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LUMINOSITY VARIATIONS OF 3C 345: IS THERE ANY EVIDENCE OF LOW-DIMENSIONAL CHAOS?

A. PROVENZALE,[1] R. VIO,[2] AND S. CRISTIANI[3]

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ABSTRACT

We analyze the light curve of the optically violent variable quasar 3C 345, with the goal of determining the possible presence of low-dimensional chaos in the system dynamics. The results of the phase-space reconstruction and of the correlation integral analysis provide a convergent, noninteger estimate of the correlation dimension, naively suggesting the presence of deterministic chaos. However, we show that this result is generated by the long memory and by the power-law shape of the power spectrum of the signal, with no relationship to any underlying chaotic behavior. In fact, the light curve of 3C 345 is consistent with the output of a nonlinear stochastic process, and it is characterized by a well-defined intermittent nature.

Subject headings: methods: numerical — quasars: individual (3C 345)

1. INTRODUCTION

In a recent paper Vio et al. (1991) have shown that the optical light curve of the optically violent variable (OVV) quasar 3C 345 may be described in terms of a nonlinear stochastic process. This means that the light emission of this object depends, in a nonlinear way, on a potentially very large number of physical parameters. The nonlinearity of the process is necessary for explaining the "explosive" character of the light curve, whereas the stochasticity has been introduced for explaining its unpredictable temporal evolution.

In the last years, however, it has become increasingly clear that even simple, deterministic nonlinear systems with chaotic behavior may produce "noisy" time series which may display some of the properties of stochastic systems, even though their dynamics is governed by a small set of deterministic differential equations, see, e.g., Eckmann & Ruelle (1985) and Ott (1993), for an introduction to chaos in dynamical systems.

Along these lines, many works have then been devoted to the attempt of determining whether the observed variability of different astronomical objects might be ascribed to the action of a low-dimensional, deterministic nonlinear entity such as a strange attractor in the system phase space, see, e.g., Voges, Atmanspacher, & Scheingraber (1987), Cannizzo & Goodings (1988), Lochner, Swank, & Szymkowiak (1989), Norris & Matilsky (1989), Cannizzo, Goodings, & Mattei (1990), Harding, Shinbrot, & Cordes (1990), Kollath (1990), Krolik, Done, & Madejsky (1993). In particular, the visible and X-ray variability of several different quasars has been analyzed by the most common methods developed in dynamical system theory, and in some cases claims on the presence of low-dimensional chaos in the quasar system have been advanced (see for example Letho, Czerny, & McHardy 1993).

In order to assess the properties and limitations of phase-space analysis when applied to astronomical data, in the present work we reconsider the light curve of the OVV quasar 3C 345 and we test whether its unpredictable nature may be associated with an underlying low-dimensional chaotic behavior. To this end, we use the procedures recently discussed by Theiler et al. (1992) and Provenzale et al. (1992), that constitute a sort of "reliability test" to be applied to any system which is candidate to an interpretation in terms of low-dimensional dissipative chaos. The basics of this approach are to create appropriate surrogate stochastic time series which have to be analyzed and compared to the original signal.

The remainder of this paper is as follows. Section 2 provides an introduction to the standard methods to search for chaos in measured signals; here we also review the appropriate tests for chaoticity to be used on the light curve of 3C 345. Section 3 reports the analysis of the luminosity variations of the OVV quasar 3C 345; here we show that this light curve does not display any evidence of low-dimensional chaos. Finally, § 4 gives conclusions and perspectives.

2. THE SEARCH FOR CHAOS IN MEASURED SIGNALS

Among the various methods now available for testing the chaoticity of a measured time series (see Drazin & King 1992 for an up-to-date view of the field), here we just recall the evaluation of the correlation dimension D_2 and of the K_2 entropy (Grassberger & Procaccia 1983, 1984), the determination of the Lyapunov exponents (Sano & Sawada 1985; Eckmann et al. 1986; Abarbanel, Brown, & Kadtke 1990) and of the approximate number of excited empirical modes (Broomhead & King 1986), as well as the various predictive algorithms based on appropriate modifications of classical AR approaches or on neural network algorithms (Farmer & Sidorovich 1987; Casdagli 1989; Sugihara & May 1990; Elsner & Tsonis 1992; Smith 1992). In this section we provide a brief introduction to the procedures of phase-space reconstruction and correlation dimension estimate, for further details see, e.g., Eckmann & Ruelle (1985), Provenzale et al. (1992), and references therein.

In general, the first step in practically all these analysis methods is a procedure of phase-space reconstruction which is commonly known as the "time embedding" technique (Takens 1981). In short, this is based on conceptually substituting the true phase-space variables of the system with the time derivatives of increasing order of the signal under study, and then, from a practical standpoint, to substitute the time derivatives with time-delayed values of the measured variable. Given the scalar time series $x(t_i)$, $t_i = t_0 + i\,\Delta t$, $i = 1, \ldots, N$, where Δt is

[1] Istituto di Cosmogeofisica del CNR Corso Fiume 4, I-10133 Torino, Italy.
[2] ESA, *IUE* Observatory, Villafranca del Castillo satellite tracking station, Apartado 50727, 28080 Madrid, Spain.
[3] Dipartimento di Astronomia dell' Universita' Vicolo dell'Osservatorio 5, I-35122 Padova, Italy.

the sampling interval and N is the number of points in the signal, the vector signal $\mathbf{x}(t_i)$ in the reconstructed space is obtained as

$$\mathbf{x}(t_i) = \{x(t_i), x(t_i + \tau), \ldots, x[t_i + (M-1)\tau]\}, \quad (1)$$

where τ is an appropriate multiple of the sampling interval, $\tau = m\,\Delta t$, and M is the dimensionality of the reconstructed phase space. The choice of the value of τ is not devoid of danger; see, e.g., Grassberger, Schreiber, & Schraffrath (1991) for a discussion on the optimal choice (if any) of the time delay. As a minimal requirement, the value of τ should be taken in an interval such that variation of the time delay inside this interval does not modify the results of the analysis. One of the most common choices is to take a value of τ which is close to the first zero of the autocorrelation of the scalar signal $x(t_i)$. In the reconstructed space, the correlation integral may then be defined as

$$C_M(r) = \frac{1}{N'^2} \sum_{i,j=1}^{N'} \Theta[r - |\mathbf{x}(t_i) - \mathbf{x}(t_j)|], \quad (2)$$

where Θ is the Heaviside step function, $N' = N - (M-1)m$ and the vertical bars indicate the norm of the vector. For a deterministic attractor in its true phase-space (or in a properly reconstructed space when M is large enough), for small values of r the correlation integral has a power-law behavior

$$C_M(r) \propto r^{D_2}, \quad (3)$$

where D_2 is the correlation dimension of the attractor; the value of D_2 provides an approximation to its fractal (Hausdorff) dimension. For nonfractal attractors the correlation dimension coincides with the topological dimension. A power-law behavior of the correlation integral is observed also for fractal, nondeterministic objects such as the trajectories produced by fractional Brownian motions in the reconstructed space; in this case, D_2 is the correlation dimension of the trajectory (see, eg., Osborne & Provenzale 1989). An alternative technique for evaluating the attractor dimension has been proposed by Termonia & Alexandrowicz (1983); in general, several algorithms are now available for the evaluation of the entire multifractal spectrum of generalized dimensions, see, e.g., Borgani et al. (1993) for a critical analysis of the properties and pitfalls of the various methods.

If the dimension M of the reconstructed space is smaller than the fractal dimension of the attractor, then the correlation integral has a power-law behavior $C_M(r) \propto r^M$. When M has been chosen sufficiently large (typically, $M > 2D_2 + 1$), the slope of the correlation integral saturates to the constant (M-independent) value D_2. Thus, in the analysis of time series produced by a low-dimensional attractor, the correlation integrals corresponding to increasing values of M display an increasing logarithmic slope until saturation at the value D_2 is attained. After saturation, the estimated value of D_2 provides a reliable measure of the attractor dimension.

In the study of a measured signal with a poorly understood dynamical origin (i.e., without an explicit knowledge of the low-dimensional deterministic nature of the generating system), the same procedure is usually followed; the correlation integrals $C_M(r)$ for increasing values of M are evaluated and their possible scaling nature is determined. If the integrals $C_M(r)$ display a power-law behavior, i.e., if

$$C_M(r) \propto r^{D_2(M)} \quad (4)$$

in an appropriate range of scales, the following step is to evaluate the behavior of $D_2(M)$ at increasing M's. By analogy with the behavior of systems whose dynamics is governed by a strange attractor, a convergence of the scaling exponent $D_2(M)$ to a finite, noninteger value D_2 is usually taken as an indication that the system under study is dominated by low-dimensional chaos; the value of D_2 is then taken as an estimate of the dimension of the strange attractor. By contrast, a nonsaturation of $D_2(M)$ is taken as an indication that the dimensionality of the system (related to the number of excited modes in the dynamics) has not yet been attained. For example, a white noise signal induces a nonsaturation of the slope of $C_M(r)$ for every M.

The main problem with the above approach (apart from the possibility of spurious results due to limited statistics, see, e.g., Smith 1988; Eckmann & Ruelle 1992), is due to the fact that a convergence of $D_2(M)$ to a finite value D_2 may be generated not only by low-dimensional chaos but also from various types of stochastic processes with long time correlations and power-law power spectra (Osborne & Provenzale 1989; Provenzale, Osborne, & Soj 1991; Provenzale et al. 1992; Vio et al. 1992). For this reason, simply observing a convergence of the correlation dimension to a finite value cannot be taken as a reliable indication of low-dimensional chaos.

To overcome the above difficulty, in the last few years various specific tests have been developed (Osborne et al. 1986; Theiler et al. 1992; Provenzale et al. 1992). In general, these tests are based on the concept of "surrogate data" (Theiler et al. 1992; Smith 1992), i.e., on the idea of appropriately modifying the original signal in order to determine whether the convergence of D_2 (or any other result of the analysis) is destroyed together with the property which has been modified or whether its origin lies in some other (untouched) characteristics of the data.

Among the various methods, a useful and effective approach is that based on the procedure of Fourier phase randomization (Osborne et al. 1986; Theiler et al. 1992). This approach consists in substituting the original Fourier phases of the time series with random, uniform distributed phases and then in inverting the phase randomized Fourier spectrum. In this way it is possible to obtain a surrogate stochastic time series which has the same power spectrum (autocorrelation) of the original signal but no phase correlations.

In the case of a signal produced by a low-dimensional chaotic system, the phase randomization destroys the convergence of the correlation dimension to a finite value. This implies that the convergence of the dimension for a nonlinear deterministic system is generated by the Fourier phase correlations, which are in turn related to the higher order moments of the probability distribution of $x(t_i)$. For a low-dimensional chaotic system, the Fourier phase-correlations at small frequency may be thought of as being generated by the existence of "close returns" in phase space, while correlations at large frequencies are induced by the differentiable nature of the signal $x(t_i)$.

Conversely, the convergence of the dimension for long-memory stochastic signals is due to the shape of the power spectrum, i.e., to the form of the autocorrelation, which is not changed by phase randomization. The procedure of phase randomization has thus proven to be an effective way for distinguishing between a convergence of the dimension generated by the presence of low-dimensional chaos and that induced by the possible fractal nature of the signal under study. In case the results provided by the correlation integral analysis do not change under phase randomization, then it is clear that the interpretation of the system dynamics in terms of a low-dimensional deterministic attractor should be drastically revised.

In case the results of the correlation integral analysis do *significantly* change under phase randomization [i.e., $D_2(M)$ does not converge anymore for the phase-randomized surrogate signal], then there is some chance that a low dimensional attractor may govern the system dynamics. However, in this case further tests are needed in order to assess the robustness of the results (see, e.g., the "Algorithm 2" in Theiler et al. 1992).

In addition to phase randomization, a different test to verify the presence of low-dimensional deterministic dynamics is based on repeating the analysis on the time series obtained by taking the first differences of the original signal (Provenzale et al. 1992). For a signal produced by a low-dimensional attractor, both the original and the first-differenced data give approximately the same estimate of the correlation dimension. By contrast, in the case of signals generated by a stochastic process with long time correlations ("colored noise"), even though the original data give a finite estimate of the dimension, the differenced data provide a nonsaturating value of $D_2(M)$. Again, this is due to the fact that for stochastic systems the convergence of the estimated dimension is forced by the power-law shape of the power spectrum which, conversely to the Fourier phases, is modified by differentiating the signal. The main drawback of this technique is its sensitivity to measurement errors in the data. Consequently, this method may be safely used only on data characterized by a good signal-to-noise ratio.

3. ANALYSIS OF THE LUMINOSITY VARIATIONS OF 3C 345

We now consider the phase-space reconstruction and analysis of the luminosity variations of the OVV 3C 345. The light curve of this object has already been studied by several authors; see, e.g., Vio et al. (1991).

3.1. *Phase Space Reconstruction and Analysis*

In the case of 3C 345, the main problem in the correlation integral analysis is the discontinuous and irregular sampling of the signal. This fact complicates the direct use of the phase-space reconstruction procedure discussed in the previous section. To overcome this difficulty, a possible approach (and the one most commonly used) is the interpolation of the gaps of the time series. In the following we use this procedure, by considering a simple linear interpolation. Of course, such a simple operation may influence the results of the analyses. However, in the present case, the use of more sophisticated smooth interpolators (e.g., cubic spline interpolation) does not improve the situation, since without "a priori" information there is no reason to prefer an oscillating interpolation to a simpler one. On the other hand, the techniques for the reconstruction of uneven time series, such as those proposed by Scargle (1989) or Roberts, Lehar, & Dreher (1987), are useless here because they are able to deal only with linear signals. The only possibility to reliably quantify the effects of "filling the gaps" is by means of numerical simulations, as it is discussed below.

Figure 1 shows the light curve of 3C 345 (*a*) and the power spectrum of the linearly interpolated curve (*b*). Note the power-law shape of the power spectrum, approximately $P(f) \propto f^{-1.5}$, over a large frequency range.

In order to proceed with the phase space reconstruction, a value of the time delay τ to be used in the embedding procedure has to be chosen. Here we use a value of τ which is close to the first zero of the autocorrelation function, namely $\tau = 80\,\Delta t$. Other values of τ around the chosen value provide analogous results. Figures 2*a* and 2*b* report the correlation integrals $C_M(r)$ versus r, for $M = 1, \ldots, 6$ and the corresponding average logarithmic slopes $D_2(M)$ versus M. The average slopes have been obtained from least-square-fits of log $C_M(r)$ versus log r over the scaling range for $0.005 < C_M(r) < 0.5$. From Figure 2, a neat saturation of $D_2(M)$ to a value $D_2 \simeq 2.4$ is observed, naively suggesting the possible presence of low-dimensional chaos. To verify this inference, in the following we apply the phase-randomization and differentiation tests discussed in the previous section.

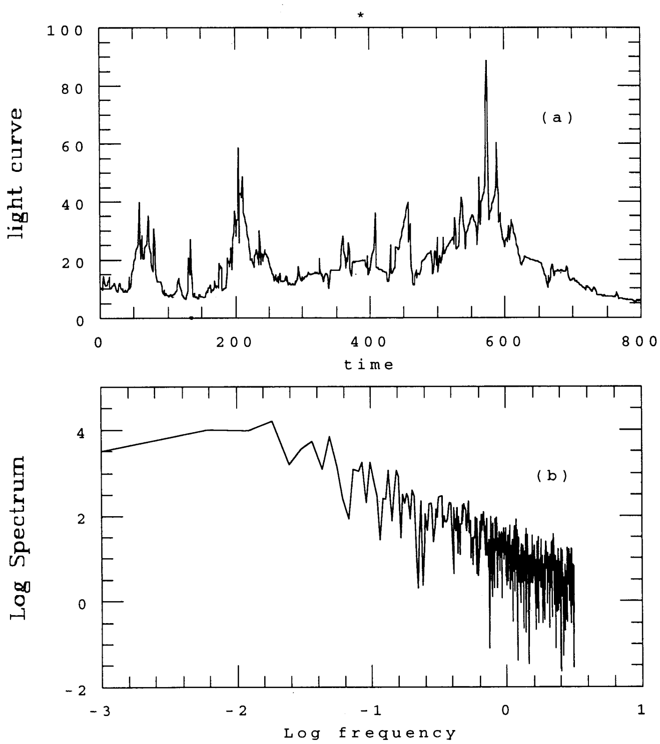


FIG. 1.—Panel (*a*) shows the light curve of the OVV 3C 345. Time is in unit of the sampling period $\Delta t = 11$ days. Panel (*b*) shows the power spectrum of the light curve of 3C 345.

3.2. *Phase Randomization*

The procedure of Fourier phase randomization provides the time series shown in Figure 3*a*. Such a signal is now consistent with the output of a linear stochastic process with the same second order moment as the original signal. Figure 3*b* shows the correlation integrals for this time series, and Figure 3*c* reports the average slopes $D_2(M)$ versus M. Again, a clear saturation, now at $D_2 \simeq 3.1$, is visible, suggesting that the system is characterized by stochastic dynamics. The results provided by a low-dimensional deterministic system are in fact completely different. To give an example of this latter behavior, in Figure 4*a* we show a 800 point time series obtained from the well-known Lorenz model (Lorenz 1963). This time series has been obtained by taking $\{[20z(t) + x(t)^2]/100\}^2$, where x and z are the x and z components of the Lorenz system, in order to obtain a postive-defined signal with enhanced bursts of activity. To illustrate the effect of phase randomization on this time series, in Figure 4*b* we show the surrogate signal obtained by

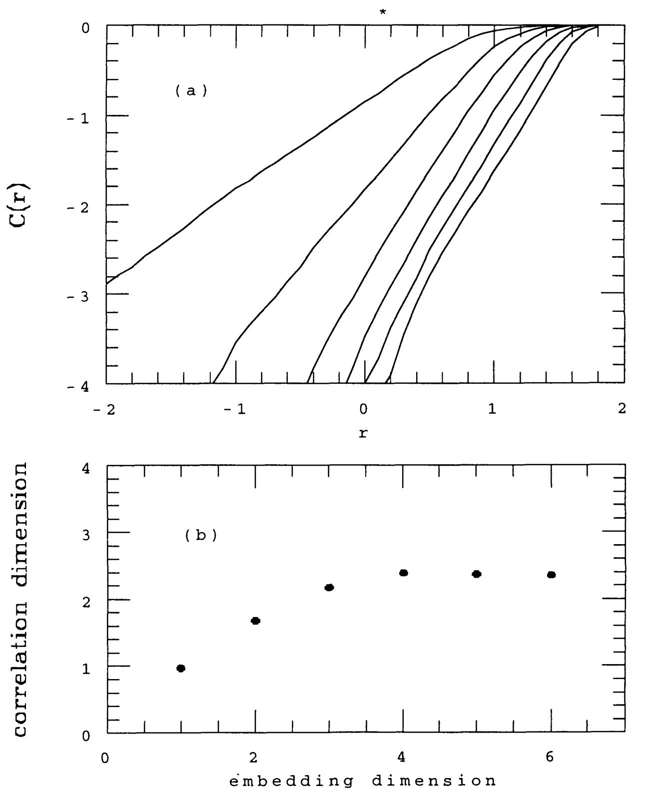


FIG. 2.—Panel (*a*) shows the correlation integrals $C_M(r)$, $M = 1, \ldots, 6$, in log-log coordinates for the light curve of 3C 345. The time delay is $\tau = 80\Delta t$. Panel (*b*) shows the quantities $D_2(M)$ vs. M as obtained by linear least-square fits of $\log C_M(r)$ vs. $\log r$ in the scaling range $0.005 < C_M(r) < 0.5$.

phase randomizing the chaotic time series. Figure 4*c* shows $D_2(M)$ versus M for the two signals; note that the phase randomized time series generates a nonconvergent correlation dimension. Thus, the results obtained from the analysis of the light curve of 3C 345 suggest that this signal should be better described in terms of a nonlinear stochastic process, and confirm that obtaining a convergent estimate of D_2 in the analysis of a long-memory signal is not sufficient to infer the presence of a strange attractor.

As a further comment on phase randomization, we note that the value of D_2 obtained from the phase-randomized light curve of 3C 345 is slightly larger than the value found for the original data. This effect, although small, may be ascribed to linear interpolation. To verify this, we have considered the phase randomized signal shown in Figure 3*a* with the same temporal sampling as the original time series of 3C 345 and with the gaps filled by linear interpolation, as shown in Figure 5*a*. In this case, the correlation dimension converges to the value $D_2 \simeq 2.4$, as indicated by Figure 5*b* where we show $D_2(M)$ versus M for this signal (*solid points*) and for the original phase-randomized data (*open circles*). This result indicates that the interpolation and filtering procedures have forced the estimated dimension to a smaller value. Note that this is opposite to what is observed for low-dimensional attractors, where the filtering may force the estimated dimension to *larger* values.

In fact, the interpolation effects observed for 3C 345 are consistent with the behavior of a stochastic system. To illustrate this, in Figure 6*a* we show a 800 point time series generated by the nonlinear stochastic process

$$\frac{dy}{dt} = (\alpha - 0.5)\beta - y(t) + [2\beta y(t)]^{1/2} w(t) , \tag{5}$$

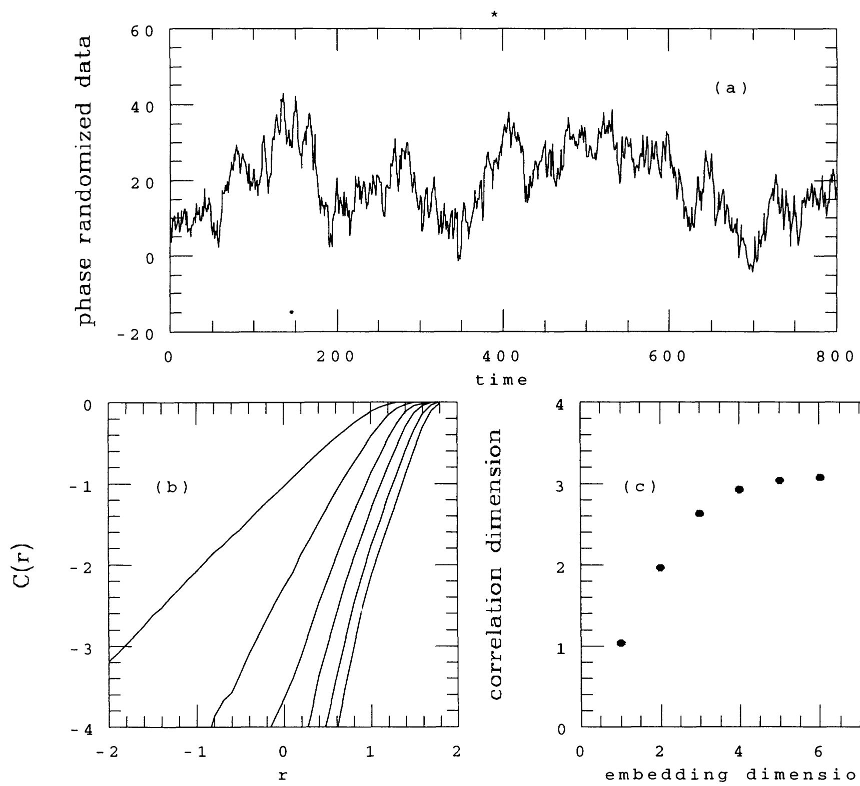
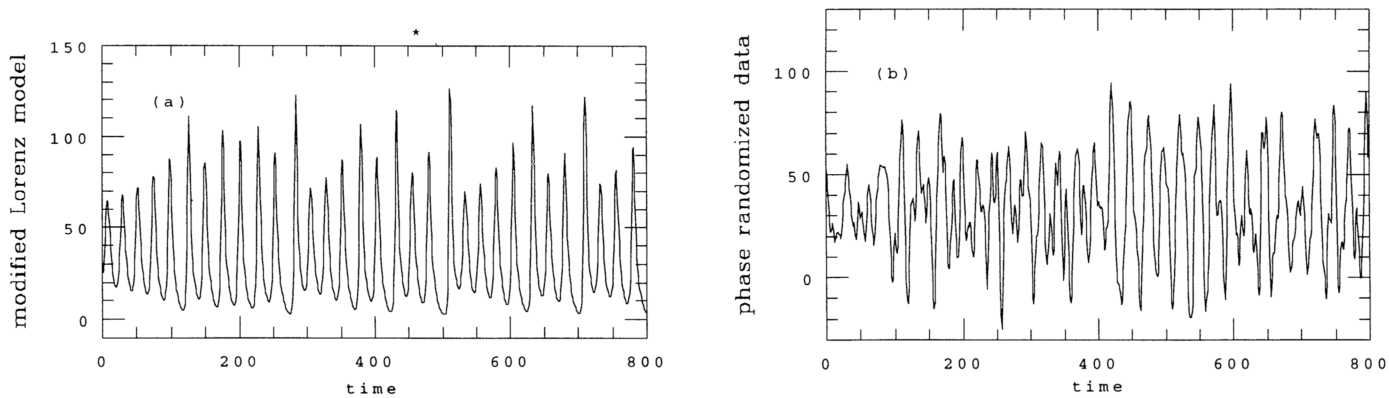


FIG. 3.—Panel (*a*) shows the phase-randomized light curve of 3C 345. Panels (*b*) and (*c*) show, respectively, the correlation integrals and the values of $D_2(M)$ vs. M for the phase-randomized signal.



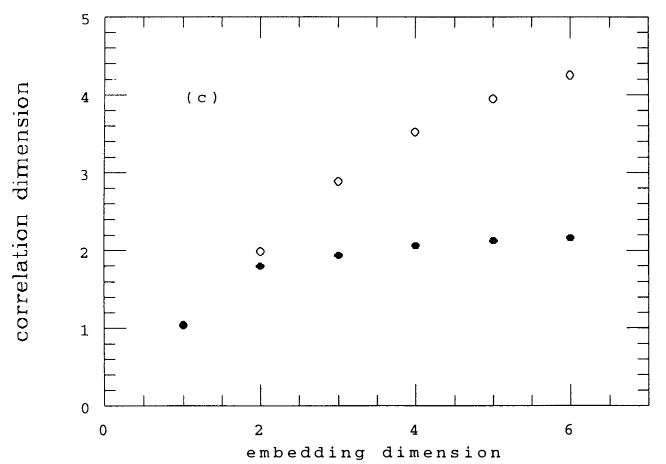


FIG. 4.—Panel (*a*) shows a deterministic time series, obtained from the Lorenz model as discussed in the text. Panel (*b*) shows the corresponding phase-randomized signal, and panel (*c*) shows $D_2(M)$ vs. M for the two time series. The solid points are for the original data and the open circles for the phase-randomized signal.

where $\alpha = \beta = 1$, $w(t)$ is a standard Gaussian white noise process and the sampling time is $\Delta t = 0.01$; see Vio et al. (1992) for a thorough discussion of this model. The signal $y(t)$ has a power-law power spectrum $P(\omega) \propto \omega^n$ with $n \approx -2$ for angular frequencies larger than $\omega \approx 1$. The signal shown in Figure 6*a* has been sampled with the same distribution of gaps as the 3C 345 light curve and linear interpolation has been used in order to "fill the gaps."

To apply the time embedding procedure to the time series shown in Figure 6*a*, we have again taken a time delay close to the first zero of the signal autocorrelation; in this case we use $\tau = 100\,\Delta t$. The correlation integrals corresponding to different embedding dimensions display a good scaling behavior; Figure 6*b* shows the values of $D_2(M)$ versus M for this stochastic time series (*solid circles*). As a comparison, the same panel shows also the values of $D_2(M)$ for the original time series $y(t)$ produced by equation (5), when no gaps and linear interpolation are made (*open circles*). These latter are larger than the corresponding values of $D_2(M)$ obtained for the linearly interpolated data, confirming that linear interpolation tends to lower the estimated dimension of a stochastic signal.

By phase randomizing the signal shown in Figure 6*a*, a linear Gaussian time series is obtained. In Figure 6*c* we show the values of $D_2(M)$ versus M for the phase-randomized data (*solid circles*). The estimated dimension still saturates for the stochastic signal; however, the saturation value of D_2 is now slightly larger than that obtained for the signal shown in Figure 5*a* and it is comparable with the value obtained for the signal without gaps. Again, the difference between these two saturation values may be entirely ascribed to the effects of interpolation: the values of $D_2(M)$ obtained by sampling the phase-randomized signal with the same gaps as the original time series, and by filling the gaps by linear interpolation, saturate to a lower value (*open circles*). In this latter case, the values of $D_2(M)$ are the same as those obtained for the signal in Figure 6*a*. In general, the behavior of this time series is quite similar to that observed for the light curve of 3C 345, confirming the non–low-dimensional nature of this data set.

3.3. *Signal Differentiation*

To further determine the absence of low-dimensional chaos in 3C 345, we have applied the differentiation test to the mea-

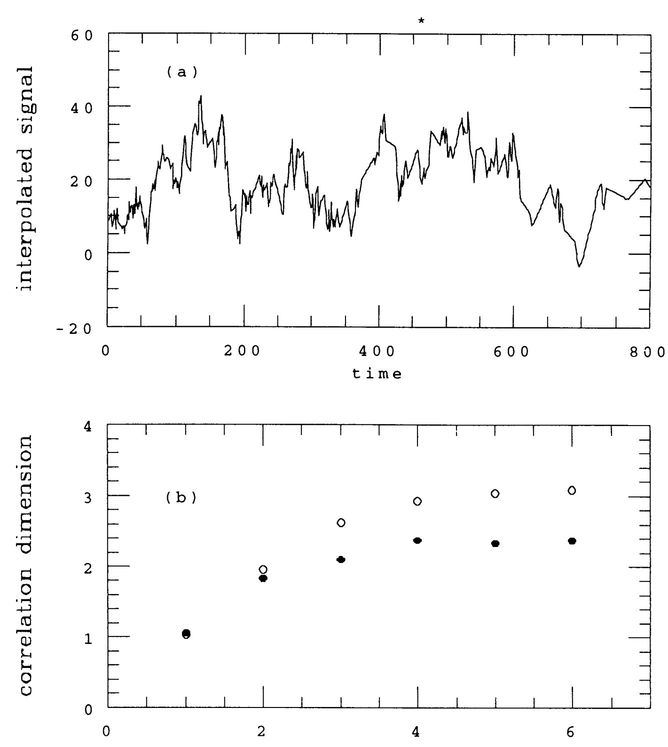


FIG. 5.—Panel (*a*) shows the linearly interpolated, phase-randomized light curve of 3C 345. Panel (*b*) shows the values of $D_2(M)$ for the interpolated signal (*solid points*) together with the corresponding values obtained for the phase-randomized signal shown in Fig. 3*a* (*open cirles*).

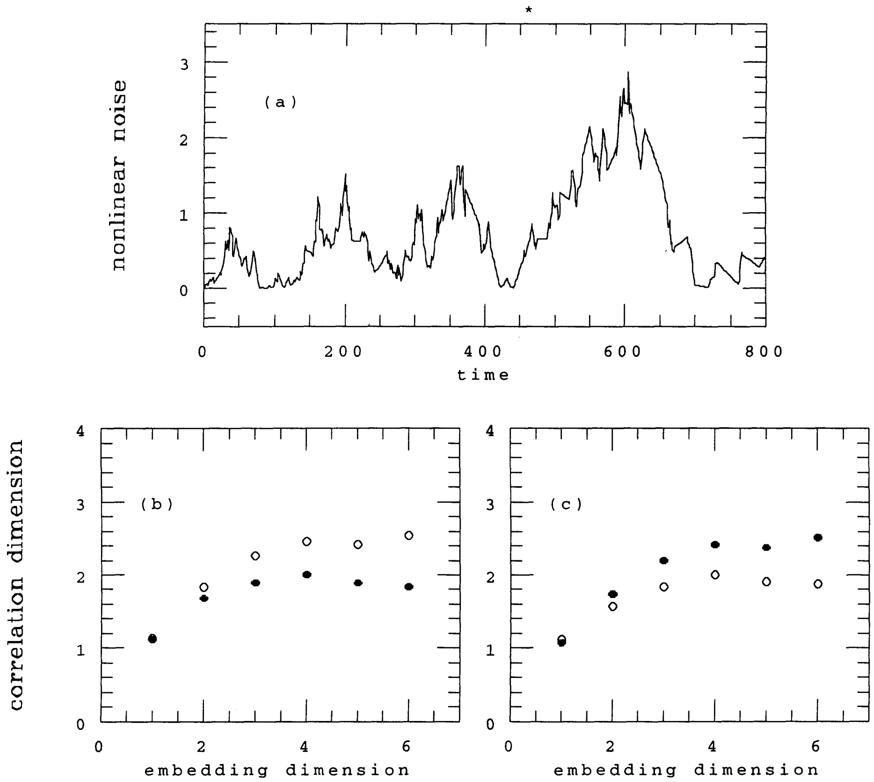


FIG. 6.—Panel (*a*) shows a time series obtained from the nonlinear stochastic process discussed in the text. Time is in unit of $\Delta t = 0.01$ and the series have been sampled and linearly interpolated as done for the 3C 345 data. Panel (*b*) shows the values of $D_2(M)$ for the original, noninterpolated signal (*open circles*) and for the linearly interpolated data (*solid points*). Panel (*c*) shows the values of $D_2(M)$ for the signal obtained by phase-randomizing the signal in panel (*a*). The solid points are for a purely phase-randomized time series and the open points are for a phase-randomized signal with sampling and linear interpolation as for 3C 345.

sured data. Figure. 7*a* shows the first differenced signal and Figure 7*b* shows the corresponding autocorrelation function overlapped to that of the original time series. As one can see from this figure, a remarkable difference exists between the two signals and the corresponding autocorrelation functions. As a comparative example, in Figure 7*c* we show the first differences for the stochastic signal shown in Figure 6*a*, and in Figure 7*d* we show the autocorrelation for the original and the first differenced stochastic time series. The behavior observed in this example is very similar to that obtained for 3C 345, again pointing toward a nondeterministic nature of the system. As an opposite example, in Figure 7*e* we show the first differenced signal obtained from the chaotic time series reported in Figure 4*a*, and in Figure 7*f* we show the autocorrelation functions of the original signal and of its first difference. For a chaotic system, the original and differenced signals have basically the same properties.

Figure 8 shows the values of $D_2(M)$ versus M for the differenced signal shown in Figure 5*a* (*solid points*), together with the values obtained for the original light curve (*open circles*). As one can see, no saturation is present in $D_2(M)$ for the differenced signal. From the results reported in this section, one has thus to conclude that the apparent convergence of the estimated dimension for the OVV 3C 345 cannot be considered as an evidence of low-dimensional chaos. Conversely, the luminosity variations of 3C 345 are consistent with the output of a nonlinear, intermittent stochastic process, as already discussed by Vio et al. (1991, 1992).

4. SUMMARY AND CONCLUSIONS

In this work we have applied the time embedding procedure and the correlation integral analysis to the light curve of the OVV 3C 345, finding a convergence of the estimated correlation dimension to a finite, noninteger value. By properly applying the phase randomization and differentiation tests, we have shown that such a result, which could have been naively interpreted as an evidence for low-dimensional chaos in the dynamics of this system, must be ascribed to the long correlation time and to the power-law shape of the power spectrum of the signal under study. In other words, we have to conclude that the present analysis does not indicate any evidence of low-dimensional chaotic dynamics in the variability of 3C 345. The properties of the time series are consistent with the output of a *nonlinear* stochastic process, as formerly discussed by Vio et al. (1991, 1992).

We stress the fact that simply determining a finite, convergent estimate of the correlation dimension in a measured signal should not be taken as a serious indication of the presence of low-dimensional chaos. In past years, such a simplistic view has led in fact to several uncorrect conclusions and to oversimplified pictures of the variability of natural systems. For these reasons, the use of appropriate tests (usually based on the concept of surrogate data) for verifying the inferences drawn from standard methods of phase space analysis should be considered as a necessary step for properly understanding the system dynamics.

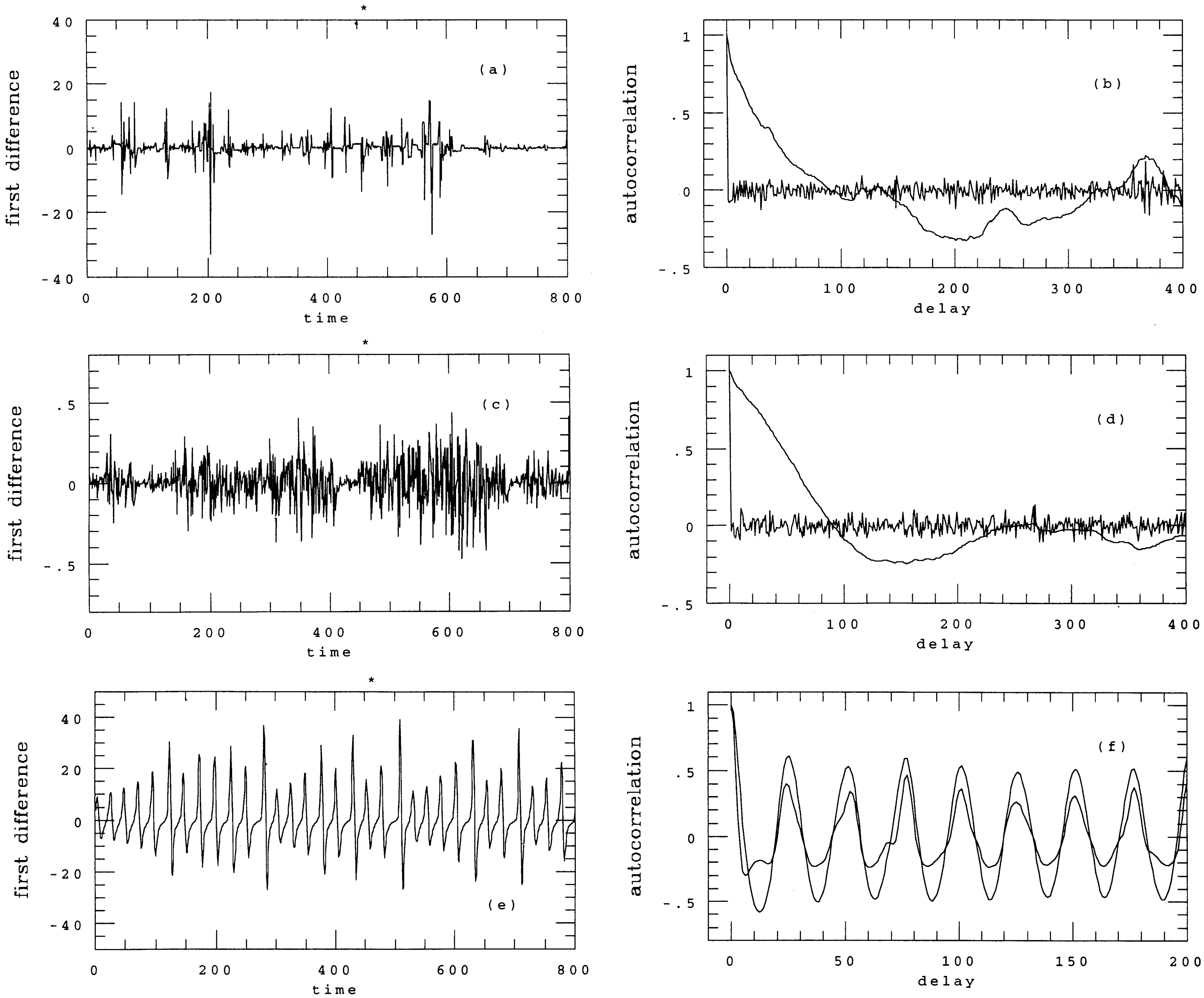


FIG. 7.—Panel (*a*) shows the first difference signal obtained from the light curve of 3C 345 and panel (*b*) shows the autocorrelation functions for the original signal and for the first differences. Panels (*c*) and (*d*) show respectively the first difference for the nonlinear noise and the autocorrelations for the original and the differenced signals. Panels (*e*) and (*f*) show the differenced data and the autocorrelations for the deterministic time series shown in Fig. 4*a*.

Before concluding, we want to venture a last remark: undoubtedly, the question on whether low-dimensional chaos may be a relevant paradigm for the understanding of *natural* astrophysical and geophysical systems, must be considered as a challenging problem. In fact, low-dimensional chaotic dynamics and strange attractors have proven to be extremely important from a conceptual point of view; they have also been shown to be relevant in the behavior of controlled laboratory systems, where the access to external control parameters allows for determining the bifurcation sequence and the properties of the system attractors. However, it is necessary to stress that it is by far less clear whether the notion of *low-*

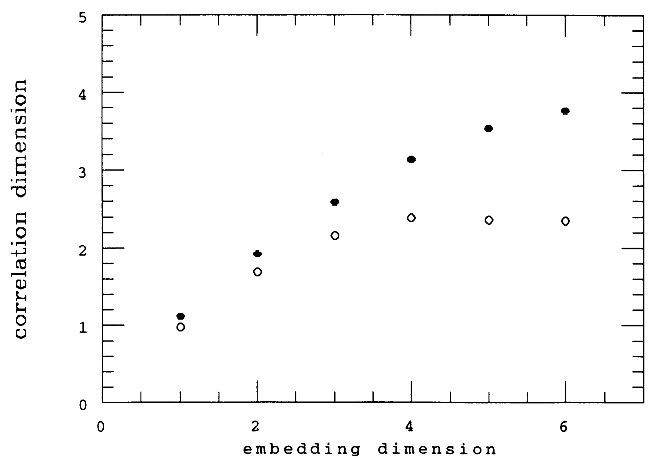


FIG. 8.—Values of $D_2(M)$ for the first differences obtained from the 3C 345 light curve (*solid points*) compared to those obtained for the original data (*open circles*).

dimensional strange attractor is relevant to the behavior of uncontrolled geophysical and astrophysical systems, outside laboratory conditions: very few, if any, presumed discoveries of strange attractors in natural systems have survived more refined verifications. Therefore, if from one hand it would be of extreme interest to discover a "genuine" low-dimensional attractor in astrophysics, on the other hand it is necessary to consider any assertion in this sense with great care.

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