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ELEMENT DIFFUSION IN THE SOLAR INTERIOR

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ABSTRACT

We study the diffusion of helium and other heavy elements in the solar interior by solving exactly the set of flow equations developed by Burgers for a multicomponent fluid, including the residual heat-flow terms. No approximation is made concerning the relative concentrations, and no restriction is placed on the number of elements considered. We give improved diffusion velocities for hydrogen, helium, oxygen, and iron, in the analytic form derived previously by Bahcall & Loeb. These expressions for the diffusion velocities are simple to program in stellar evolution codes and are expected to be accurate to ~15%. We find that the inclusion of the residual heat flow terms leads to an increase in the hydrogen diffusion velocity. We compare our numerical results with those obtained analytically by Bahcall & Loeb using a simplified treatment, as well as with those derived numerically by Michaud & Proffitt. We find that for conditions characteristic of the Sun, the results of Bahcall & Loeb for the hydrogen diffusion velocity are smaller than our more accurate numerical results by ~30%, except very near the center where the error becomes larger. The Michaud & Proffitt results differ from the numerical results derived here by $\lesssim 15\%$. Our complete treatment of element diffusion can be directly incorporated in a standard stellar evolution code by means of an exportable subroutine, but, for convenience, we also give simple analytical fits to our numerical results.

Subject headings: diffusion — stars: abundances — stars: interiors — Sun: interior

1. INTRODUCTION

Precise solar evolution calculations must be carried out to compare model results with observations of solar neutrino fluxes and of *p*-mode oscillation frequencies. In particular, element diffusion affects the element abundances, the mean molecular weight, and the radiative opacity in the core of the Sun, and therefore affects the calculated neutrino fluxes and oscillation frequencies. The characteristic time for elements to diffuse a solar radius under solar conditions is of the order of 6×10^{13} yr, much larger than the age of the Sun. Element diffusion therefore introduces only a small correction to standard solar model calculations. Bahcall & Pinsonneault (1992a, b) showed that helium diffusion increases the predicted event rates by ~11% in the chlorine solar neutrino experiment, by 3% in the gallium experiment, and by 12% in the Kamiokande and SNO experiments, while increasing the inferred primordial helium abundance by 0.4% and decreasing the calculated depth of the convection zone by 2%. Christensen-Dalsgaard, Proffitt, & Thompson (1993) calculated the sound speed as a function of radius in the solar model and concluded that helium diffusion causes a significant difference in the computed radial profile of the sound speed. Guenther, Pinsonneault, & Bahcall (1993) demonstrated that the helioseismological effects of helium diffusion depend upon the degree and frequency of the *p*-mode being discussed and have typical amplitudes of order 1–3 MHz.

Since the effects of diffusion are small, there is in principle no need for very high accuracy in its treatment. However, discrepancies appear between various results in the literature, depending on the approximations made. Previous studies of element diffusion in the Sun (Aller & Chapman 1960; Michaud et al. 1976; Noerdlinger 1977, 1978; Cox, Guzik, & Kidman 1989; Paquette et al. 1986; Bahcall & Loeb 1990, hereafter BL; Proffitt & Michaud 1991; Michaud & Proffitt 1992; Bahcall & Pinsonneault 1992a, b; Christensen-Dalsgaard et al. 1993; Guenther et al. 1993; Vauclair & Vauclair 1982 and references therein) have usually included one or more of the following simplifying assumptions: neglecting thermal diffusion, or treating it using a simplified empirical formula; neglecting the presence of heavy elements when calculating helium diffusion; assuming a negligible helium abundance when calculating the diffusion of heavier elements; adopting a single constant value for all Coulomb logarithms. In this paper, we provide a simple but complete treatment of the problem, making none of the above approximations, and we compare our results with those obtained under different simplifying assumptions. In particular, we compare our results with those obtained by BL and those obtained by Michaud & Proffitt (1992, hereafter MP). BL made most of the above simplifying assumptions; they used empirical results for the thermal diffusion coefficients and a single value for the Coulomb logarithms, equal to 2.2. MP solved the Burgers equations and then represented the effects of the residual heat flow vectors by an ad hoc correction to the results obtained when neglecting those heat fluxes. The principal difference between this work and most previous studies is that we solve the Burgers equations exactly and then represent the numerical results by simple analytic functions, rather than trying to obtain analytic solutions by various approximations.

Element diffusion in stars is driven by pressure gradients (or gravity), temperature gradients, composition gradients, and radiation pressure. Gravity tends to concentrate the heavier elements toward the center of the star. In a pure hydrogen-helium plasma, helium diffuses toward the center of the star, while hydrogen diffuses outward. The local rate of change of the hydrogen mass fraction is equal and opposite to the rate of change of the helium mass fraction. The light electrons also tend to rise, but are held back by an electric field which counteracts gravity. Temperature gradients lead to thermal diffusion, which tends to concentrate more highly charged and more massive species toward the hottest region of the star, its center. Concentration gradients oppose the above processes. Radiation pressure causes negligible diffusion in the solar core and will be neglected in this paper.

We study the relative diffusion of hydrogen, helium, and heavier elements, such as oxygen and iron. In contrast to many previous studies, no approximation is made concerning the relative concentrations of the various species, and no restriction is placed on the number of elements considered. Our method is therefore applicable to a wide variety of astrophysical problems, such as the diffusion of elements in white dwarf envelopes (see, e.g., Fontaine & Michaud 1979; Pelletier et al. 1986) and in globular cluster stars (see, e.g., Chaboyer et al. 1992). In this paper, we concentrate on calculating the diffusion velocities in the temperature and density ranges relevant to the Sun, although our exportable subroutine can be used to calculate diffusion velocities in red giants and in white dwarfs.

Burgers (1969) has provided a complete and straightforward set of equations to describe the evolution of a multicomponent fluid. In order to include the effects of thermal diffusion, he introduced the so-called "residual heat flow vectors." Here we will use the Burgers equations, including the residual heat fluxes, to describe the plasma in the solar interior. Even though these equations can in principle be solved analytically, the algebraic complexity increases rapidly with the number of species considered. For example, because of computational limitations, Noerdlinger (1977) included only three species (hydrogen, helium, and electrons) and adopted a single constant for all the Coulomb logarithms. In contrast, we solve the full set of Burgers equations numerically, and place no restriction on the number of species. The Coulomb logarithm is obtained by calculating the collision integrals using a pure Coulomb potential with a long-range cutoff at the Debye length. However, the result obtained for the Coulomb logarithm is valid only for plasmas that are sufficiently hot and rarefied, i.e., such that the plasma parameter Λ is much larger than unity. For conditions characteristic of the solar interior, the Coulomb logarithms are small, and can even become negative for collisions between heavy elements. For such plasmas, the collision integrals can be calculated numerically using a screened Debye-Huckel Coulomb potential. The results can then be fitted to simple analytic functions. We adopt an expression for the "effective" Coulomb logarithm obtained by Iben & MacDonald (1985) by fitting numerical results for Fontaine & Michaud (1979b).

It should be relatively easy to incorporate our complete treatment of element diffusion into any standard solar evolution code.[1] However, we have obtained simple analytic fits to the exact results, which can provide a convenient alternative. These fits can be expressed as follows: following BL's notations (see footnotes 2 and 4) and using BL's dimensionless variable (see § 2), the mass fraction of element s satisfies the equation

$$\frac{\partial X_s}{\partial t} = -\frac{1}{\rho r^2}\frac{\partial}{\partial r}\left[r^2 X_s T^{5/2}\xi_s(r)\right] , \tag{1}$$

where the partial derivatives are evaluated in the local rest frame of a mass shell in the star, i.e., in Lagrangian coordinates. The function $\xi_s(r)$ is related to the diffusion velocity w_s of species s through

$$\xi_s(r) = w_s(r)\rho(r)/T^{5/2}(r) . \tag{2}$$

It is convenient to express ξ_s in terms of the logarithmic deviates of the local pressure, temperature, and hydrogen concentration:

$$\xi_s(r) = A_p(s)\frac{\partial \ln p}{\partial r} + A_T(s)\frac{\partial \ln T}{\partial r} + A_{\rm H}(s)\frac{\partial \ln C_{\rm H}}{\partial r} . \tag{3}$$

Convenient expressions for A_p, A_T, and $A_{\rm H}$ are given for the important species of hydrogen, oxygen, and iron in equations (56)–(58) of § 7.

We ignore in this work the effects of meridional circulation and of turbulent mixing below the base of the convection zone. For this outer region of the Sun, effects not discussed here are probably required in order to account for the depletion of lithium (see, however, Proffitt & Michaud 1991 for a strong upper limit to the effect of turbulent mixing on the surface helium abundance). In standard solar models computed by different groups, there is no significant mixing in the solar interior over essentially the full main-sequence lifetime of the Sun (see, e.g., Bahcall 1990). Meridional circulation velocities are estimated to be several orders of magnitude smaller than diffusion velocities in the solar interior (see, e.g., Michaud & Vauclair 1991). There is also some helioseismological evidence that suggests that there has not been substantial mixing in the solar core (see Elsworth et al. 1990). Our results can, therefore, be applied self-consistently to the calculations of the solar interior of standard solar models.

This paper is organized as follows. In § 2, we introduce the notation and basic equations. In § 3, we describe the method of solution. In § 4, we give the results for the hydrogen and helium diffusion coefficients, for a fixed value of the temperature and density, characteristic of the solar core. We compare these results with those obtained by BL and MP. In § 5, we give the results for the heavy element diffusion coefficients, obtained under the same conditions. In § 6, we give the diffusion velocities in the Sun, and again we compare our results with those obtained by BL and MP. In § 7, we give analytical expressions for our numerical results. In § 8, we compare our expression for the electric field with the value obtained by Braginskii (1965, p. 205). Finally, in § 9, we give a summary of the most important results.

2. BASIC EQUATIONS

Each species of particles s is described by a distribution function $F_s(\boldsymbol{x}, \boldsymbol{v}, t)$ normalized to unit integral, a mean number density n_s, an ionic charge $q_s \equiv Z_s e$, and a mass m_s. All species are assumed to be at the same temperature T and in an overall hydrostatic equilibrium, since the temperature and pressure equilibration timescales are much shorter than the diffusion times. The mass and charge densities are $\rho_s = n_s m_s$ and $\rho_{es} = n_s q_s$. The mean fluid velocity of each species is defined by

$$\boldsymbol{u}_s = \int \boldsymbol{v} F_s d\boldsymbol{v} . \tag{4}$$

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[1] Our FORTRAN routine will be made available upon request.

The mean fluid velocity is given by

$$\boldsymbol{u} = \frac{1}{\rho} \sum_s \rho_s \boldsymbol{u}_s \,, \tag{5}$$

where $\rho = \sum_s \rho_s$ is the total mass density. The diffusion velocity for species s is defined by

$$\boldsymbol{w}_s = \boldsymbol{u}_s - \boldsymbol{u} \tag{6}$$

and is therefore measured relative to the mean velocity of the fluid as a whole. We define the "residual heat flow vector" for species s by (Burgers 1969):

$$\boldsymbol{r}_s = \left[\frac{m_s}{2k_\mathrm{B} T} \int F_s(\boldsymbol{v} - \boldsymbol{u}) |\boldsymbol{v} - \boldsymbol{u}|^2 d\boldsymbol{v} - \frac{5}{2} \boldsymbol{w}_s\right] , \tag{7}$$

where k_B is Boltzmann's constant. The cross section for Coulomb scattering between particles of species s and of species t (s can be equal to t) is given by

$$\sigma_{st} = 2\sqrt{\pi}\, e^4 Z_s^2 Z_t^2 (k_\mathrm{B} T)^{-2} \ln \Lambda_{st} \,, \tag{8}$$

where $\ln \Lambda_{st}$ is the Coulomb logarithm (Λ_{st} is the "plasma parameter"), a correction factor taking into account the logarithmic contribution of binary encounters with impact parameters up to the Debye shielding length. For the Coulomb logarithm, we adopt the following expression, obtained by Iben & MacDonald (1985) using numerical results for Fontaine & Michaud (1979b),

$$\ln \Lambda_{st} = \frac{1.6249}{2} \ln\left[1 + 0.18769\left(\frac{4k_\mathrm{B} T\lambda}{Z_s Z_t e^2}\right)\right] , \tag{9}$$

where $\lambda = \max(\lambda_\mathrm{D}, a_0)$, $\lambda_\mathrm{D} = (k_\mathrm{B} T/4\pi e^2 \sum_s n_s Z_s^2)^{1/2}$ is the Debye length, and $a_0 = (3/4\pi \sum_\mathrm{ions} n_i)^{1/3}$ is the interionic distance. The friction coefficient between species s and t is

$$K_{st} = (2/3)\mu_{st}(2k_\mathrm{B} T/\mu_{st})^{1/2} n_s n_t \sigma_{st} \,, \tag{10}$$

where $\mu_{st} \equiv m_s m_t/(m_s + m_t)$ is the reduced mass for species s and t.

The Burgers equations for mass, momentum, and energy conservation can then be written as

$$\frac{\partial n_s}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n_s w_s) = \left(\frac{\partial n_s}{\partial t}\right)_\mathrm{nucl.} , \tag{11}$$

$$\frac{dp_s}{dr} + \rho_s g - \rho_{es} E = \sum_{t \neq s} K_{st}[(w_t - w_s) + 0.6(x_{st} r_s - y_{st} r_t)] \,, \tag{12}$$

and

$$\frac{5}{2} n_s k_\mathrm{B} \frac{dT}{dr} = \sum_{t \neq s} K_{st}\left\{\frac{3}{2} x_{st}(w_s - w_t) - y_{st}[1.6x_{st}(r_s + r_t) + Y_{st} r_s - 4.3x_{st} r_t]\right\} - 0.8K_{ss} r_s \,. \tag{13}$$

In these equations, $\boldsymbol{g} \equiv -[GM(r)/r^2]\hat{e}_r$ is the gravitational acceleration, $g \equiv |\boldsymbol{g}|$, $\boldsymbol{E}$ is the electric field, $E \equiv |\boldsymbol{E}|$, $x_{st} = \mu_{st}/m_s$, $y_{st} = \mu_{st}/m_t$, and $Y_{st} = 3y_{st} + 1.3x_{st} m_t/m_s$. The numerical coefficients in equations (12) and (13) are related to the collision integrals and were obtained using a pure Coulomb potential with a long-range cutoff at the Debye length. More accurate results can be obtained by using numerical values derived from calculations using a screened Coulomb potential. We have assumed spherical symmetry and included a term for composition changes due to nuclear burning in equation (11). Using equation (12), it is straightforward to show that

$$\sum_s \left(\frac{dp_s}{dr} + \rho_s g - \rho_{es} E\right) = 0 \,, \tag{14}$$

or

$$\frac{dp}{dr} + \rho g - \rho_e E = 0 \,, \tag{15}$$

where $p \equiv \sum_s p_s$ and $\rho_e \equiv \sum_s \rho_{es}$ are the total pressure and total charge density. The departure from local charge neutrality is very small, with $\rho_e E/\rho g \sim G m_p^2/e^2 \sim 10^{-37}$ (see discussion and eqs. [22]–[23] in BL). Equation (15) therefore reduces to the familiar equation of hydrostatic equilibrium,

$$dp/dr = -\rho g \,. \tag{16}$$

In addition, the following constraints must be satisfied: charge neutrality,

$$\sum_s q_{es} n_s = 0 \,; \tag{17}$$

current neutrality,

$$\sum_s q_{es} n_s w_s = 0 \; ; \tag{18}$$

and local mass conservation,

$$\sum_s m_s n_s w_s = 0 \, . \tag{19}$$

Note that equation (19) follows from the fact that the diffusion equations are solved in the rest frame of the plasma. The set of linear equations (12)–(13) and (18)–(19) forms a closed system for the diffusion velocities $\boldsymbol{w}_s$, the residual heat flow vectors $\boldsymbol{r}_s$, the gravitational acceleration $\boldsymbol{g}$ and the electric field $\boldsymbol{E}$ in terms of the pressure, temperature, and concentration gradients. Since we already know the value of $g = GM(r)/r^2$, this relation provides a useful check on the numerical results.

If we ignore thermal diffusion, the electric field is given by $E = -(1/en_e)(\partial p_e/\partial r)$. When thermal effects are included, the electric field can be written as (see eq. [57] in BL)

$$eE = -\frac{1}{n_e}\frac{\partial p_e}{\partial r} - \alpha_e k_{\rm B}\frac{\partial T}{\partial r} \, . \tag{20}$$

Using his two-component fluid equations, Braginskii (1965) has obtained the values $\alpha_e \approx 0.71$ for a pure hydrogen plasma and $\alpha_e \approx 0.9$ for a pure helium plasma.

3. METHOD OF SOLUTION

The system of equations (12)–(13) and (18)–(19) can be solved numerically. If there are S species in the system ($S - 1$ ions plus electrons), there are S momentum equations (12) and S energy equations (13). The unknowns are the S drift velocities w_s and the S heat fluxes r_s. The gravitational acceleration and the electric field are also treated as unknowns, and we use the two additional equations for mass and charge conservation, equations (18) and (19), to help determine g and E. Note that the hydrostatic equilibrium condition (eq. [16]) should be satisfied automatically, providing a useful check on the numerical results.

We now rewrite the basic equations in a dimensionless form that is better suited to a numerical treatment. The radius r is expressed in units of $R_\odot$, the mass density ρ in units of 100 g cm^{-3} and temperature T in the units of 10^7 K, characteristic values at the center of the Sun, and the time t is in units of $\tau_0 = 6 \times 10^{13}$ yr, a characteristic diffusion time in the Sun (see, e.g., Kippenhahn & Weigert, p. 60, or eq. [9] in BL). We write the Burgers equations (12)–(13) and the constraints (18)–(19) as

$$\frac{p}{K_0}\left(\alpha_i \frac{d\ln p}{dr} + \nu_i \frac{d\ln T}{dr} + \sum_{\substack{j=1 \\ j\neq e,2}}^{S} \gamma_{ij}\frac{d\ln C_j}{dr}\right) = \sum_{j=1}^{2S+2} \Delta_{ij} W_j \, , \tag{21}$$

where the following notations have been introduced. The concentration of species s is defined by

$$C_s \equiv n_s/n_e \, . \tag{22}$$

It is related to the mass fractions $X_s \equiv m_s n_s/\rho$ by

$$C_s = \frac{X_s/A_s}{\sum_i Z_i X_i/A_i} \tag{23}$$

or inversely

$$X_s = \frac{A_s C_s}{\sum_i A_i C_i} \, , \tag{24}$$

where A_s is the atomic number of species s, and the sum is over all species, including the electrons. For the electrons,

$$A_e \equiv m_e/m_0 \, , \tag{25}$$

where m_e and m_0 are the electron and atomic masses, and

$$C_e \equiv 1 \, . \tag{26}$$

The constant K_0 is given by

$$K_0 = 1.144 \times 10^{-40} T^{-3/2} n_e^2 \, , \tag{27}$$

where T and ρ are expressed in the characteristic units defined above. We use the ideal gas equation of state, $p_s = n_s k_{\rm B} T$, and equations (22), (25), and (26) to write

$$\frac{p}{K_0} = 2.00 \, \frac{T^{5/2}}{\rho}\left(\sum_s C_s\right)\left(\sum_s A_s C_s\right) , \tag{28}$$

1994ApJ...421..828T

where we have written the electron number density in terms of the mass density, $\rho = m_0 n_e \sum_s A_s C_s$. The variables W_i are

$$W_i = \begin{cases} w_i & \text{for } i = 1, \dots S , \\ r_{i-S} & \text{for } i = S + 1, \dots 2S , \\ K_0^{-1} n_e eE & \text{for } i = 2S + 1 , \\ K_0^{-1} n_e m_0 g & \text{for } i = 2S + 2 . \end{cases} \tag{29}$$

If we define $C \equiv \sum_i C_i$, the coefficients on the left-hand side of equation (21) are given by

$$\alpha_i = \begin{cases} \dfrac{C_i}{C} & \text{for } i = 1, 2, \dots S , \\ 0 & \text{for } i = S + 1, \dots 2S + 2 , \end{cases} \tag{30}$$

$$\nu_i = \begin{cases} 2.5 \dfrac{C_{i-S}}{C} & \text{for } i = S + 1, \dots 2S , \\ 0 & \text{for } i = 1, \dots, S \text{ and } i = 2S + 1, 2S + 2 , \end{cases} \tag{31}$$

$$\gamma_{ij} = \begin{cases} \dfrac{C_i}{C} \left[\left(\delta_{ij} - \dfrac{C_j}{C} \right) - \left(\delta_{i2} - \dfrac{C_2}{C} \right) \dfrac{Z_j C_j}{Z_2 C_2} \right] & \text{for } i = 1, \dots, S , \\ 0 & \text{for } i = S + 1, \dots, 2S + 2 . \end{cases} \tag{32}$$

The coefficients on the right-hand side of equation (21) are given by

$$\Delta_{ij} = \begin{cases} -\sum\limits_{k \neq i} \kappa_{ik} & \text{for } j = i , \\ \kappa_{ij} & \text{for } j = 1, \dots, S \text{ and } j \neq i , \\ \sum\limits_{k \neq i} 0.6 \kappa_{ik} x_{ik} & \text{for } j = i + S , \\ -0.6 \kappa_{i,j-S} y_{i,j-S} & \text{for } j = S + 1, \dots, 2S \text{ and } j \neq i + S , \\ Z_i C_i & \text{for } j = 2S + 1 , \\ -A_i C_i & \text{for } j = 2S + 2 , \end{cases} \tag{33}$$

for $i = 1, \dots, S$, by

$$\Delta_{ij} = \begin{cases} \sum\limits_{k \neq j} 1.5 \kappa_{i-S,k} x_{i-S,k} & \text{for } j = i - S , \\ -1.5 \kappa_{i-S,j} x_{i-S,j} & \text{for } j = 1, \dots, S \text{ and } j \neq i - S , \\ -\sum\limits_{k \neq i} \kappa_{i-S,k} y_{i-S,k} (1.6 x_{i-S,k} + Y_{i-S,k}) - 0.8 \kappa_{i-S,i-S} & \text{for } j = i , \\ 2.7 \kappa_{i-S,j-S} y_{i-S,j-S} x_{i-S,j-S} & \text{for } j = S + 1, \dots, 2S \text{ and } j \neq i , \\ 0 & \text{for } j = 2S + 1, 2S + 2 , \end{cases} \tag{34}$$

for $i = S + 1, \dots 2S$; by

$$\Delta_{ij} = \begin{cases} Z_j C_j & \text{for } j = 1, \dots, S , \\ 0 & \text{for } j = S + 1, \dots, 2S + 2 , \end{cases} \tag{35}$$

for $i = 2S + 1$; and finally by

$$\Delta_{ij} = \begin{cases} A_j C_j & \text{for } j = 1, \dots, S , \\ 0 & \text{for } j = S + 1, \dots, 2S + 2 , \end{cases} \tag{36}$$

for $i = 2S + 2$. In these expressions, the coefficient κ_{st} is defined by

$$\kappa_{st} = \left(\frac{A_s A_t}{A_s + A_t} \right)^{1/2} C_s C_t Z_s^2 Z_t^2 \ln \Lambda_{st} . \tag{37}$$

It is related to the friction coefficient through $K_{st} = K_0 \kappa_{st}$. We have used the constraint of charge neutrality to eliminate the concentration gradient of species 2 in equation (21),

$$\frac{d \ln C_2}{dr} = - \sum_{\substack{j=1 \\ j \neq e, 2}}^{S} \frac{Z_j C_j}{Z_2 C_2} \frac{d \ln C_j}{dr} . \tag{38}$$

Since p/K_0 is proportional to $T^{5/2}/\rho$ (see eq. [28]), all the velocities will be proportional to $T^{5/2}/\rho$. Therefore, we introduce the function ξ_s (following BL), such that

$$w_s = (T^{5/2}/\rho)\xi_s\ . \tag{39}$$

The rate of change of the element mass fractions due to diffusion is now written in dimensionless form as

$$\frac{\partial X_s}{\partial t} = -\frac{1}{\rho r^2}\frac{\partial}{\partial r}[r^2 X_s T^{5/2}\xi_s(r)]\ , \tag{40}$$

the generalization of equation (1) of BL to the case of arbitrary concentrations and a more accurate treatment of the plasma physics.

Equations (21) are linear. Therefore, we can combine linearly the solutions obtained by keeping only one of the gradients different from zero. We write the results in terms of generalized diffusion coefficients $A_p(s)$, $A_T(s)$, and $A_t(s)$ for species s, as

$$\xi_s(r) = A_p(s)\frac{\partial \ln p}{\partial r} + A_T(s)\frac{\partial \ln T}{\partial r} + \sum_{t \neq e, 2} A_t(s)\frac{\partial \ln C_t}{\partial r}\ . \tag{41}$$

If ln Λ is assumed identical for all the interactions, the coefficients A_p, A_T, and A_t are functions of the mass fractions only. If ln Λ is defined by equation (9), these coefficients also depend on the charges, the temperature, and the density.

4. HYDROGEN AND HELIUM DIFFUSION

First, we consider the diffusion of hydrogen and helium, neglecting the presence of heavier elements. We calculate the hydrogen diffusion velocity. The helium diffusion velocity is then simply obtained from the constraint that there is no mean fluid velocity, $\sum_s X_s w_s = 0$. Neglecting the electron mass compared to the proton mass, we have $w_\alpha = -(X/Y)w_{\rm H}$. The rate of change of the helium number density is therefore equal and opposite to the rate of change of the hydrogen number density, $(\partial Y/\partial t) = -(\partial X/\partial t)$. Helium diffuses toward the center of the star, whereas hydrogen diffuses outward.

In the absence of heavy elements, the function $\xi_{\rm H}$ is given by[2]

$$\xi_{\rm H} = A_p({\rm H})\frac{\partial \ln p}{\partial r} + A_T({\rm H})\frac{\partial \ln T}{\partial r} + A_{\rm H}({\rm H})\frac{\partial \ln C_{\rm H}}{\partial r}\ . \tag{42}$$

We have chosen helium as element number 2; i.e., we write the diffusion velocity in terms of the hydrogen concentration gradient, using equation (38) to eliminate the helium concentration gradient.

Two major simplifications are usually made when calculating the hydrogen and helium diffusion velocities in the absence of heavy elements. It is usually assumed that the Coulomb logarithms $\ln \Lambda_{ij}$ are identical for all interactions. This allows the factorization of ln Λ outside the function $\xi_{\rm H}$ (see, e.g., Noerdlinger 1977; Bahcall & Loeb 1990). In that case, the coefficients A_p, A_T, and $A_{\rm H}$ depend only on the hydrogen (or helium) concentration, not on density, temperature, and ionic charges. The second simplification is to ignore the residual heat fluxes $\boldsymbol{r}_s$. Then, the diffusion velocities are easier to calculate analytically, since there is no need for the heat equations (13) and the number of variables and equations is reduced by a factor of 2. However, these simplifications can lead to large relative errors in the diffusion velocities. In particular, it has been argued by MP that thermal diffusion can increase the diffusion velocities by 30%.

In Figures 1*a*–1*c*, we show the variation of the coefficients A_p, A_T, and $A_{\rm H}$ with the hydrogen mass fraction X. To obtain these results, we have assumed $T = 10^7$ K and $\rho = 100$ g cm^{-3}, typical values in the core of the Sun. The exact results are represented with solid lines, the results obtained neglecting the heat fluxes are represented with short-dashed lines, and the results obtained by keeping the heat fluxes but using $\ln \Lambda = 2.2$ for all interactions[3] are represented with long-dashed lines. If the heat fluxes are totally neglected, the two coefficients A_p and $A_{\rm H}$ are underestimated, and $A_T = 0$. In Figure 1*d*, we show the relative errors on the coefficients due to these approximations. The short-dashed line represents the error on A_p and $A_{\rm H}$ when the heat fluxes are neglected. It can be as high as 45% for small values for the hydrogen mass fraction (not relevant to the Sun). The long-dashed lines represent the error due to $\ln \Lambda = 2.2$. The errors on A_p, $A_{\rm H}$, and A_T are smaller than 20%, except when $X \sim 1$. In the interior of the Sun, X varies approximately between 0.3 and 0.7. For these values of X, the error does not exceed 20%.

4.1. *Comparison with Bahcall & Loeb*

In order to keep the analytical calculations simple without neglecting the thermal effects one can use "effective" thermal diffusion coefficients (obtained through fits to the exact numerical results). In Figure 2*a*, we show the ratio between the "exact" coefficients and those obtained by BL who neglected the residual heat fluxes, assumed a Coulomb logarithm of 2.2 for all the interactions, and used an "effective" thermal diffusion coefficient. The expressions obtained by BL are

$$\xi_{\rm H}^{\rm BL}(r) = A_p^{\rm BL}\frac{\partial \ln p}{\partial r} + A_T^{\rm BL}\frac{\partial \ln T}{\partial r} + A_{\rm H}^{\rm BL}\frac{\partial \ln C_{\rm H}}{\partial r} \tag{43}$$

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[2] Note that BL define $\xi_{\rm H}$ with the opposite sign. They also write $\xi_{\rm H}$ in terms of the mass fraction gradient instead of the number concentration gradients. These are simply related by $\partial \ln C_s/\partial r = \partial \ln X_s/\partial r - (\sum_i Z_i X_i/A_i)^{-1} \sum_j (Z_j X_j/A_j)\partial \ln X_j/\partial r$ or inversely, $\partial \ln X_s/\partial r = \partial \ln C_s/\partial r - (\sum_i A_i C_i)^{-1} \sum_j A_j C_j \partial \ln C_j/\partial r$. If hydrogen and helium are the only elements, $\partial \ln X/\partial r = (1 + X)\partial \ln C_{\rm H}/\partial r$.

[3] This value is usually considered representative of the Coulomb logarithms in the solar interior (see, e.g., Noerdlinger 1977 and BL).

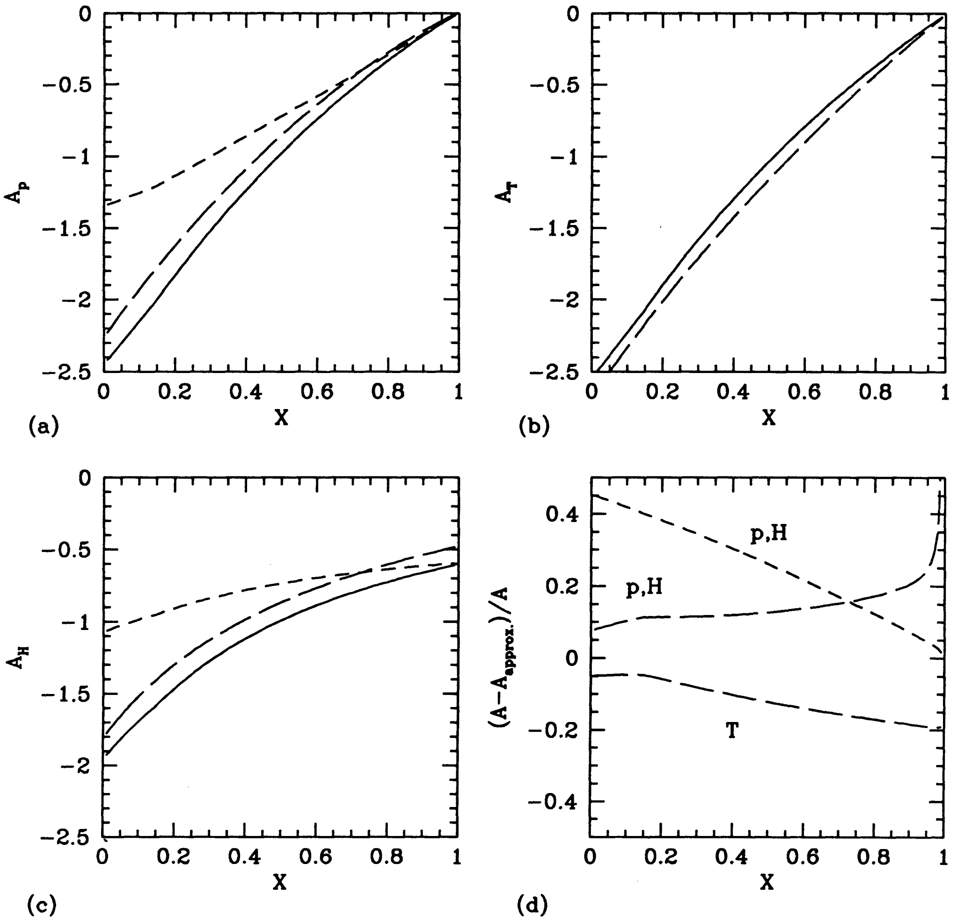

FIG. 1.—Variation of the hydrogen diffusion coefficients with the hydrogen mass fraction in a pure hydrogen-helium plasma, with $T = 10^7$ K and $\rho = 100$ g cm^{-3}. The solid lines represent the results obtained using Burgers equations, with no approximation. The short-dashed lines represent the results obtained when neglecting the heat fluxes. The long-dashed lines are the results obtained by using a single value for all the Coulomb logarithms, equal to 2.2. The dash-dotted lines are the results obtained when neglecting the heat fluxes *and* using a single value for all the Coulomb logarithms, equal to 2.2. (*a*) Pressure gradient coefficient A_p. (*b*) Temperature gradient coefficient A_T. (*c*) Hydrogen concentration gradient coefficient A_H. (*d*) Relative errors due to the approximations. The short-dashed line is the error on A_p and A_H when the heat fluxes are neglected. The long-dashed lines are the errors made by using $\ln \Lambda = 2.2$, and the dashed-dotted line is the error on A_p and A_H when both approximations are made.

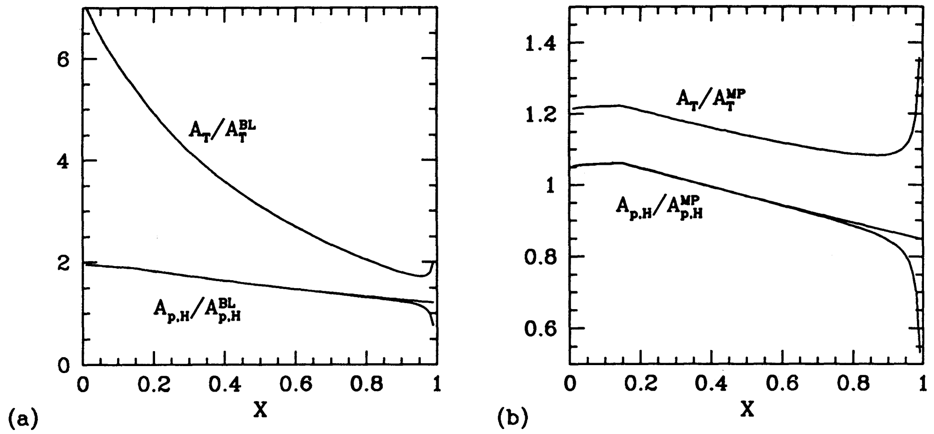

FIG. 2.—(*a*) Ratio of the exact hydrogen diffusion coefficients and those obtained by various approximations, in terms of the hydrogen mass fraction X, with $T = 10^7$ K and $\rho = 100$ g cm^{-3}. (*a*) Comparison with Bahcall & Loeb (1990). (*b*) Comparison with Michaud & Proffitt (1992). The solid lines and the dashed lines are the results obtained using eqs. (51) and (9), respectively, for the Coulomb logarithms in the Michaud & Proffitt formulae (48)–(50).

1994ApJ...421..828T

with

$$A_p^{\mathrm{BL}} = -5(1 - X)/4 \, , \tag{44}$$

$$A_T^{\mathrm{BL}} = -6(1 - X)(X + 0.32)/(1.8 - 0.9X)(3 + 5X) \, , \tag{45}$$

$$A_{\mathrm{H}}^{\mathrm{BL}} = -(\mathrm{X} + 3)/(3 + 5X) \, . \tag{46}$$

The result for the thermal diffusion coefficient was obtained by fitting values obtained previously by Aller & Chapman (1960), Montmerle & Michaud (1976), and Noerdlinger (1978). For values of X between 0.3 and 0.7, the error made by BL on A_p and A_{H} is smaller than 40%, whereas the error on A_T is as large as 70%. However, as we will show in § 6, large errors on A_T do not necessarily lead to large errors on the diffusion velocities, since the temperature gradient in the Sun is much smaller than the pressure and concentration gradients.

It is important to notice that the heat fluxes affect not only the thermal diffusion coefficients, but also the pressure and composition gradient coefficients.

4.2. *Comparison with Michaud & Proffitt*

MP solved the Burgers equations with and without including the heat fluxes, then represented the effects of the heat fluxes by an ad hoc correlation to the results obtained when neglecting those heat fluxes. In our dimensionless variables, their results can be written as

$$\xi_{\mathrm{H}}^{\mathrm{MP}}(r) = A_p^{\mathrm{MP}} \frac{\partial \ln p}{\partial r} + A_T^{\mathrm{MP}} \frac{\partial \ln T}{\partial r} + A_{\mathrm{H}}^{\mathrm{MP}} \frac{\partial \ln C_{\mathrm{H}}}{\partial r} \tag{47}$$

with

$$A_p^{\mathrm{MP}} = -\frac{5}{4} \frac{(1 - X)}{(0.7 + 0.3X)(\ln \Lambda_{xy}/2.2)} \, , \tag{48}$$

$$A_T^{\mathrm{MP}} = -\frac{9}{8} \frac{(1 - X)}{(0.7 + 0.3X)(\ln \Lambda_{xy}/2.2)} \, , \tag{49}$$

$$A_{\mathrm{H}}^{\mathrm{MP}} = -\frac{(X + 3)}{(3 + 5X)(0.7 + 0.3X)(\ln \Lambda_{xy}/2.2)} \, , \tag{50}$$

where

$$\ln \Lambda_{xy} = -19.95 - \frac{1}{2} \ln \rho - \frac{1}{2} \ln \frac{X + 3}{2} + \frac{3}{2} \ln T \, . \tag{51}$$

In Figure 2*b*, we show in solid lines the ratio of our coefficients and those obtained by MP. The difference between our results and those obtained by MP is smaller than 15% for A_T, and smaller than 5% for A_p and A_{H}. This small discrepancy will be discussed in § 6.

5. HEAVY ELEMENT DIFFUSION

Because of the complexity caused by the addition of heavy elements, this problem has always been approached with additional simplifications (Vauclair, Vauclair, & Pamjatnikh 1974; Noerdlinger 1978; BL; MP). One common simplification is to assume a negligible helium concentration, therefore reducing the problem to a two-species situation. However, this assumption is not valid in the interior of the Sun, where the characteristic mass fractions of hydrogen, helium, and oxygen are of the order of 0.34, 0.64, and 0.01, respectively (see, e.g., Bahcall 1990).

The functions ξ_s for the hydrogen and oxygen are now written as

$$\xi_{\mathrm{H}} = A_p(\mathrm{H}) \frac{d \ln p}{dr} + A_T(\mathrm{H}) \frac{d \ln T}{T} + A_{\mathrm{H}}(\mathrm{H}) \frac{d \ln C_{\mathrm{H}}}{dr} + A_{\mathrm{O}}(\mathrm{H}) \frac{d \ln C_{\mathrm{O}}}{dr} \, , \tag{52}$$

and[4]

$$\xi_{\mathrm{O}} = A_p(\mathrm{O}) \frac{\partial \ln p}{\partial r} + A_T(\mathrm{O}) \frac{\partial \ln T}{\partial r} + A_{\mathrm{H}}(\mathrm{O}) \frac{\partial \ln C_{\mathrm{H}}}{\partial r} + A_{\mathrm{O}}(\mathrm{O}) \frac{\partial \ln C_{\mathrm{O}}}{\partial r} \, . \tag{53}$$

It is interesting to show the variation of the diffusion coefficients as a function of the hydrogen mass fraction for fixed values of the temperature, density, and heavy element mass fraction. Indeed, as we will show in the next section, in the Sun these parameters all vary with radius, and it is more difficult to extract the hydrogen mass fraction dependence itself. In Figure 3, we show the four coefficients $A_p(\mathrm{H})$, $A_T(\mathrm{H})$, $A_{\mathrm{H}}(\mathrm{H})$, and $A_{\mathrm{O}}(\mathrm{H})$ as a function of the oxygen mass fraction X. These results were obtained using

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[4] Note that BL define the function ξ_{A} such that $\partial Z/\partial t = -(1/\rho r^2)\partial[r^2 X Z T^{5/2} \xi_A/(2 - X)] \, \partial r$. The function ξ_A is related to the function ξ_{O} through $\xi_A = \xi_{\mathrm{O}}(2 - X)/X$.

$T = 10^7$ K, $\rho = 100$ g cm^{-3} and $Z = 0.01$, were Z is the oxygen mass fraction. These values are typical in the solar interior. Note that now X has a maximum value determined by $X_{\max} = 1 - Z$, because of the charge neutrality constraint. We notice that the coefficient $A_{\rm O}({\rm H})$ is two orders of magnitude smaller than the other coefficients. This was expected, since we have chosen $K_{\rm HO}/K_{\rm H\alpha} \sim Z/Y \sim 10^{-2}$. The error made by neglecting the presence of oxygen when calculating the hydrogen diffusion velocity is smaller than 2%.

In Figure 4 we show the oxygen diffusion coefficients $A_p({\rm O})$, $A_T({\rm O})$, $A_{\rm H}({\rm O})$, and $A_{\rm O}({\rm O})$. Again, the coefficient $A_{\rm O}({\rm O})$ is much smaller than the other three coefficients and can be neglected to the level of precision desired in these calculations.

6. DIFFUSION VELOCITIES IN THE SUN

We now calculate the diffusion velocities of hydrogen, helium, and heavier elements in the present solar interior ($r < 0.7\, R_\odot$). We use values for the pressure, temperature, density, and mass fractions of the contemporary Sun, obtained from the standard solar model (Table 4.4, in Bahcall 1990). Since we have the radial profile of the pressure, temperature, and mass fractions, we can calculate their gradients. The coefficients A_p, A_T, $A_{\rm H}$, and $A_{\rm O}$ are computed using the tabulated values of T, ρ, X, Y, and Z. The iron abundance is assumed to be uniform and given by $\log_{10}(n_{\rm Fe}/n_{\rm H}) = 6.82–12$, and its ionization is 21.

In Figure 5, we show the radial variation of the hydrogen diffusion coefficients. In Figure 6, we show the relative importance of the different terms in the hydrogen diffusion velocity. The oxygen concentration gradient gives a negligible contribution to $w_{\rm H}$. The temperature term is not negligible, but is smaller than the pressure term (between 25% and 50% of the pressure term). Therefore, a large error on the temperature diffusion coefficient does not necessarily lead to a large error for the diffusion velocities.

The timescale for a change in the element abundances can be characterized by $t \equiv r/w_{\rm H}$. To obtain the time t in units of $t_0 = 5 \times 10^9$ yr, the age of the Sun, we simply multiply the dimensionless time by t_0/τ_0, where τ_0 is the characteristic diffusion time defined in § 3. The smaller the time t, the faster the element concentrations change. In Figure 7, we show the variation of t with the radius. The fastest changes in the hydrogen concentration occurs at $\sim 0.05\, R_\odot$, where $t_{\min} \sim 70 t_0$.

6.1. *Comparison with Bahcall & Loeb*

In Figure 8*a*, we compare our exact results with those obtained using the analytic BL formulae, equations (44)–(46) (eqs. [1]–[5] in BL). The BL formula underestimate the diffusion coefficients. The error on the pressure and concentration diffusion coefficients is smaller than 30%. The error on the temperature diffusion coefficient is of the order of 50%, except near the center where it becomes very large.

In Figures 9*a*–9*c*, we show in solid lines the results for the diffusion velocities of hydrogen, oxygen, and iron. The helium diffusion velocity w_α is related to the hydrogen and oxygen diffusion velocities through the zero mean velocity constraint, equation (19),

$$w_\alpha = -(X_{\rm H} w_{\rm H} + X_{\rm O} w_{\rm O} + X_{\rm Fe} w_{\rm Fe})/X_{\rm He}\,. \tag{54}$$

We also show in short-dashed lines the BL results for the hydrogen and oxygen diffusion velocities, given by equations (44)–(46) for the hydrogen velocity, and equations (2)–(5) in BL for the oxygen velocity. In the BL approximation, the helium velocity is given by $w_\alpha = -(X_{\rm H}/X_{\rm He}) w_{\rm H}$. The error in $w_{\rm H}$ due to the BL approximations is smaller than 30%, except near the center, where the error is as

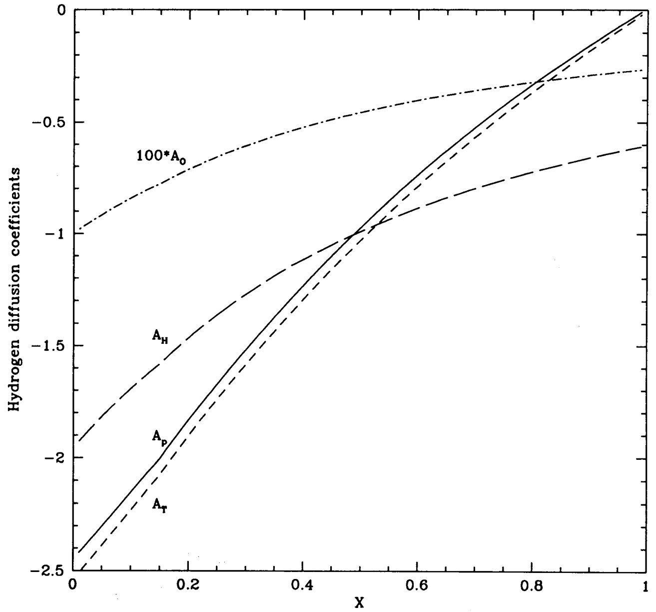


FIG. 3

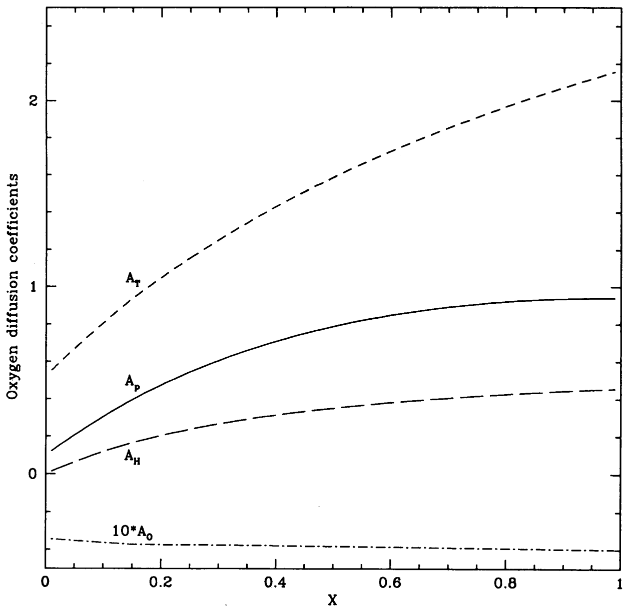


FIG. 4

FIG. 3.—Variation of the hydrogen diffusion coefficients with the hydrogen mass fraction in a hydrogen-helium-oxygen plasma, with $T = 10^7$ K, $\rho = 100$ g cm^{-3}, and $Z = 0.01$.

FIG. 4.—Variation of the oxygen diffusion coefficients with the hydrogen mass fraction in a hydrogen-helium-oxygen plasma, with $T = 10^7$ K, $\rho = 100$ g cm^{-3}, and $Z = 0.01$.

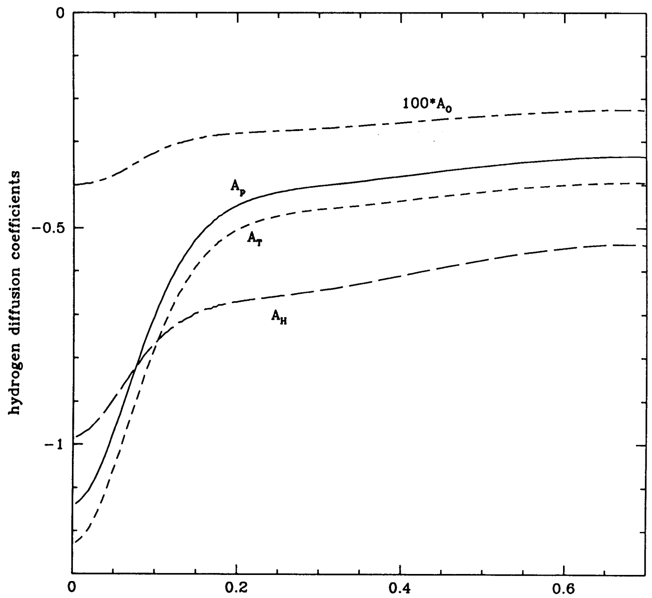


FIG. 5

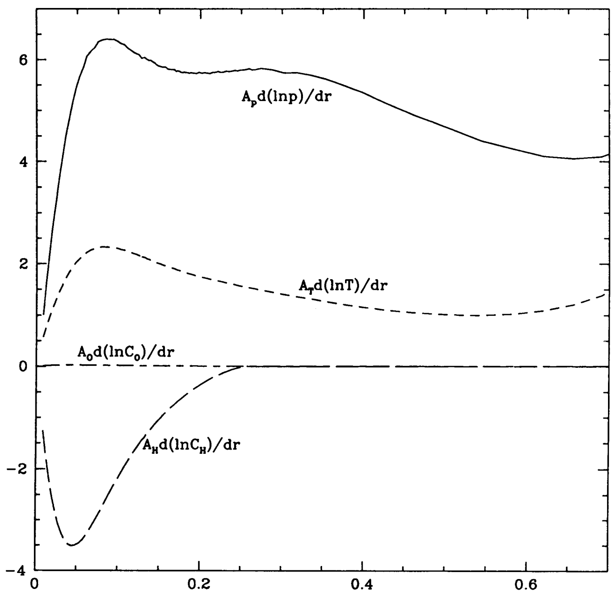


FIG. 6

FIG. 5.—Hydrogen diffusion coefficients in the present Sun, as a function of radius

FIG. 6.—Contributions to the hydrogen diffusion velocity in the Sun due to each gradient

high as 70%. However, the error for the oxygen diffusion velocity is very large. This was expected, since BL neglected completely the presence of helium when calculating w_O. The BL results for w_O are therefore only valid when $X \approx 1$.

6.2. *Comparison with Michaud & Proffitt*

In Figure 8*b*, we compare our results with those obtained using the MP formula, equations (48)–(50). The difference between our results and the results obtained by MP for the diffusion coefficients is smaller than 5% for A_p and A_H. Our thermal diffusion coefficient A_T is larger by ~20%. This discrepancy may result from the fact that we have used $z_{st} = 0.6$ for the heat flux coefficient in equations (12) and (13). If the collision integrals are calculated using a screened Coulomb potential, the value obtained for z_{st} is $\sim \frac{2}{3}$ smaller (Proffitt 1993).

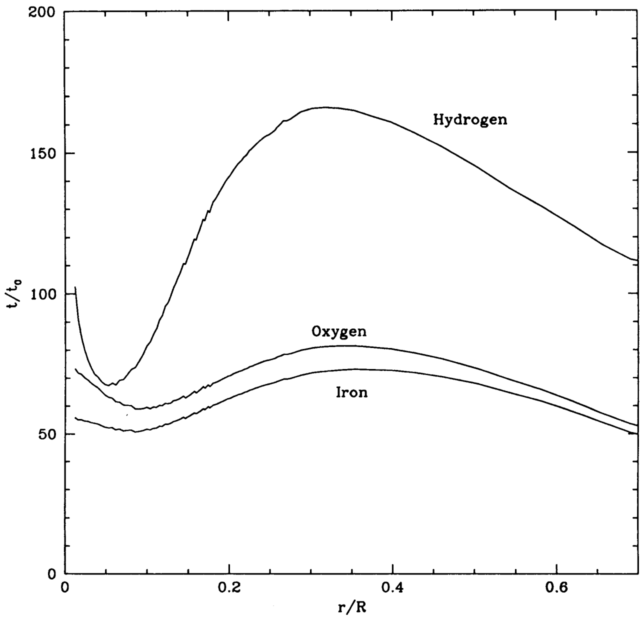


FIG. 7.—Local diffusion time of hydrogen, oxygen, and iron as a function of radius, in units of the age of the Sun

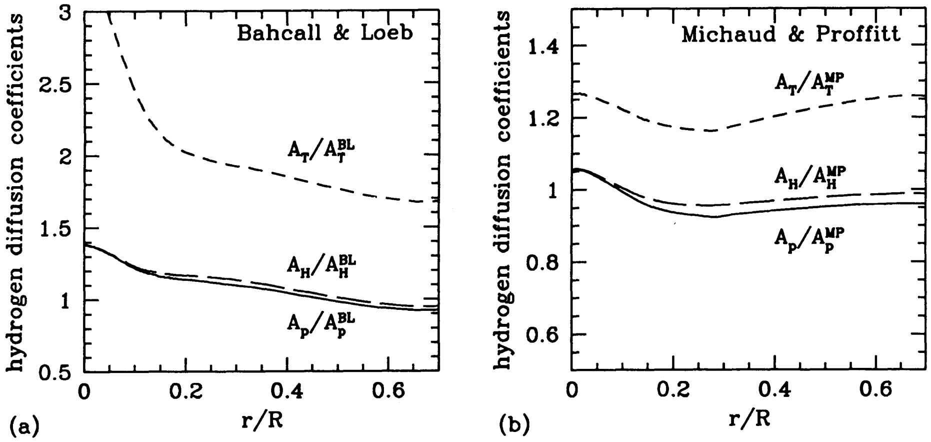
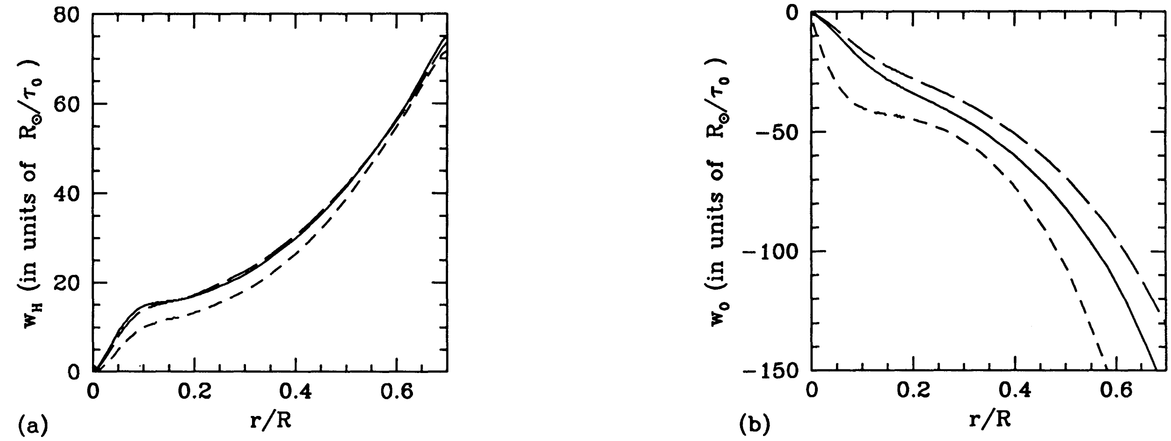


FIG. 8.—(*a*) Ratio of the exact hydrogen diffusion coefficients and those obtained by Bahcall & Loeb (1990) as a function of the radius. (*b*) Ratio of the exact hydrogen diffusion coefficients and those obtained by Michaud & Proffitt (1992) as a function of the radius.

In Figures 9*a*–9*c*, we show the results for the diffusion velocity of hydrogen, oxygen, and iron. The MP result for the heavy elements diffusion velocities are given by

$$\xi_i = -\frac{2}{\sqrt{5}\,Z_i^2}\left(\frac{d/dr[\![\ln\{[X_i/(5X+3)][(1+X)/(5X+3)]^{Z_i}\}]\!] + \{1+Z_i-A_i[(5X+3)/4]\}d\ln p/dr}{X(A_{ix}^{1/2}C_{ix}-A_{iy}^{1/2}C_{iy})+A_{iy}^{1/2}C_{iy}}\right)$$
$$+ X\xi_{\rm H}\left[\frac{(A_{ix}^{1/2}C_{ix}-A_{iy}^{1/2}C_{iy})}{X(A_{ix}^{1/2}C_{ix}-A_{iy}^{1/2}C_{iy})+A_{iy}^{1/2}C_{iy}} - 0.23\right] + \frac{0.54(4.75X+2.25)}{(\ln\Lambda_{xy}+5)}\frac{d\ln T}{dr}\,, \qquad (55)$$



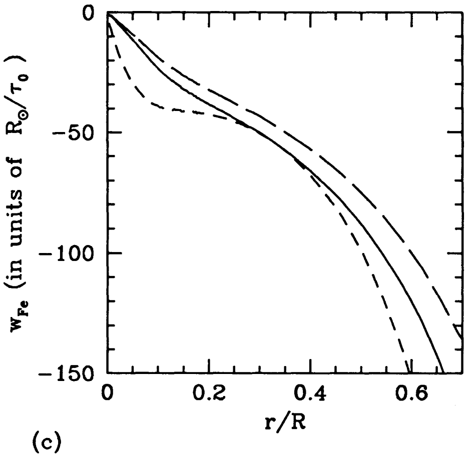


FIG. 9.—Diffusion velocities in the contemporary Sun. The solid lines are the exact results; the short-dashed lines are the results of Bahcall & Loeb (1990) (see eqs. [1]–[5] in BL); The long-dashed lines are the results of Michaud & Proffitt (1992). (*a*) Hydrogen diffusion velocity. (*b*) Oxygen diffusion velocity. (*c*) Iron diffusion velocity.

where $A_{ij} \equiv A_i A_j/(A_i + A_j)$ is the reduced mass in atomic number units, $C_{ij} = \ln[\exp(1.2 \ln \Lambda_{ij}) + 1]/1.2$, and $\ln \Lambda_{ij}$ is given by equation (51). The difference between our results and those obtained by MP is smaller than 15% for the hydrogen diffusion velocity, and smaller than 20% for the oxygen diffusion velocity.

7. ANALYTICAL FITS

All the results shown above were obtained using the expression (9) for the "effective" Coulomb logarithms. If we use (as in the previous results) the correct charge-, temperature-, and density-dependent expression for the Coulomb logarithms, we cannot give a simple analytical fit, as in BL, for the diffusion coefficients in terms of the hydrogen mass fraction. Even though one could, in principle, incorporate a subroutine which solves the problem of element diffusion in a standard solar model evolution code, it is useful to give a simple analytical fit of the results obtained here in the solar interior. It is convenient to provide to stellar evolution programs other types of input physics, such as opacities or equations of state, in the form of tabulated values, or in terms of approximate analytical fits. In order to provide a similarly convenient service for diffusion, we adopted a constant for each Coulomb logarithm, the value it has at the center of the Sun. In Figure 10, we show the values of the various Coulomb logarithms in the Sun computed using equation (9). In Figures 11*a*–11*c*, we compare the diffusion coefficients obtained with these Coulomb logarithms with those obtained using a constant value of Λ (equal to its central value). In Figure 11*d*, we show that the error on $w_{\rm H}$ is smaller than 4% in the solar core ($r/R_\odot \leq 0.4$), and remains smaller than 15% up to the convection zone. The error on the heavy elements diffusion velocities are $\lesssim 6\%$ in the solar core, and remain $\lesssim 20\%$ up to the convection zone. We can now fit these results to second-order polynomials. Since the presence of oxygen and iron have very little influence on the hydrogen velocity, we can assume that the diffusion coefficients depend only on X. We find

$$\begin{cases} A_p(\mathrm{H}) = -2.09 + 3.15X - 1.07X^2 , \\ A_T(\mathrm{H}) = -2.18 + 3.12X - 0.96X^2 , \\ A_{\mathrm{H}}(\mathrm{H}) = -1.51 + 1.85X - 0.85X^2 , \end{cases} \tag{56}$$

for the hydrogen diffusion coefficients;

$$\begin{cases} A_p(\mathrm{O}) = 0.15 + 1.34X - 0.89X^2 , \\ A_T(\mathrm{O}) = 0.53 + 1.99X - 0.72X^2 , \\ A_{\mathrm{H}}(\mathrm{O}) = 0.08 + 0.58X - 0.28X^2 , \end{cases} \tag{57}$$

for the oxygen diffusion coefficients; and

$$\begin{cases} A_p(\mathrm{Fe}) = 0.25 + 1.31X - 0.87X^2 , \\ A_T(\mathrm{Fe}) = 0.65 + 1.99X - 0.75X^2 , \\ A_{\mathrm{H}}(\mathrm{Fe}) = 0.09 + 0.53X - 0.27X^2 , \end{cases} \tag{58}$$

for the iron diffusion coefficients. The error due to the polynomial fits are of the order of 0.2%. There is no need to use higher order

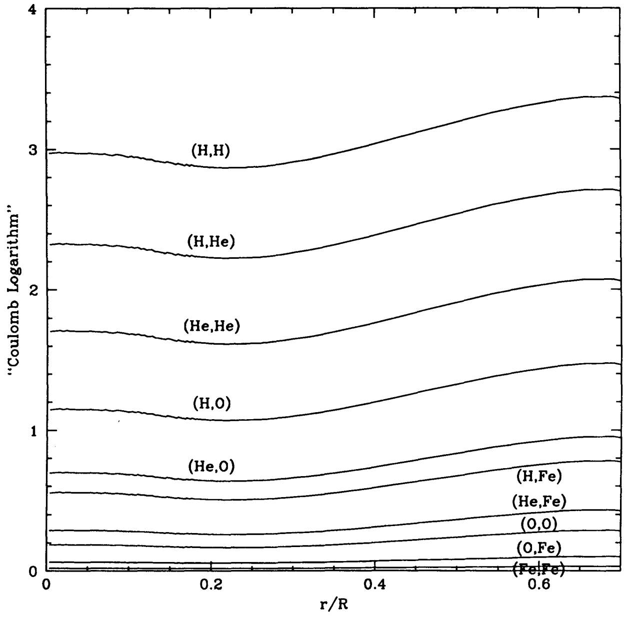


FIG. 10.—Coulomb logarithms in the Sun

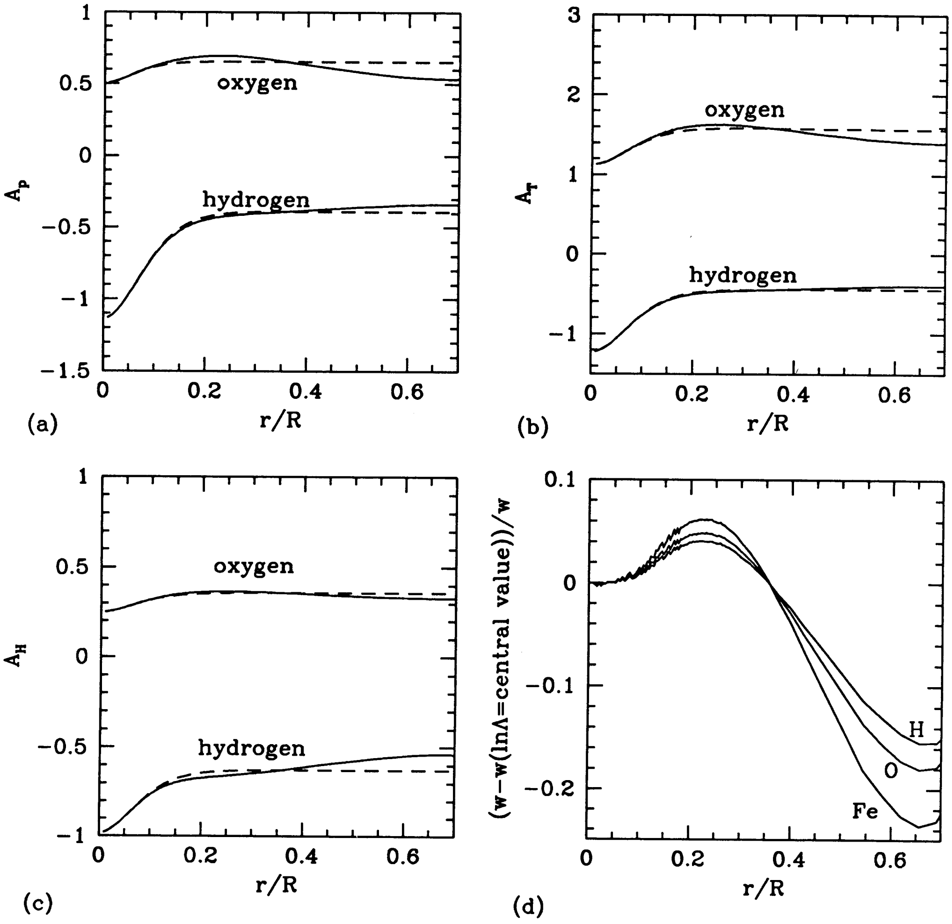


FIG. 11.—Diffusion coefficients in the sun, as a function of radius. The solid lines represent the results obtained using Burgers equations, with no approximation. The dashed lines represent the results obtained using $\ln \Lambda = 3.2$ for all the interactions. (*a*) Pressure gradient coefficient A_p. (*b*) Temperature gradient coefficient A_T. (*c*) Hydrogen concentration gradient coefficient A_H. (*d*) Relative error on the hydrogen and oxygen diffusion velocities made by using $\ln \Lambda = 3.2$.

polynomials, since the error made with these second-order polynomials are already much smaller than the errors introduced by the simplified Coulomb logarithm.

For an easy comparison with BL and MP results, we can factorize the BL results and write the numerical fits as

$$A_p(\mathrm{H}) = -\tfrac{5}{4}(1 - X)a_p\,, \tag{59}$$

$$A_T(\mathrm{H}) = -\frac{6(1 - X)(X + 0.32)}{(1.8 - 0.9X)(3 + 5X)}\,a_t\,, \tag{60}$$

$$A_{\mathrm{H}}(\mathrm{H}) = -\frac{(X + 3)}{(3 + 5X)}\,a_{\mathrm{H}}\,. \tag{61}$$

First-order polynomial fits give the following analytical results:

$$\begin{cases} a_p = 1.66 - 0.82X\,, \\ a_T = 4.46 - 3.65X\,, \\ a_{\mathrm{H}} = 1.63 - 0.74X\,. \end{cases} \tag{62}$$

The error introduced by the polynomial fit is much smaller than the error due to the simplified Coulomb logarithm. Equations (59)–(62) can be used to improve existing diffusion subroutines that are based on the BL formalism.

It is important to remember that these fits are made for the standard parameters of the solar interior.

8. HYDROSTATIC EQUILIBRIUM AND ELECTRIC FIELD

As explained in § 2, we solve the system of equations (21) for the diffusion velocities, the residual heat flow vectors, the gravitational acceleration, and the electric field. On the other hand, the equation of hydrostatic equilibrium given by equation (16)

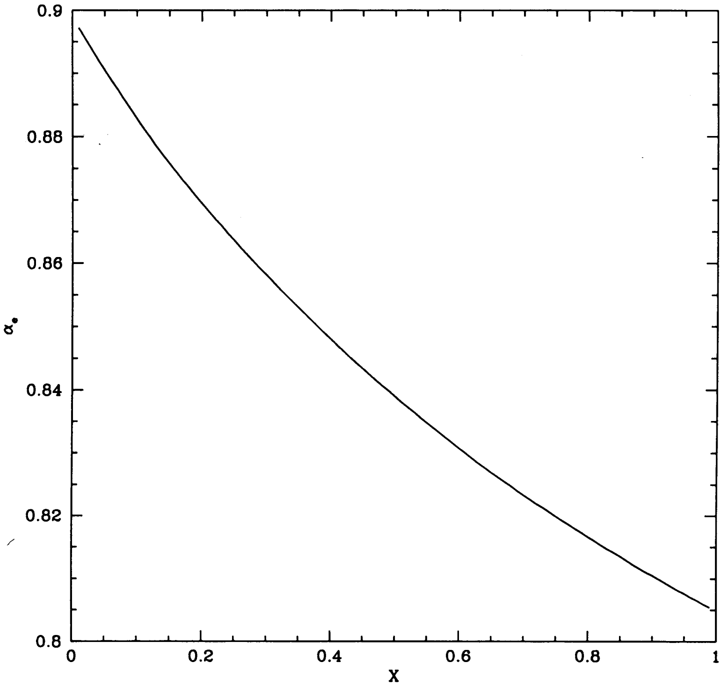


FIG. 12.—Thermal coefficient for the electric field (see eq. [20])

must be satisfied. The numerical results for g can be written as

$$g = -\frac{p}{\rho}\left[A_p(g)\frac{d\ln p}{dr} + A_T(g)\frac{d\ln T}{dr} + A_{\rm H}(g)\frac{d\ln C_{\rm H}}{dr}\right]. \tag{63}$$

In hydrostatic equilibrium, $A_T = A_{\rm H} = 0$ and $A_p = 1$. The equation of hydrostatic equilibrium is satisfied to 10^{-7} by the numerical results.

Similarly, we write the results for the electric field as

$$E = \frac{-p_e}{en_e}\left[A_p(E)\frac{d\ln p}{dr} + A_T(E)\frac{d\ln T}{dr} + A_{\rm H}(E)\frac{d\ln C_{\rm H}}{dr}\right]. \tag{64}$$

The coefficients in equation (64) should be compared with equation (20). We find an excellent agreement between our numerical results and equation (20) for the coefficients in front of the pressure and concentration gradients (with an error smaller than 0.2×10^{-2}). Our values for the coefficient $A_T(E)$ are slightly larger than the results obtained by Braginskii (1965) for α_e. As shown in Figure 12, we observe, however, the same tendency for this coefficient to increase slightly with an increase in the helium concentration. We get $\alpha_e = 0.8$ for $X = 1$, and $\alpha_e = 0.9$ for $X = 0$.

9. SUMMARY AND CONCLUSIONS

We have developed a Fortran program to solve numerically the Burgers equations for an arbitrary number of species, without any approximation. For the discussion of solar conditions given here, we have neglected the radiative forces, but these forces could easily be incorporated in the numerical routine. The accuracy of the results for the diffusion velocities is limited only by the validity of the expression for the Coulomb logarithm. The diffusion velocities of hydrogen, oxygen, and iron were calculated in the solar interior and compared with the results of Bahcall & Loeb (1990) and by Michaud & Proffitt (1992). The results of BL for the hydrogen diffusion velocity are smaller by $\sim 30\%$, except near the center, where the error is much larger. The results obtained by MP for the hydrogen and oxygen diffusion velocities differ by $\lesssim 15\%$.

We provide analytical fits of our numerical results for the diffusion coefficients as a function of the hydrogen mass fraction in the solar interior (eqs. [56]–[62]). These fits were obtained by assuming fixed values for the Coulomb logarithms, equal to their values at the center of the Sun.

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REFERENCES

Aller, L. H., & Chapman, S. 1960, ApJ, 132, 461

Bahcall, J. N. 1990, Neutrino Astrophysics (Cambridge University Press)

Bahcall, J. N., & Loeb, A. 1990, ApJ, 360, 267 (BL)

Bahcall, J. N., & Pinsonneault, M. H. 1992a, ApJ, 395, L119

———. 1992b, Rev. Mod. Phys., 64, 885

Braginskii, S. I. 1965, Reviews of Plasma Physics, Vol. 1 (New York: Consultants Bureau)

Burgers, J. M. 1969, Flow Equations for Composite Gases (New York: Academic)

Chaboyer, B., Deliyannis, C. P., Demarque, P., Pinsonneault, M. H., & Sarajedini, A. 1992, ApJ, 388, 372

Christensen-Dalsgaard, J., Proffitt, C. R., & Thompson, M. J. 1993, preprint

Cox, A. N., Guzik, J. A., & Kidman, R. B. 1989, ApJ, 342, 1187

Elsworth, Y., Howe, R., Isaak, G. R., McLeod, C. P., & New, R. 1990, Nature, 347, 536

Fontaine, G., & Michaud, G. 1979, ApJ, 231, 826

———. 1979b, IAU Colloq. 59, Effects of Mass Loss on Stellar Evolution, ed. C. Chiosi & R. Stalio (Dordrecht: Reidel), 192

Guenther, D. B., Pinsonneault, M. H., & Bahcall, J. N. 1993, preprint

Iben, I., Jr., & MacDonald, J. 1985, ApJ, 296, 540

Kippenhahn, R., & Weigert, A. 1991, Stellar Structure and Evolution (Berlin: Springer)

Michaud, G., Charland, Y., Vauclair, S., & Vauclair, G. 1976, ApJ, 210, 447

Michaud, G., & Proffitt, C. R. 1992, preprint (MP)

Michaud, G., & Vauclair, S. 1991, in Solar Interior and Atmosphere, ed. A. N. Cox, W. C. Livingston, & M. S. Matthews (Tucson: University of Arizona Press), 312

Montmerle, T., & Michaud, G. 1976, ApJS, 31, 489

Noerdlinger, P. D. 1977, A&A, 57, 407

———. 1978, ApJS, 36, 259

Paquette, C., Pelletier, C., Fontaine, G., & Michaud, G. 1986, ApJS, 61, 177

Pelletier, C., Fontaine, G., Wesemael, F., Michaud, G., & Wegner, G. 1986, ApJ, 307, 242

Proffitt, C. R. 1993, personal communication

Proffitt, C. R., & Michaud, G. 1991, ApJ, 380, 238

Vauclair, S., & Vauclair, G. 1982, ARA&A, 20, 37

Vauclair, G., Vauclair, S., & Pamjatnikh, A. 1974, A&A, 31, 63