THE ASTROPHYSICAL JOURNAL, 407:266–275, 1993 April 10

THREE-BODY EFFECTS IN THE PSR 1257+12 PLANETARY SYSTEM

RENU MALHOTRA
Lunar and Planetary Institute, 3600 Bay Area Boulevard, Houston, TX 77058
Received 1992 August 20; accepted 1992 October 13

ABSTRACT

We present a detailed theoretical analysis of the three-body effects in the putative planetary system of PSR 1257+12. We discuss how these effects are manifested in the pattern of pulse arrival times; the dominant perturbation can be described as a modulation of the phases of the near-sinusoidal signals of the two planetary companions. We provide explicit formulas for the time dependence of the osculating orbital elements that are needed for an improved timing model for this system.

If a timing model with fixed, independent Keplerian orbits continues to be used for the timing analysis, and if two planets are indeed orbiting this pulsar, then the three-body effects should become detectable by means of a *growth* in the postfit residuals as more observations are accumulated. If the typical error in the pulse arrival time measurements is $\sim 10\ \mu$s, the amplitude of the postfit residuals will increase beyond this level with three to five years of timing observations. Their detection will place the planetary interpretation on firmer ground, and the improved timing model will yield the orbital inclinations and the masses of the planets relative to the mass of the pulsar. An absence of this signal will make the presence of these planets highly unlikely.

Subject headings: celestial mechanics, stellar dynamics — planetary systems — pulsars: individual (PSR 1257+12)

1. INTRODUCTION

The 6.2 ms pulsar 1257+12, discovered in 1990 February (Wolszczan 1990) is reported to have two planet mass companions in gravitationally bound orbits about it (Wolszczan & Frail 1992, hereafter W&F). Since pulsars are very reliable clocks, the signatures of orbital companions can be rather easily detected by the long period modulation of the pulse arrival times at the solar system barycenter. For a pulsar with a single low-mass companion, the Newtonian point-mass gravitational interaction is a good approximation; the pulsar moves about the system barycenter on a fixed Keplerian ellipse:

$$\boldsymbol{r}_*(t) = \frac{-m_1}{m_* + m_1}\,\boldsymbol{r}(t)\,, \tag{1}$$

where

$$r = \frac{a(1-e^2)}{1 + e\cos f}\,, \tag{2}$$

m_* and m_1 are the masses of the pulsar and "planet," respectively, $\boldsymbol{r}$ is the position vector of m_1 relative to m_*, and $\boldsymbol{r}_*$ is the instantaneous position of the pulsar referred to the barycenter of the system. Equation (2) is the usual formula for the instantaneous radial distance on a Keplerian ellipse of semimajor axis a and eccentricity e, and f is the instantaneous true anomaly.

With two low-mass companions, m_1 and m_2, the motion of the pulsar about the system barycenter is given by (Malhotra 1992)

$$\boldsymbol{R}_* = -\kappa_1 \boldsymbol{r}_1 - \kappa_2 \boldsymbol{r}_2\,, \tag{3}$$

where

$$\kappa_1 = \frac{m_1}{m_* + m_1}\,, \quad \text{and} \quad \kappa_2 = \frac{m_2}{m_* + m_1 + m_2}\,, \tag{4}$$

and, in the first approximation, $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ describe the motion on two fixed Keplerian ellipses. Using essentially this approximation in their timing model, W&F deduced five parameters for each Keplerian orbit. These are reproduced in Table 1. Assuming that the pulsar has a mass of 1.4 $M_\odot$, the observations and the model yield companion masses of 3.4 $M_\oplus/\sin i$ and 2.8 $M_\oplus/\sin i$, where i is the as yet unknown inclination of the orbital plane to the plane of the sky. The putative planets move in nearly circular orbits (eccentricities of $\sim$2%), with mean angular velocities, n_i, that are close to a 3:2 commensurability.

While the observations are consistent with the effects of planet-like companions to the star, other interpretations are conceivable (Dolginov & Stepinski 1992; Gil & Jessner 1992). It is clearly imperative to find additional evidence to support or refute the planetary interpretation.

One promising avenue is to look for the signatures of the mutual gravitational perturbations of the planets. The gravitational interactions of low-mass companions can be analyzed by considering the pulsar's orbit to be a superposition of two *osculating* ellipses (i.e., orbits that are instantaneously elliptical, but whose orbital parameters are time dependent). The dynamics of the

TABLE 1

ORBITAL PARAMETERS OF THE PSR 1257+12 PLANETARY SYSTEM[a]
EPOCH: JD 2448088.9

Parameter	Planet 1	Planet 2
Orbital period, P_j	5751011.0 ± 800.0	8487388.0 ± 1800.0
Projected semimajor axis,		
$\kappa_j a_j \sin i_j$ (light-ms)	1.31 ± 0.01	1.41 ± 0.01
Eccentricity	0.022 ± 0.007	0.020 ± 0.006
Epoch of periastron, T_{0j}	JD 2448105.3 ± 1.0	JD 2447998.6 ± 1.0
Longitude of periastron, ω_j	252° ± 20°	107° ± 20°

[a] From Wolszczan & Frail 1992.

three-body system then predicts a specific time dependence of the osculating orbital elements. It was first shown by Rasio et al. (1992a) that, as a consequence of the near-commensurability of the mean orbital frequencies, $n_j = 2\pi/P_j$, the orbital semimajor axes, eccentricities, and longitudes of periastron suffer periodic variations. They calculated that the variations are sinusoidal with period $2\pi/(2n_1 - 3n_2) \approx 5.6$ yr, and the amplitudes are proportional to the mass ratios m_j/m_*. However, as pointed out in Malhotra et al. (1992), and expanded upon by Rasio et al. (1992b), this analysis holds only for masses near the lower limits of 3.4 and 2.8 $M_\oplus$ (which correspond to $\sin i = 1$, or equivalently, the orbits being viewed edge-on). If the planet masses are appreciably larger than the lower limits (i.e., if $1/\sin i \gg 1$), then it is possible for the orbital motions to be exactly commensurable. The *resonant* perturbations are then qualitatively and quantitatively different: the period and amplitudes are much larger, and the variations are markedly non-sinusoidal. The critical value of $1/\sin i$ above which it is possible for the two planets to be "locked" in the 3:2 resonance is about 10 (Malhotra et al. 1992). Since the probability[1] that such might be the case is only about 0.5%, in this paper we address only the nonresonant case. (The analytical solutions given are adequate for a preliminary analysis of the resonant case also.)

Our purpose is twofold. Although the effects of the mutual perturbations of the planets on their orbital elements have been discussed previously, how exactly these effects would appear in the pattern of pulse arrival times and their detectability above observational errors have not been fully investigated. Second, in order to take account of the three-body effects in the timing model used by W&F for their data analysis, there is a need for an explicit set of formulas describing the time dependence of the osculating elements. We hope to satisfy both these needs by expanding on the preliminary results of a previous paper (Malhotra 1992).

At a less ambitious level, we note that if PSR 1257+12 does indeed have two planetary companions, then merely using the fixed-orbit solution in the timing model for future observations should result in a postfit residual with a pattern characteristic of the three-body effects. This pattern is also quasi-periodic and similar to the residuals that obtain from planets on fixed Keplerian orbits; the amplitude is small (on the order of a few to 10's of microseconds), *but grows with time.* Assuming the typical error in the pulse arrival time measurements is 10 μs, the additional residuals due to the three-body effects can be expected to become detectable with three to five years of observations. If a growing postfit residual is actually detected, the improved timing model presented here can then be used to extract the orbital inclinations and the masses of the planets relative to the mass of the star.

The rest of this paper is organized as follows. Section 2 describes the kinematical model for the pulsar's motion. In § 3 we give the equations for the time derivatives of the osculating orbital elements as well as their solutions in terms of an explicit time dependence. (Readers who are not interested in the mathematical details can skip § 3.) In § 4 we discuss the signature of these three-body effects in the pattern of pulse arrival times.

2. KINEMATICS

Without loss of generality, we may define a barycentric coordinate system with origin at the barycenter of the PSR 1257+12 system such that the (x, y)-plane is the plane of the sky and the z-axis points toward the solar system barycenter. Let $\boldsymbol{R}_*$, $\boldsymbol{R}_1$, and $\boldsymbol{R}_2$ be the barycentric position vectors of the pulsar, planet 1, and planet 2, respectively. Then, by definition,

$$\boldsymbol{R}_* = -\frac{m_1}{m_*}\boldsymbol{R}_1 - \frac{m_2}{m_*}\boldsymbol{R}_2 . \tag{5}$$

The solar system barycentric pulse time of arrival residual, Δt, due to the motion of the pulsar about the system barycenter is given by

$$c\,\Delta t = -\hat{z}\cdot\boldsymbol{R}_* . \tag{6}$$

For the perturbative analysis of the nearly Keplerian motion of the planets around the pulsar, it is more convenient to use the Jacobi coordinates, $\boldsymbol{r}_1$ (the position of m_1 relative to m_*) and $\boldsymbol{r}_2$ (the position of m_2 relative to the center of mass of m_* and m_1):

$$\boldsymbol{r}_1 = \boldsymbol{R}_1 - \boldsymbol{R}_* , \qquad \text{and} \qquad \boldsymbol{r}_2 = \boldsymbol{R}_2 - \frac{m_*\boldsymbol{R}_* + m_1\boldsymbol{R}_1}{m_* + m_1} . \tag{7}$$

Then, in terms of the Jacobi coordinates, the instantaneous position of the pulsar, $\boldsymbol{R}_*$ is as given in equation (3).

[1] For random orientations of an orbit plane, the probability of observing with $i < i_0$ is $(1 - \cos i_0)$.

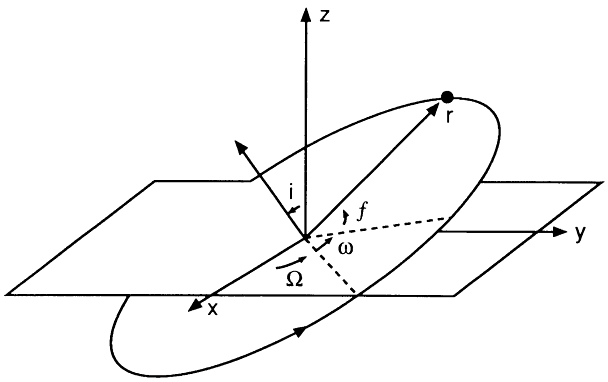

FIG. 1.—Schematic diagram illustrating the Keplerian orbital elements

To first approximation, the solution $\boldsymbol{r}_j(t)$ is an elliptical trajectory (see Fig. 1):

$$\boldsymbol{r} = r\{[\cos(f+\omega)\cos\Omega - \sin(f+\omega)\sin\Omega\cos i]\hat{x} + [\cos(f+\omega)\sin\Omega + \sin(f+\omega)\cos\Omega\cos i]\hat{y} + \sin(f+\omega)\sin i\,\hat{z}\}\,, \quad (8)$$

where r is as given in equation (2), and we also have the following relations:

$$f = M + 2e\sin M + \tfrac{5}{4}e^2\sin 2M + \mathcal{O}(e^3)\,,$$
$$M = nt + \sigma\,,$$
$$n^2a^3 = Gm_*/(1-\kappa)\,. \quad (9)$$

In keeping with celestial mechanics notation, n is the mean motion and M is the mean anaomaly; Ω is the longitude of ascending node, and ω is the argument of periastron; σ is related to the time of periastron. In equations (8) and (9), the subscript $j = 1, 2$ is implicit for the orbital elements as well as κ.

Thus it is evident that in the first approximation, the motion of the pulsar, $\boldsymbol{R}_*(t)$, about the system barycenter is a superposition of two elliptical motions. However, note that it differs from the superposition of the independent solutions to the two 2-body problems of (m_*, m_1) and (m_*, m_2) by terms of $\mathcal{O}(m_j/m_*)^2$.

Using equations (3), (8), and (9) in equation (6), we have

$$c\,\Delta t = c(\Delta t_1 + \Delta t_2)\,, \quad (10)$$

where

$$c\,\Delta t_j = \kappa_j a_j \sin i_j(1-e_j^2)\frac{\sin(f_j+\omega_j)}{1+e_j\cos f_j}\,, \qquad j = 1, 2\,. \quad (11)$$

For circular orbits, $e_1 = e_2 = 0, f_j = M_j = n_j t + \sigma_j$, so that

$$c\,\Delta t_j = \kappa_j a_j \sin i_j \sin(n_j t + \sigma_j)\,.$$

Thus, in this case, the TOA residuals would be precisely the superposition of two sines with frequencies equal to the mean motions, n_1 and n_2, and amplitudes $\mathscr{A}_j = \kappa_j a_j \sin i_j$. Using the Keplerian relationship between n and a (see eq. [9]), we get

$$\left(\frac{Gm^*}{1-\kappa_j}\right)^{1/3}\kappa_j \sin i_j = \mathscr{A}_j n_j^{2/3}\,. \quad (12)$$

The above equation is valid in the case of noncircular orbits as well, except that $\mathscr{A}_j$ would then be identified with the amplitude of that component in the TOA residuals which has frequency n_j. Using W&F's reported values of periods and amplitudes (Table 1), equation (12) yields the following estimates for the planet masses:

$$m_1 \sin i_1 = 3.4\ M_\oplus\left(\frac{m_* + m_1}{1.4\ M_\odot}\right)^{2/3}\,, \quad (13)$$

$$m_2 \sin i_2 = 2.8\ M_\oplus\left(\frac{m_* + m_1}{m_*}\right)\left(\frac{m_* + m_1 + m_2}{1.4\ M_\odot}\right)^{2/3}\,. \quad (14)$$

The above equations show explicitly how the estimates of the planet masses depend upon the unknown orbital inclinations and the unknown mass of the star.

For the general case of noncircular orbits, to first order in the orbital eccentricities, we have

$$c\,\Delta t_j = \mathscr{A}_j \sin(M_j + \omega_j) + \tfrac{1}{2}\mathscr{A}_j e_j[\sin 2M_j \cos\omega_j + (\cos 2M_j - 3)\sin\omega_j] + \mathcal{O}(e^2)\,. \quad (15)$$

We note that in this first approximation, the orbital inclinations, i_j, and the longitudes of the ascending nodes, Ω_j, remain indeterminate from the above kinematical considerations. As discussed in the Introduction, dynamical considerations can, in principle, yield further information on the companion masses and the orbital inclinations, while simultaneously providing a test for the planetary hypothesis. The cost of obtaining this information lies in the introduction of two additional parameters in the multiparameter timing model used in W&F; these two parameters can be conveniently taken to be the mass ratios, m_1/m_* and m_2/m_*.

3. PERTURBATION THEORY

Qualitatively, there are three types of significant perturbations in the PSR 1257+12 planetary system; (1) those related to conjunctions, or "close encounters," of the planets, (2) secular effects that lead to precession of the apsides and nodes, and (3) the effects of the 3:2 near-commensurability of the mean motions. The analysis proceeds by assuming that the solutions, $\boldsymbol{r}_j = \boldsymbol{r}_j(t)$, are osculating Keplerian ellipses. To first order in the product of the masses $m_1 m_2$ the perturbations are described by the interaction Hamiltonian (Malhotra 1992)

$$\mathscr{H}' = -Gm_1 m_2 \left(\frac{1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} - \frac{\boldsymbol{r}_1 \cdot \boldsymbol{r}_2}{r_2^3} \right) . \tag{16}$$

$\mathscr{H}'$ can be developed in terms of the osculating orbital elements, and the principal perturbation components are given by

$$\mathscr{H}' = -\frac{Gm_1 m_2}{a_2} \left\{ [P(\psi, \alpha) - \alpha \cos \psi] + \frac{1}{2} A_1(\alpha)(e_1^2 + e_2^2) + A_2(\alpha) e_1 e_2 \cos(\omega_1 - \omega_2) + C_1(\alpha) e_1 \cos(\phi - \omega_1) + C_2(\alpha) e_2 \cos(\phi - \omega_2) \right\} , \tag{17}$$

where

$$\begin{aligned} \alpha &= a_1/a_2 , \\ \psi &= \lambda_1 - \lambda_2 , \\ \phi &= 3\lambda_2 - 2\lambda_1 , \\ \lambda_j &= M_j + \omega_j , \\ P(\psi, \alpha) &= (1 - 2\alpha \cos \psi + \alpha^2)^{-1/2} , \\ A_1(\alpha) &= +\frac{1}{4} \alpha b_{3/2}^{(1)}(\alpha) , \\ A_2(\alpha) &= -\frac{1}{4} \alpha b_{3/2}^{(2)}(\alpha) , \\ C_1(\alpha) &= -\frac{1}{2} \left(6 + \alpha \frac{d}{d\alpha} \right) b_{1/2}^{(3)}(\alpha) , \\ C_2(\alpha) &= +\frac{1}{2} \left(5 + \alpha \frac{d}{d\alpha} \right) b_{1/2}^{(2)}(\alpha) . \end{aligned} \tag{18}$$

The $b_{s/2}^{(j)}(\alpha)$ are Laplace coefficients. The expression for $\mathscr{H}'$ above assumes coplanar orbits, and ω_j are measured from the ascending node.[2] The term $[P(\psi, \alpha) - \alpha \cos \psi]$ describes the perturbations at conjunctions of the planets; the terms with coefficients A_1 and A_2 describe the secular effects responsible for the slow precession of the apsides; and the remaining terms decribe the effects of the 3:2 near-commensurability of the mean motions.

The first time derivatives of the orbital elements induced by these perturbations are as follows (for the general perturbation equations, the reader is referred to Brouwer & Clemence 1961):

the semimajor axes:

$$\begin{aligned} \frac{\dot{a}_1}{a_1} &= +2 \frac{m_2}{m_*} n_1 \alpha \left[\frac{\partial}{\partial \psi} P(\psi, \alpha) + \alpha \sin \psi + 2C_1(\alpha) e_1 \sin(\phi - \omega_1) + 2C_2(\alpha) e_2 \sin(\phi - \omega_2) \right] , \\ \frac{\dot{a}_2}{a_2} &= -2 \frac{m_1}{m_*} n_2 \left[\frac{\partial}{\partial \psi} P(\psi, \alpha) + \alpha \sin \psi + 3C_1(\alpha) e_1 \sin(\phi - \omega_1) + 3C_2(\alpha) e_2 \sin(\phi - \omega_2) \right] , \end{aligned} \tag{19}$$

[2] If the orbits have a small mutual inclination δi, then there are corrections of $\mathcal{O}(\delta i)^2$ to the coefficients of the secular and resonant terms, and there is also a small phase shift in the arguments of the cosines, since ω_j must then be replaced by Π_j, the longitude of pericenter measured from the mutual node of the two orbits. The general effect of noncoplanarity is to reduce the strength of the interaction.

the longitudes:

$$\dot{\lambda}_1 = n_1 - \frac{m_2}{m_*} n_1\alpha\left[2\alpha \frac{\partial}{\partial\alpha} P(\psi, \alpha) - 2\alpha \cos\psi - \frac{1}{2} A_1(\alpha)e_1^2 + \alpha \frac{d}{d\alpha} A_1(\alpha)(e_1^2 + e_2^2) - \left(\frac{1}{2} - 2\alpha \frac{d}{d\alpha}\right)A_2(\alpha)e_1e_2 \cos(\omega_1 - \omega_2) - \left(\frac{1}{2} - 2\alpha \frac{d}{d\alpha}\right)C_1(\alpha)e_1 \cos(\phi - \omega_1) + 2\alpha \frac{d}{d\alpha} C_2(\alpha)e_2 \cos(\phi - \omega_2)\right],$$
$$\dot{\lambda}_2 = n_2 + \frac{m_1}{m_*} n_2\left[2\left(1 + \alpha \frac{\partial}{\partial\alpha}\right)P(\psi, \alpha) - 4\alpha \cos\psi + \frac{1}{2} A_1(\alpha)e_2^2 + \left(1 + \alpha \frac{d}{d\alpha}\right)A_1(\alpha)(e_1^2 + e_2^2) + \left(\frac{5}{2} + 2\alpha \frac{d}{d\alpha}\right) \times A_2(\alpha)e_1e_2 \cos(\omega_1 - \omega_2) + \left(2 + 2\alpha \frac{d}{d\alpha}\right)C_1(\alpha)e_1 \cos(\phi - \omega_1) + \left(\frac{5}{2} + 2\alpha \frac{d}{d\alpha}\right)C_2(\alpha)e_2 \cos(\phi - \omega_2)\right]. \quad (20)$$

For the variations of the eccentricities and pericenters, it is convenient to use the variables $h_j = e_j \sin \omega_j$ and $k_j = e_j \cos \omega_j$. Then,

$$\dot{h}_j = + \sum_{l=1,2} \mathbf{A}_{jl} k_l + B_j \cos\phi ,$$
$$\dot{k}_j = - \sum_{l=1,2} \mathbf{A}_{jl} h_l - B_j \sin\phi , \quad (21)$$

where the coefficient matrix **A** is

$$\mathbf{A} = \begin{bmatrix} \frac{m_2}{m_*} n_1\alpha A_1(\alpha) & \frac{m_2}{m_*} n_1\alpha A_2(\alpha) \\ \frac{m_1}{m_*} n_2 A_2(\alpha) & \frac{m_1}{m_*} n_2 A_1(\alpha) \end{bmatrix}, \quad (22)$$

and

$$B_1 = \frac{m_2}{m_*} n_1\alpha C_1(\alpha) , \qquad B_2 = \frac{m_1}{m_*} n_2 C_2(\alpha) . \quad (23)$$

We solve first the equations for a_j by setting in the right-hand sides a_j, n_j, h_j, and k_j equal to their initial values (denoted by the subscripts $0j$), and $\lambda_j = n_{0j} t + \lambda_{0j}$. Then, with $\alpha = a_{01}/a_{02} = 0.7715$, the numerical values of the various coefficients are

$$A_1(\alpha) = +2.50 , \qquad \alpha \frac{d}{d\alpha} A_1(\alpha) = +19.22 ,$$
$$A_2(\alpha) = -2.19 , \qquad \alpha \frac{d}{d\alpha} A_2(\alpha) = -18.43 ,$$
$$C_1(\alpha) = -2.13 , \qquad \alpha \frac{d}{d\alpha} C_1(\alpha) = -10.00 ,$$
$$C_2(\alpha) = +2.59 , \qquad \alpha \frac{d}{d\alpha} C_2(\alpha) = +9.60 .$$

A straightforward integration of equations (19) then yields

$$\frac{a_1(t) - a_{01}}{a_{01}} = +2 \frac{m_2}{m_*} \alpha\left[\left[\frac{n_{01}}{n_{01} - n_{02}} \{P(\psi(t), \alpha) - P(\psi_0, \alpha) - \alpha[\cos\psi(t) - \cos\psi_0]\} - \frac{2n_{01}}{3n_{02} - 2n_{01}} \{(C_1 k_{01} + C_2 k_{02})[\cos\phi(t) - \cos\phi_0] + (C_1 h_{01} + C_2 h_{02})[\sin\phi(t) - \sin\phi_0]\}\right]\right],$$
$$\frac{a_2(t) - a_{02}}{a_{02}} = -2 \frac{m_1}{m_*} \left[\left[\frac{n_2}{n_{01} - n_{02}} \{P(\psi(t), \alpha) - P(\psi_0, \alpha) - \alpha[\cos\psi(t) - \cos\psi_0]\} + \frac{3n_2}{3n_{02} - 2n_{01}} \{(C_1 k_{01} + C_2 k_{02})[\cos\phi(t) - \cos\phi_0] + (C_1 h_{01} + C_2 h_{02})[\sin\phi(t) - \sin\phi_0]\}\right]\right], \quad (24)$$

where

$$\psi(t) = (n_{01} - n_{02})t + \psi_0 , \qquad \psi_0 = \lambda_{01} - \lambda_{02} ,$$
$$\phi(t) = (3n_{02} - 2n_{01})t + \phi_0 \qquad \phi_0 = 3\lambda_{02} - 2\lambda_{01} . \quad (25)$$

The mean motion variations are related to the semimajor axes variations by

$$\frac{n_j(t) - n_{0j}}{n_{0j}} \simeq -\frac{3}{2}\frac{a_j(t) - a_{0j}}{a_{0j}} . \tag{26}$$

Since the pulse arrival time residuals are quite sensitive to the relative phases of the planets, we use this solution for the n_j to obtain the following improved time derivatives for the λ:

$$\dot{\lambda}_j = \tilde{n}_{0j} + \delta n_j(t) + \delta\dot{\lambda}_j(t) , \tag{27}$$

where

$$\tilde{n}_{01} = n_{01}\Bigg[\Bigg[1 + \frac{m_2}{m_*}\alpha\Bigg\{\frac{3n_{01}}{n_{01} - n_{02}}[P(\psi_0, \alpha) - \alpha\cos\psi_0] + \frac{1}{2}A_1(\alpha)e_{01}^2 - \alpha\frac{d}{d\alpha}A_1(\alpha)(e_{01}^2 + e_{02}^2)$$
$$+ \left(\frac{1}{2} - 2\alpha\frac{d}{d\alpha}\right)A_2(\alpha)(k_{01}k_{02} + h_{01}h_{02}) - \frac{6n_{01}}{3n_{02} - 2n_{01}}[(C_1k_{01} + C_2k_{02})\cos\phi_0 + (C_1h_{01} + C_2h_{02})\sin\phi_0]\Bigg\}\Bigg]\Bigg], \tag{28}$$

$$\tilde{n}_{02} = n_{02}\Bigg[\Bigg[1 - \frac{m_1}{m_*}\Bigg\{\frac{3n_{02}}{n_{01} - n_{02}}[P(\psi_0, \alpha) - \alpha\cos\psi_0] - \frac{1}{2}A_1(\alpha)e_{02}^2 - \left(1 + \alpha\frac{d}{d\alpha}\right)A_1(\alpha)(e_{01}^2 + e_{02}^2)$$
$$- \left(\frac{5}{2} + 2\alpha\frac{d}{d\alpha}\right)A_2(\alpha)(k_{01}k_{02} + h_{01}h_{02}) - \frac{9n_{02}}{3n_{02} - 2n_{01}}[(C_1k_{01} + C_2k_{02})\cos\phi_0 + (C_1h_{01} + C_2h_{02})\sin\phi_0]\Bigg\}\Bigg]\Bigg],$$

$$\delta n_1(t) = -\frac{m_2}{m_*}\alpha\Bigg[\Bigg[\frac{3n_{01}}{n_{01} - n_{02}}\{P(\psi(t), \alpha) - \alpha\cos\psi(t)\} - \frac{6n_{01}}{3n_{02} - 2n_{01}}[(C_1k_{01} + C_2k_{02})\cos\phi(t) + (C_1h_{01} + C_2h_{02})\sin\phi(t)]\Bigg]\Bigg], \tag{29}$$

$$\delta n_2(t) = +\frac{m_1}{m_*}\Bigg[\Bigg[\frac{3n_{02}}{n_{01} - n_{02}}\{P(\psi(t), \alpha) - \alpha\cos\psi(t)\} - \frac{9n_{02}}{3n_{02} - 2n_{01}}[(C_1k_{01} + C_2k_{02})\cos\phi(t) + (C_1h_{01} + C_2h_{02})\sin\phi(t)]\Bigg]\Bigg],$$

$$\delta\dot{\lambda}_1(t) = -\frac{m_2}{m_*}n_{01}\alpha\Bigg\{2\alpha\frac{\partial}{\partial\alpha}P(\psi(t), \alpha) - 2\alpha\cos\psi(t) - \left(\frac{1}{2} - 2\alpha\frac{d}{d\alpha}\right)C_1(\alpha)[k_{01}\cos\phi(t) + h_{01}\sin\phi(t)]$$
$$+ 2\alpha\frac{d}{d\alpha}C_2(\alpha)[k_{02}\cos\phi(t) + h_{02}\sin\phi(t)]\Bigg\}, \tag{30}$$

$$\delta\dot{\lambda}_2(t) = +\frac{m_1}{m_*}n_{02}\Bigg[\Bigg[2\Bigg\{\left(1 + \alpha\frac{\partial}{\partial\alpha}\right)P(\psi(t), \alpha) - 2\alpha\cos\psi(t)\Bigg\} + 2\left(1 + \alpha\frac{d}{d\alpha}\right)C_1(\alpha)[k_{01}\cos\phi(t) + h_{01}\sin\phi(t)]$$
$$+ \left(\frac{5}{2} + 2\alpha\frac{d}{d\alpha}\right)C_2(\alpha)[k_{02}\cos\phi(t) + h_{02}\sin\phi(t)]\Bigg]\Bigg].$$

Upon integrating, we find the following solution for the mean longitudes:

$$\lambda_j(t) = \lambda_{0j} + \tilde{n}_{0j}t + \Delta\lambda_j(t) , \tag{31}$$

where

$$\Delta\lambda_1(t) = -\frac{m_2}{m_*}\alpha\Bigg(3\left(\frac{n_{01}}{n_{01} - n_{02}}\right)^2\{Q(\psi(t), \alpha) - Q(\psi_0, \alpha) - \alpha[\sin\psi(t) - \sin\psi_0]\}$$
$$+ 2\frac{n_{01}}{n_{01} - n_{02}}\Bigg[\Bigg[\alpha\frac{\partial}{\partial\alpha}\{Q(\psi(t), \alpha) - Q(\psi_0, \alpha)\} - \alpha[\sin\psi(t) - \sin\psi_0]\Bigg]\Bigg]$$
$$- 6\left(\frac{n_{01}}{3n_{02} - 2n_{01}}\right)^2\{(C_1k_{01} + C_2k_{02})[\sin\phi(t) - \sin\phi_0] - (C_1h_{01} + C_2h_{02})[\cos\phi(t) - \cos\phi_0]\}$$
$$- \frac{n_{01}}{3n_{02} - 2n_{01}}\Bigg[\Bigg[\left(\frac{1}{2} - 2\alpha\frac{d}{d\alpha}\right)C_1(\alpha)\{k_{01}[\sin\phi(t) - \sin\phi_0] - h_{01}[\cos\phi(t) - \cos\phi_0]\}$$
$$- 2\alpha\frac{d}{d\alpha}C_2(\alpha)\{k_{02}[\sin\phi(t) - \sin\phi_0] - h_{02}[\cos\phi(t) - \cos\phi_0]\}\Bigg]\Bigg]\Bigg),$$

$$\Delta\lambda_2(t) = +\frac{m_1}{m_*}\left(3\left(\frac{n_{02}}{n_{01}-n_{02}}\right)^2\left\{Q(\psi(t),\alpha) - Q(\psi_0,\alpha) - \alpha[\sin\psi(t) - \sin\psi_0]\right\}\right.$$
$$+ 2\frac{n_{02}}{n_{01}-n_{02}}\left[\left[\left(1+\alpha\frac{\partial}{\partial\alpha}\right)\{Q(\psi(t),\alpha) - Q(\psi_0,\alpha)\} - 2\alpha[\sin\psi(t) - \sin\psi_0]\right]\right]$$
$$- 9\left(\frac{n_{02}}{3n_{02}-2n_{01}}\right)^2\{(C_1k_{01} + C_2k_{02})[\sin\phi(t) - \sin\phi_0] - (C_1h_{01} + C_2h_{02})[\cos\phi(t) - \cos\phi_0]\}$$
$$+ \frac{n_{02}}{3n_{02}-2n_{01}}\left[\left[2\left(1+\alpha\frac{d}{d\alpha}\right)C_1(\alpha)\{k_{01}[\sin\phi(t) - \sin\phi_0] - h_{01}[\cos\phi(t) - \cos\phi_0]\}\right.\right.$$
$$\left.\left.\left. + \left(\frac{5}{2} + 2\alpha\frac{d}{d\alpha}\right)C_2(\alpha)\{k_{02}[\sin\phi(t) - \sin\phi_0] - h_{02}[\cos\phi(t) - \cos\phi_0]\}\right]\right]\right), \quad (32)$$

where

$$Q(\psi, \alpha) = \int_0^{\psi} d\psi' P(\psi', \alpha) . \quad (33)$$

The calculation of $Q(\psi, \alpha)$ and its α-derivative is given in the Appendix.

Finally, we solve the $\{h_j, k_j\}$ equations. Equations (21) can be recognized as the equations for a pair of forced oscillators. (A similar situation arises in the secular variations of the Uranian satellites as a result of the near-resonances among Umbriel, Titania, and Oberon; Malhotra et al. 1989.) If we use the unperturbed solution for the arguments of the forcing terms (see eq. [25]) then, to a good approximation, the evolution of $\{h_j, k_j\}$ is a superposition of free oscillations and forced oscillations:

$$h_i(t) = \sum_{j=1,2} E_i^{(j)} \sin(g_j t + \beta_j) + F_i \sin\phi(t) ,$$
$$k_i(t) = \sum_{j=1,2} E_i^{(j)} \cos(g_j t + \beta_j) + F_i \cos\phi(t) , \quad (34)$$

where g_j are eigenvalues of the coefficient matrix $\mathbf{A}$:

$$g_{1,2} = \tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{A} \pm \sqrt{\tfrac{1}{4}(\mathrm{Tr}\,\mathbf{A})^2 - \det\mathbf{A}} , \quad (35)$$

and $\mathbf{E}^{(j)}$ are the corresponding eigenvectors:

$$\frac{\mathbf{E}_2^{(j)}}{\mathbf{E}_1^{(j)}} = \frac{g_j - \mathbf{A}_{11}}{\mathbf{A}_{12}} . \quad (36)$$

The phases, β_j, and the normalization of the eigenvectors, $\mathbf{E}^{(j)}$, are determined by the initial conditions, $\{e_{0j}, \omega_{0j}\}$ and ϕ_0. The amplitude of the forcing is given by

$$\mathbf{F} = -[\mathbf{A} - (3n_{02} - 2n_{01})\mathbf{I}]^{-1} \cdot \mathbf{B} , \quad (37)$$

where $\mathbf{I}$ is the 2×2 identity matrix. Clearly, the solution given in equations (34) is valid only if the matrix $\mathbf{A} - (3n_{02} - 2n_{01})\mathbf{I}$ is non-singular, that is, only if $(3n_{02} - 2n_{01}) \neq g_1$ or g_2. Actually, the condition is more stringent: $g_j \ll |3n_{02} - 2n_{01}|$, and is related to the possibility of resonance "locking" (Malhotra et al. 1992).

We note that the eigenfrequencies, g_j, are simply proportional to the mass ratios m_i/m_*, and thus inversely proportional to sin i. Furthermore, if $g_j \ll |3n_{02} - 2n_{01}|$, then $\mathbf{F}_j$ are also simply proportional to the mass ratios m_i/m_*. On the other hand, the phases, β_j and the amplitudes, $\mathbf{E}_i^{(j)}$ are only weakly dependent on the mass ratios. Therefore, to a good approximation (for $[\sin i]^{-1} < 10$), the parameters of the $\{h_j, k_j\}$ solution, with initial conditions of Table 1, are given in Table 2. The parameters of the "free" oscillations, g_j, β_j and $\mathbf{E}_i^{(j)}$ given here are only slightly different from those derived in Rasio et al. (1992a). As pointed out in that paper, the current state of the putative planetary system is apparently dominated by the g_1 eigenmode.

4. SIGNATURE OF THREE-BODY EFFECTS IN THE PULSE ARRIVAL TIME RESIDUALS

The analytical solution for the time dependence of the osculating orbital elements can be used to predict the pulse arrival time residuals. This is shown in Figure 2, in which we have also plotted W&F's data. This data was kindly provided by Alex Wolszczan.

TABLE 2

PARAMETERS FOR THE $\{h_j, k_j\}$ SOLUTION

j	g_j sin i (deg yr^{-1})	β (deg)	$\mathbf{E}_1^{(j)}$	$\mathbf{E}_2^{(j)}$	F_j sin i
1.........	4.45×10^{-2}	268.7	1.98×10^{-2}	-2.10×10^{-2}	-3.04×10^{-4}
2.........	2.98×10^{-3}	193.0	6.52×10^{-3}	6.49×10^{-3}	3.90×10^{-4}

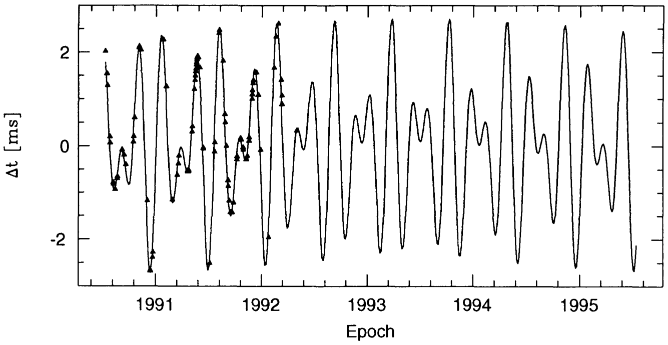

FIG. 2

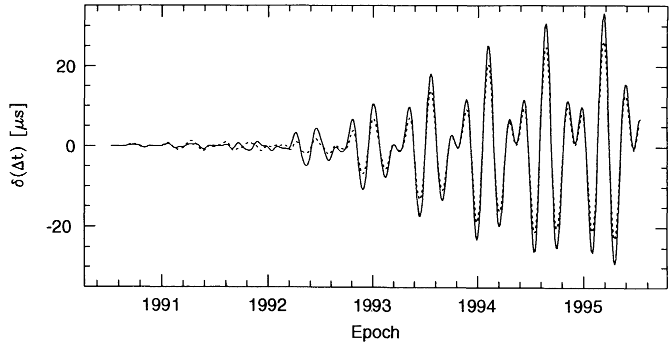

FIG. 3

FIG. 2.—Solid line shows the predicted pulse arrival time residuals (for a period of 5 yr) due to the perturbations of two planetary companions in orbit about the pulsar. The points are Wolzczan & Frail's (1992) observations over a period of ~2 yr.

FIG. 3.—Pulse arrival time residuals attributed to the three-body effects (i.e., due to the mutual perturbations of the planets). The solid line shows the residuals predicted by the analytical solution of § 3; the dotted line shows the same predicted by a full numerical integration of the three-body equations. The initial conditions are those deduced from the best-fit orbital parameters of Wolszczan & Frail (1992). The results shown are for the case of coplanar, edge-on orbits.

It shows the pulse arrival time residuals after fitting for a pulsar model that includes as free parameters P, $\dot{P}$, and the position and proper motion of the pulsar; the last was not included in the originally published data.

In order to discern the three-body effects in these arrival time residuals, we show in Figure 3 the difference in the arrival time residuals that obtain from the three-body solution and those that obtain from a model with independent, fixed Keplerian orbits. (This figure also shows that our analytical solution compares well with the results of a full numerical integration of the three-body equations.) As anticipated, the differences are quasi-periodic, and of quite small amplitude. However, the phase residual begins to accummulate an amplitude greater than 10 μs after ~2.5 yr.

Qualitatively, the characteristic pattern of growth of the post-fit residuals shown in Figure 3 alone would support the planetary interpretation of W&F. If such a pattern is not found in the postfit residuals, then the presence of two planetary companions would be highly unlikely; the planetary interpretation would be ruled out, barring perhaps the possibility of nearly edge-on orbits with significant mutual inclination.

We have given in § 3 the ingredients for an improved model for analyzing the pulse arrival times. These improvements take account of the three-body effects in the system and require two additional parameters—the masses of the two planets relative to the star. By including these in the currently used multiparameter least-squares analysis, m_1/m_* and m_2/m_* (and, hence, the orbital inclinations) could be determined.

An obviously relevant question here is whether the current level of observational errors allows detection of such a phase residual with a least-squares fit to a timing model. We note that an error ϵ in any phase angle (ω_j or σ_j for either planet) produces strictly periodic residuals with period equal to the orbital period $P_j = 2\pi/n_j$, and of amplitude $\epsilon\kappa_j a_j \sin i_j$. Thus, neglecting uncertainties in other parameters, a 10 μs error in the pulse arrival time residual corresponds to typical phase uncertainties at the level of $\epsilon \approx \pm 0^\circ.4$. The analytic solution of § 3 is not sensitive to such uncertainties, although they would obviously be reflected in the pattern of phase residuals which would not appear as "clean" as in Figure 3.

Finally, a few words are in order regarding the effects of a mutual inclination of the orbits. As mentioned in § 3, a small mutual inclination will not substantially affect our analytical solution. We use the results of numerical integrations to illustrate this. Figure 4 compares the arrival time residuals to be expected from coplanar orbits with inclination 60° to the plane of the sky, and those expected from orbits with a 60° average inclination but which have a mutual inclination of about 13°. The differences are less than 5 μs at any epoch over the 5 yr period.

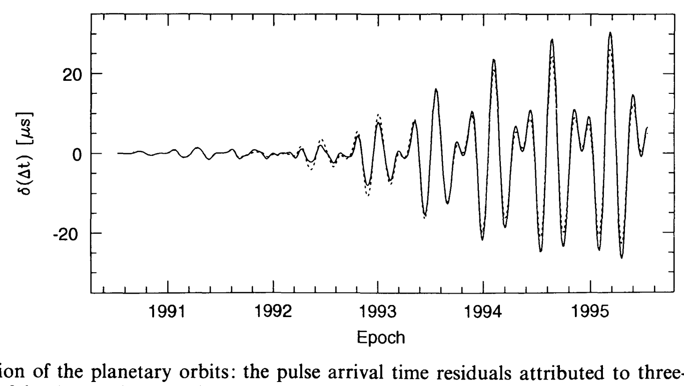

FIG. 4.—Effects of a mutual inclination of the planetary orbits: the pulse arrival time residuals attributed to three-body effects for the case of (1) (*solid line*) coplanar orbits inclined 60° to the plane of the sky and (2) (*dotted line*) orbits with a mutual inclination of about 13° ($i_1 = 55°$, $i_2 = 65°$, $\Omega_2 - \Omega_1 = 10°$).

I thank Alex Wolszczan for providing the improved data for Figure 2 and for discussions. I also thank David Black for discussions. This research was done while the author was a Staff Scientist at the Lunar and Planetary Institute which is operated by the Universities Space Research Association under contract NASW-4574 with the National Aeronautics and Space Administration. This paper is Lunar and Planetary Institute Contribution no. 801.

APPENDIX

EVALUATION OF $Q(\psi, \alpha)$

From the definition of $P(\psi, \alpha)$ (see eq. [18]), we have

$$P(\psi, \alpha) = (1 + \alpha^2 - 2\alpha \cos \psi)^{-1/2} ,$$
$$= \frac{1}{1+\alpha}(1 - k^2 \sin^2 \theta)^{-1/2} \tag{A1}$$

where

$$\theta = \frac{\psi + \pi}{2} \quad \text{and} \quad k^2 = \frac{4\alpha}{(1+\alpha)^2} . \tag{A2}$$

Therefore,

$$Q(\psi, \alpha) \equiv \int_0^{\psi} d\psi' P(\psi', \alpha)$$
$$= \frac{2}{1+\alpha}\left[\int_0^{\theta} \frac{d\theta'}{(1 - k^2 \sin^2 \theta)^{1/2}} - \int_0^{\pi/2} \frac{d\theta'}{(1 - k^2 \sin^2 \theta)^{1/2}}\right] ,$$
$$= \frac{2}{1+\alpha}\left[\int_0^{\theta} \frac{d\theta'}{(1 - k^2 \sin^2 \theta)^{1/2}} - \mathbf{K}(k)\right] , \tag{A3}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

Figure 5 shows a sketch of the integrand $q(\theta) = (1 - k^2 \sin^2 \theta)^{-1/2}$. Let us define $\tilde{\theta} \in [0, \pi/2]$ as follows:

$$\theta = N \frac{\pi}{2} + \tilde{\theta} , \qquad N \text{ is an integer} . \tag{A4}$$

Then

$$\int_0^{\theta} d\theta'(1 - k^2 \sin^2 \theta')^{-1/2} = \begin{cases} N\mathbf{K}(k) + F(\tilde{\theta}, k) , & \text{if } N \text{ is even} , \\ (N+1)\mathbf{K}(k) - F\left(\frac{\pi}{2} - \tilde{\theta}, k\right) , & \text{if } N \text{ is odd} . \end{cases} \tag{A5}$$

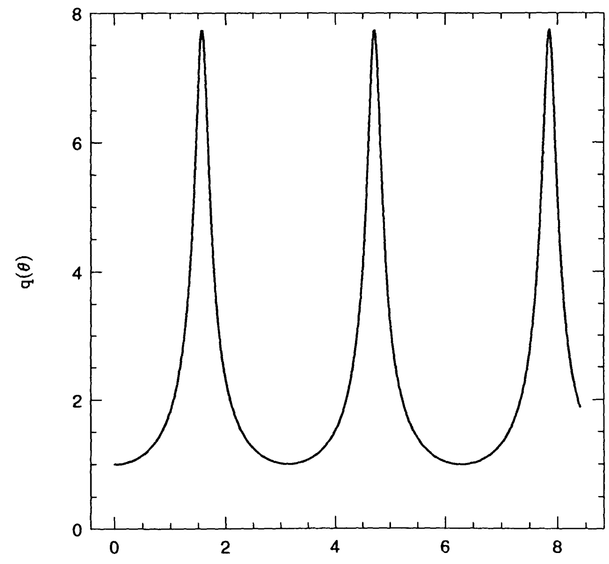

FIG. 5.—Sketch of the integrand, $q(\theta) = (1 - k^2 \sin^2 \theta]^{-1/2}$, for $k = 0.9815$

where $F(\theta, k)$ is the elliptic integral of the first kind. Therefore, it follows that

$$Q(\psi, \alpha) = \frac{2}{1+\alpha} \begin{cases} (N-1)\mathbf{K}(k) + F(\tilde{\theta}, k)\,, & \text{if } N \text{ is even}\,, \\ N\mathbf{K}(k) - F\left(\frac{\pi}{2} - \tilde{\theta}, k\right), & \text{if } N \text{ is odd}\,. \end{cases} \tag{A6}$$

The derivative with respect to α is

$$\alpha \frac{\partial}{\partial \alpha} Q(\psi, \alpha) = -\frac{\alpha}{1+\alpha} Q(\psi, \alpha) + \frac{2(1-\alpha)}{(1+\alpha)^3\sqrt{\alpha}} \begin{cases} (N-1)\frac{d\mathbf{K}(k)}{dk} + \frac{\partial F(\tilde{\theta}, k)}{\partial k}\,, & \text{if } N \text{ is even}\,, \\ N\frac{d\mathbf{K}(k)}{dk} - \frac{\partial F(\pi/2 - \tilde{\theta}, k)}{\partial k}\,, & \text{if } N \text{ is odd}\,. \end{cases} \tag{A7}$$

The following identities are useful in evaluating the k-derivatives above:

$$\frac{d\mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k) - (1-k^2)\mathbf{K}(k)}{k}\,,$$

$$\frac{\partial F(\theta, k)}{\partial k} = \frac{1}{k(1-k^2)}\left[E(\theta, k) - (1-k^2)F(\theta, k) - \frac{k^2 \sin\theta\cos\theta}{(1-k^2\sin^2\theta)^{1/2}}\right], \tag{A8}$$

where $E(\theta, k)$ is the elliptic integral of the second kind, and $\mathbf{E}(k) = E(\pi/2, k)$ is the complete elliptic integral of the second kind.

REFERENCES

Brouwer, D., & Clemence, G. M. 1961, Methods of Celestial Mechanics (New York: Academic)
Dolginov, A., & Stepinski, T. 1992, in Planets around Pulsars, ed. J. A. Phillips, S. E. Thorsett, & S. R. Kulkarni, in press
Gil, J. A., & Jessner, A. 1992, in Planets around Pulsars, ed. J. A. Phillips, S. E. Thorsett, & S. R. Kulkarni, in press
Malhotra, R. 1992, in Planets around Pulsars, ed. J. A. Phillips, S. E. Thorsett, & S. R. Kulkarni, in press
Malhotra, R., Black, D., Eck, A., & Jackson, A. 1992, Nature, 335, 583
Malhotra, R., Fox, K., Murray, C. D., & Nicholson, P. D. 1989, A&A, 221, 348
Rasio, F. A., Nicholson, P. D., Shapiro, S. L., & Teukolsky, S. A. 1992a, Nature, 355, 325
———. 1992b, in Planets around Pulsars, ed. J. A. Phillips, S. E. Thorsett, & S. R. Kulkarni, in press
Wolszczan, A. 1990, IAU Circ., No. 5073
Wolszczan, A., & Frail, D. A. 1992, Nature, 355, 145 (W&F)