogonality condition $\nabla q \cdot \nabla t = 0$, i.e.,

$$\frac{\partial q}{\partial s}\frac{\partial t}{\partial s} + \frac{1}{s^2}\frac{\partial q}{\partial \vartheta}\frac{\partial t}{\partial \vartheta} = 0 . \tag{A1}$$

Considerable freedom still exists in the exact specifications for (q, t) even if equation (A1) is satisfied. To make the choice more definite, let us define

$$t(s, \vartheta) \equiv \frac{\vartheta}{\vartheta_x(s)} , \tag{A2}$$

which fixes $t = 1$ on the upper streamline. The orthogonality condition (A1) now reads

$$-\frac{\vartheta\vartheta_x'(s)}{\vartheta_x^2(s)}\frac{\partial q}{\partial s} + \frac{1}{s^2\vartheta_x(s)}\frac{\partial q}{\partial \vartheta} = 0 , \tag{A3}$$

which we may regard as a linear PDE of first order to solve for q. The method of characteristics yields the solution,

$$q = \text{constant} \quad \text{on} \quad -\frac{\vartheta_x(s)}{s^2\vartheta_x'(s)}ds = \vartheta\, d\vartheta . \tag{A4}$$

Equation (A4) may be integrated to give

$$I(s) + \tfrac{1}{2}\vartheta^2 = \text{constant} , \tag{A5}$$

where $I(s)$ is the integral function

$$I(s) \equiv \int_{s_0}^{s} \frac{\vartheta_x(\sigma)d\sigma}{\sigma^2\vartheta_x'(\sigma)} , \tag{A6}$$

with s_0 being an arbitrary positive constant. The solution for q therefore takes the form

$$q = F[I(s) + \tfrac{1}{2}\vartheta^2] , \tag{A7}$$

where F is an arbitrary function, whose form is at our disposal to choose as we like.

For convenience, we choose F so that $q = s$ along $\vartheta = 0$. With $q(\vartheta = 0) = F[I(s)]$, we see that we want $F = I^{-1}$, and

$$q(s, \vartheta) = I^{-1}[I(s) + \tfrac{1}{2}\vartheta^2] , \tag{A8}$$

with I^{-1} being the inverse of the integral function defined by equation (A6). The integral I and its inverse I^{-1} can be tabulated numerically at discrete points and evaluated accurately for any value of their arguments with the help of interpolation by splines.

The metric coefficients h_q and h_t associated with the coordinates q and t can then be found from the formulae:

$$h_q^{-2} = \left(\frac{\partial q}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial q}{\partial \vartheta}\right)^2 , \tag{A9a}$$

$$h_t^{-2} = \left(\frac{\partial t}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial t}{\partial \vartheta}\right)^2 . \tag{A9b}$$

We can simplify the indicated operations by noting that equation (A8) allows us to write

$$I(q) = I(s) + \tfrac{1}{2}\vartheta^2 , \tag{A10}$$

so that $I'(q)(\partial q/\partial s) = I'(s)$, etc. Some manipulation finally obtains

$$h_q = \frac{I'(q)s^2\vartheta_x'(s)/\vartheta_x(s)}{\{1 + [st\vartheta_x'(s)]^2\}^{1/2}} , \tag{A11a}$$

$$h_t = \frac{s\vartheta_x(s)}{\{1 + [st\vartheta_x'(s)]^2\}^{1/2}} . \tag{A11b}$$

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