THE ASTROPHYSICAL JOURNAL, 394:670–687, 1992 August 1

EFFECTS OF DIFFERENTIAL ROTATION ON STELLAR OSCILLATIONS: A SECOND-ORDER THEORY

W. A. DZIEMBOWSKI
Copernicus Astronomical Center, Polish Academy of Sciences, Bartycka ul. 18, 00-716 Warsaw, Poland

AND

PHILIP R. GOODE[1]
Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge, CB4 3QF, England
Received 1991 July 22; accepted 1992 February 10

ABSTRACT

A complete formalism, valid through second order in differential rotation, is developed and applied to calculate the frequencies of stellar oscillations. We improved the derivation and generalized the asymptotic formulae for g-mode splittings.

In application to solar oscillations, we find that the second-order effects are dominated by distortion for $l < 500$. Further, these effects are sufficiently large that they must be accounted for in any effort to seismically determine the Sun's internal magnetic field. In the solar oscillation spectrum, accidental degeneracies happen but cannot lead to large frequency shifts. For evolved δ Scuti stars, calculated spectra are dense, and, under the perturbing effect of rotation, members of neighboring multiplets may overlap. The seismic potential of modes of mixed p-mode and g-mode character is emphasized for these stars.

Subject headings: stars: oscillations — stars: rotation — Sun: oscillations

1. INTRODUCTION

Helio- and asteroseismology provide us the opportunity to learn about the internal rotation and magnetism of the Sun and some stars. Such knowledge is critical in efforts to understand solar and stellar activity cycles. Further, constraints on the angular momentum evolution in the Sun and stars are important for understanding the role of mixing in stellar interiors.

Rotation and magnetism induce a fine structure in the oscillation spectrum of a star. Many fine-structure multiplets have been observed in the Sun and used to extract information about solar internal rotation and magnetism. In stars, fine structures have been observed, although more limited in scope. In the case of white dwarfs, oscillation data have been used to establish that these objects are slow rotators, McGraw & Robinson (1975), McGraw (1977), and Chlebowski (1978). Fine structure has also been observed in δ Scuti and β Cephei stars.

In stellar applications, the studies of the effect of rotation on the fine structure have been confined to rigid rotation (Saio 1981). Helioseismic applications include r- and θ-dependences but only for the linear effect of rotation. From helioseismic studies, we have learned that surface-like differential rotation persists through much of the Sun's convection zone with an abrupt transition to solid body–like rotation beneath. In spite of the slow rotation of the Sun, the importance of the second-order effects of rotation, like distortion, was realized in the context of efforts to extract magnetic field information from the oscillation data (Dziembowski & Goode 1984; Gough & Taylor 1984; Dziembowski & Goode 1989, 1991; and Gough & Thompson 1990). Except for our recent works, the θ-dependence in the second order effect of rotation was ignored. However, none of these treatments of the second-order effect of rotation was complete.

We develop here the formalism to describe the effect of rotation, through second order, on solar oscillations when the rotation law depends both on radius and latitude. This development also includes a treatment of accidental degeneracies, which, in principle, could contribute to the quadratic effect of rotation. For the Sun, the effect of distortion has been observed in oscillation data. However, most second-order effects are negligible at the current accuracy of oscillation data. For the stars, data on the fine structure will always be sparser; however, we may expect that some stars will rotate much faster than the Sun. With incomplete information, describing the departure from uniform spacing could well be critical in mode identification. In the future, we expect much richer stellar spectra data from space and ground-based observations like the Whole Earth Telescope as described in Wignet et al. (1991).

In the next section, we develop the general problem of rotation including quadratic effects. In the subsequent sections, we perform explicitly the angular integrals, leaving the radial integrals to be performed numerically employing a stellar model.

2. FORMALISM FOR PULSATIONS IN ROTATING STARS

We sketch here the general perturbative formalism describing the oscillations of rotating stars. A detailed development may be found in a standard text, e.g. Unno et al. (1989, chap. 6).

The basic variable describes the Lagrangian fluid displacement, ξ. This displacement can be defined by

$$\xi \propto \exp\left[\mathrm{i}(\omega t + m\phi)\right] , \tag{1}$$

[1] On leave from Department of Physics, New Jersey Institute of Technology, Newark, NJ 07102.

where ω is the angular frequency of the oscillation and m is its azimuthal order. The basic equation for oscillations gives

$$\rho[-(\omega + m\Omega)^2\xi + 2i(\omega + m\Omega)\Omega \boldsymbol{e}_z \times \xi + (\xi \cdot \nabla\Omega^2)r \sin\theta \boldsymbol{e}_s] + \boldsymbol{L}\xi = 0 , \tag{2}$$

where ρ and Ω are the density and rotation rate and $\boldsymbol{e}_z$ and $\boldsymbol{e}_s$ are the standardly defined unit vectors in cylindrical coordinates. The operator $\boldsymbol{L}$ is given by

$$\boldsymbol{L}\xi = \nabla p' + \rho'(\nabla\Phi + \Omega^2 r \boldsymbol{e}_s) + \rho\nabla\Phi' , \tag{3}$$

where p' and Φ' are the pressure and gravitational potential and the prime represents an Eulerian perturbation of a dynamical variable. The perturbed quantities in equation (2) are given in terms of ξ by

$$\rho' = -\nabla \cdot (\rho\xi) \tag{4}$$

and

$$p' = -p\Gamma\left(\nabla \cdot \xi + \frac{\boldsymbol{A} \cdot \xi}{r}\right) , \tag{5}$$

where Γ is the adiabatic exponent and $\boldsymbol{A}$ is

$$\boldsymbol{A} = \frac{1}{\Gamma} \nabla \ln p - \nabla \ln \rho . \tag{6}$$

The perturbation of the gravitational potential is described by

$$\nabla^2\Phi' = 4\pi G\rho' . \tag{7}$$

Assuming that the rotation rate is slow ($\Omega \ll \omega$), we perturb about the mean radial state of the star and collect terms by their order in Ω. Our approach here is a generalization of that of Saio (1981) who considered only the case of uniform rotation. In addition, the general formalism has been presented, but not developed, by Gough & Thompson (1990). Perturbing equation (2) and collecting terms by their order, we have for the unperturbed term

$$-\omega_0^2 \rho_0 \xi_0 + \boldsymbol{L}_0 \xi_0 = 0 . \tag{8}$$

The first-order term is

$$-[2\omega_0(\omega_1 + \boldsymbol{K})\xi_0 + \omega_0^2 \xi_1]\rho_0 + \boldsymbol{L}_0 \xi_1 = 0 , \tag{9}$$

and the second-order term is

$$[(-2\omega_0\omega_2 - \omega_1^2 - 2\omega_1\boldsymbol{K} - 2m\Omega\boldsymbol{K} + r \sin\theta \boldsymbol{e}_s \cdot \nabla\Omega)\rho_0 - \omega_0^2\rho_2 + \boldsymbol{L}_2]\xi_0 - 2\omega_0(\omega_1 + \boldsymbol{K})\rho_0\xi_1 - \omega_0^2\rho_0\xi_2 + \boldsymbol{L}_0\xi_2 = 0 , \tag{10}$$

where

$$\boldsymbol{K} = m\Omega - i\Omega \boldsymbol{e}_z \times . \tag{11}$$

The subscripts 1 and 2 indicate the order of the perturbation. The unperturbed fluid displacement satisfies

$$\int \xi_{0,a}^* \xi_{0,b} \rho d^3\boldsymbol{x} = \langle \xi_a | \xi_b \rangle = \delta_{a,b} , \tag{12}$$

where a and b are a shorthand for n, l, and m. The quantities n, l, and m are the radial order, angular degree, and azimuthal order of the oscillation. The perturbed eigenfunctions are required to satisfy

$$\langle \xi_0 | \xi_1 \rangle = \langle \xi_0 | \xi_2 \rangle = 0 . \tag{13}$$

Using equation (9), we obtain an expression for the first-order change in the frequency, ω_1,

$$\omega_1 = -\langle \xi_0 | \boldsymbol{K} | \xi_0 \rangle . \tag{14}$$

In a similar way from equation (10), we obtain the second-order correction ω_2. We choose to separate ω_2 into several component parts,

$$\omega_2 = \frac{\omega_1^2}{2\omega_0} + \omega_2^D + \omega_2^I + \omega_2^P + \omega_2^T . \tag{15}$$

The term resulting from centrifugal distortion is given by

$$\omega_2^D = \frac{1}{2\omega_0} \langle \xi_0 | (\boldsymbol{L}_2 - \omega_0^2 \rho_2) | \xi_0 \rangle . \tag{16}$$

We ignore here the effect of perturbing the boundary; we will return to this point in § 5. In the third term we collect all terms arising from the explicit second-order effect of inertia,

$$\omega_2^I = \frac{1}{2\omega_0} \int [-m^2\Omega^2 |\xi_0|^2 + 2im\Omega^2 \xi_0^* \cdot (\boldsymbol{e}_z \times \xi_0) + (\xi_0 \cdot \boldsymbol{e}_s)(\xi_0 \cdot \nabla\Omega^2) r \sin\theta] \rho d^3\boldsymbol{x} . \tag{17}$$

The last two terms in equation (15) arise from the first-order perturbation of the eigenfunction which is split into its toroidal and poloidal components, so that

$$\omega_2^T = -\langle \xi_0 | \boldsymbol{K} | \xi_1^T \rangle \tag{18}$$

and

$$\omega_2^P = -\langle \xi_0 | \boldsymbol{K} | \xi_1^P \rangle \; . \tag{19}$$

In the next three sections, we develop an explicit form for $\omega_1(m)$ and $\omega_2(m)$ in terms of the unperturbed eigenfunctions.

3. FIRST-ORDER EFFECT OF ROTATION

We consider a rotation law of the form

$$\Omega = \bar{\Omega}\left(1 + \sum_{s=0}^{S} \eta_s \mu^{2s}\right), \qquad \mu = \cos\theta \; , \tag{20}$$

where $\bar{\Omega}$ is a measure of the mean rotation and the η_s are each a function of radius. Such a rotation law presumes that the rotation is symmetric about the star's equator. If the rotation law had a part that were antisymmetric about the equator, then the antisymmetric terms would have their lowest order effect on frequency in ω_2. We choose the law in equation (20) because we intend to focus much of our attention on the Sun where that law is directly consistent with the observed surface rotation and indirectly consistent with the internal rotation from helioseismology.

The fluid displacement is quite generally defined by

$$\xi = r \sum_{n,l,m} [y_{n,l,m}(r) Y_l^m(\theta, \phi) \boldsymbol{e}_r + z_{n,l,m}(r) \nabla_H Y_l^m + \tau_{l,m}(r) \boldsymbol{e}_r \times \nabla_H Y_l^m] e^{i\omega t} \; . \tag{21}$$

In zeroth order, each mode is characterized by a single n, l, and m, where modes are grouped in degenerate (nl)-multiplets and all τ_l's are identically zero. Heretoforward, we shall drop the n, l, and m subscripts unless they are specifically required for clarity. With this in mind, the normalization condition is

$$\int [y^2 + \Lambda z^2] \rho r^4 dr = 1 \; , \qquad \Lambda = l(l+1) \; . \tag{22}$$

In addition to y and z, we also use the eigenfunctions w and v defined by the pertubation of the gravitational potential,

$$\nabla \Phi' = g[w(r) Y_l^m \boldsymbol{e}_r + v(r) \nabla_H Y_l^m] \; , \tag{23}$$

where

$$g = \frac{GM_r}{r^2} \; . \tag{24}$$

3.1. *Effect on Frequencies*

Inserting the rotation law of equation (20) and the eigenfunctions of equation (21) into equation (9) for the first-order change in frequency, ω_1, we obtain

$$\omega_1 = m\bar{\Omega}\left[\!\!\left[\mathrm{C_L} - 1 - \sum_{s=0} \{J_{1,s} \mathcal{Q}_s + sJ_{2,s}[(2s-1)\mathcal{Q}_{s-1} - (2s+3)\mathcal{Q}_s]\} \right]\!\!\right] , \tag{25}$$

where the Ledoux constant, $\mathrm{C_L}$, is

$$\mathrm{C_L} = \int (2yz + z^2) \rho r^4 dr \; , \tag{26}$$

and where the remaining radial integrals are given by

$$J_{1,s} = \int \eta_s (y^2 + \Lambda z^2 - 2yz - z^2) \rho r^4 \, dr \tag{27}$$

and

$$J_{2,s} = \int \eta_s z^2 \rho r^4 \, dr \; . \tag{28}$$

For high-frequency p-modes, such as 5 minute solar oscillations, we have that $\mathrm{C_L} \ll 1$ and $J_{2,s} \gg J_{1,s}$. The angular integrals are given by

$$\mathcal{Q}_s = \int \mu^{2s} | Y_l^m |^2 \, d\mu \, d\phi \; . \tag{29}$$

They are conveniently evaluated by a recursive relation (Dziembowski & Goode 1989), which makes explicit the polynomial dependence on m^2, whereby

$$\mathscr{Q}_{-1} = 0\,, \qquad \mathscr{Q}_0 = 1 \tag{30}$$

and

$$\mathscr{Q}_s = \frac{1}{4\Lambda + 1 - 4s^2} \frac{2s-1}{s} \{\mathscr{Q}_{s-1}[-2m^2 + 2\Lambda - (2s-1)^2] + \mathscr{Q}_{s-2}(2s-3)(s-1)\}\,. \tag{31}$$

3.2. *First-Order Perturbation of the Eigenfunction*

The toroidal component can be expressed in terms of y and z by taking the radial curl of equation (9). First we get

$$\tfrac{1}{2}\omega_0 \nabla \times \xi_{1,T}\Big|_r = \mathrm{i}\Omega\nabla \times (e_z \times \xi_0)\Big|_r - \nabla\Omega \times (m\xi_0 - \mathrm{i}e_z \times \xi_0)\Big|_r\,. \tag{32}$$

With the use of equation (21) and well-known relations,

$$\mu Y_l^m = \sqrt{\mathscr{I}_{l+1}}\,Y_{l+1}^m + \sqrt{\mathscr{I}_l}\,Y_{l-1}^m \tag{33}$$

and

$$\sin\theta \frac{\partial Y_l^m}{\partial\theta} = l\sqrt{\mathscr{I}_{l+1}}\,Y_{l+1}^m - (l+1)\sqrt{\mathscr{I}_l}\,Y_{l-1}^m\,, \tag{34}$$

where $\mathscr{I}_l = (l^2 - m^2)/(4l^2 - 1)$. We get

$$\tau_{l+j} = \frac{2\mathrm{i}\bar{\Omega}}{\omega_0(l+j)(l+j+1)}\,\tilde{\tau}_j\,, \tag{35}$$

where

$$\begin{aligned}\tau_j = {} & \delta_{0,k}\{\sqrt{\mathscr{I}_l}(l-1)[y + (l+1)z] - \sqrt{\mathscr{I}_{l+1}}(l+2)(y - lz)\} \\ & + \sum_{s\geq k} \eta_s\{\sqrt{\mathscr{I}_l}(l-1-2s)\mathscr{M}_{s,l-1,l+j}[y + (l+1)z] - \sqrt{\mathscr{I}_{l+1}}(l+2+2s)\mathscr{M}_{s,l+1,l+j}(y - lz) \\ & + 2s(\sqrt{\mathscr{I}_{l+1}}\mathscr{M}_{s-1,l+1,l+j} + \sqrt{\mathscr{I}_l}\mathscr{M}_{s-1,l-1,l+j})(y - m^2 z)\}\,, \qquad k \leq S\,, \qquad j = \pm(2k+1)\end{aligned} \tag{36}$$

and

$$\mathscr{M}_{s,l,k} = \int \mu^{2s} Y_l^m Y_k^{-m}\, d\mu\, d\phi\,. \tag{37}$$

Equation (33) can be used to express this integral in terms of a polynomial in m^2.

There are two ways to calculate the perturbed part of the poloidal mode eigenvector. One consists of expanding it in terms of unperturbed eigenvectors. Inserting this expansion into equation (9), one gets in a standard way,

$$\xi_{1,n,l}^P = \sum_{(\tilde{n},l)\neq(n,l)} \frac{\omega_{0,n,l}\langle\xi_0|\boldsymbol{K}|\xi_{0,\tilde{n},l}\rangle}{\omega_{0,n,l}^2 - \omega_{0,\tilde{n},l}^2}\,\xi_{0,\tilde{n},l}\,, \tag{38}$$

where

$$\langle\xi_{0,n,l}|\boldsymbol{K}|\xi_{0,\tilde{n},l}\rangle = \bar{\Omega}\int\left\{-(y\tilde{y} + \tilde{y}z + z\tilde{z}) + \sum_s \eta_s\left[\mathscr{M}_{s,l,l}\left(y\tilde{y} - y\tilde{z} - \tilde{y}z + \frac{\Lambda + \tilde{\Lambda} - 2}{2} z\tilde{z}\right) + \mathscr{N}_{s,l,l} z\tilde{z}\right]\right\}\rho r^4\, dr\,, \tag{39}$$

and

$$\mathscr{N}_{s,l,l} = s[\mathscr{M}_{s-1,l,l} - \mathscr{M}_{s,l,l}(2s+3)]\,. \tag{40}$$

The disadvantage with this method is the infinite sum. The convergence properties of the expansion are not generally clear. Furthermore, in a dense spectrum, like those for high-l p-modes or acoustic gravity modes in evolved stars, the denominator may be small. An alternative approach is to solve equation (9) directly. Here we follow Hansen, Cox, & Carroll (1978) and Saio (1981) in deriving equations for the radial eigenfunctions corresponding to ξ_1^P, but generalizing to include differential rotation. This generalization results in having to consider ξ_1^P not only being proportional to a single Y_l^m, but also depending on $Y_{l\pm 2j}^m$ for integer j. We first take the horizontal divergence of equation (9),

$$\nabla_H \cdot \left(\xi_1 - \frac{\boldsymbol{L}_0\xi_1}{\omega^2}\right) = \frac{2m}{\omega_0}\left\{\Omega[(\Lambda - 1)z - y]Y_l^m + \frac{\partial\Omega}{\partial\theta}\left(\cot\theta\, Y_l^m - \frac{\partial Y_l^m}{\partial\theta}\right) + \frac{\omega_1}{m}\Lambda z Y_l^m\right\} \tag{41}$$

and then the radial component of equation (9),

$$(\boldsymbol{L}_0\xi_1 - \omega_0^2\xi_1)\Big|_r = 2m\omega_0 r\left[\left(\frac{\omega_1}{m} + \Omega\right)y - \Omega z\right]Y_l^m\,. \tag{42}$$

We write the poloidal part in the form

$$\xi_1^P = \frac{2m\bar{\Omega}}{\omega_0} \sum_{j=-S}^{j=+S} \sqrt{\mathscr{A}_j}\,[\tilde{y}_j(r) Y_{l+2j}^m \boldsymbol{e}_r + \tilde{z}_j(r) \nabla_H Y_{l+2j}^m] . \tag{43}$$

The factor $\sqrt{\mathscr{A}_j}$ was introduced for convenience in the polynomial expansion. For the lowest order, we have

$$\mathscr{A}_0 = 1 , \qquad \mathscr{A}_{\pm 1} = \mathscr{I}_{l\pm 1}\mathscr{I}_{l+1\pm 1} , \qquad \mathscr{A}_{\pm 2} = \mathscr{A}_{\pm 1}\mathscr{I}_{l\pm 3}\mathscr{I}_{l+1\pm 3} , \text{ etc .} \tag{44}$$

Using equation (43) in equations (41) and (42) and equations (3)–(7) corresponding to ξ_1^P, we obtain

$$r\frac{d\tilde{y}_j}{dr} + (3 - V_g)\tilde{y}_j + (V_g C\sigma^2 - \Lambda_j)\tilde{z}_j - V_g \tilde{v}_j = q_j(y - \tilde{\Lambda}_j z) - s_j z - \delta_{j,0}\,\tilde{\sigma}_1 \Lambda z \tag{45}$$

and

$$r\frac{d\tilde{z}_j}{dr} + \left(\frac{A}{C\sigma^2} - 1\right)\tilde{y}_j + (2 - A)\tilde{z}_j - \frac{A}{C\sigma^2}\,\tilde{v}_j = q_j(y - z) + 2\delta_{j,0}\,\tilde{\sigma}_1\, y , \tag{46}$$

where $\tilde{v}_j$ is the eigenfunction corresponding to the first-order perturbation of the radial eigenfunctions of the potential and where

$$\tilde{\sigma}_1 = 1 + \frac{\omega_1}{m\bar{\Omega}} \tag{47}$$

and

$$\tilde{\Lambda}_j = \Lambda + l\,\frac{4j+1}{2} + j(2j+1) - 1 . \tag{48}$$

The q_j and s_j terms arise from the angular intervals and are given by

$$q_j = \frac{1}{\sqrt{\mathscr{A}_j}} \sum_s \eta_s \mathscr{M}_{s,l,l+2j} \tag{49}$$

and

$$s_j = \frac{1}{\sqrt{\mathscr{A}_j}} \sum_s \eta_s \mathscr{N}_{s,l,l+2j} , \tag{50}$$

where the q_j and s_j are polynomials in m^2 and so are the perturbed eigenfunctions $\tilde{y}_j$, etc. In particular, for the three-term rotation law typically used to describe the Sun's internal rotation ($S = 2$ in eq. [20]), we have

$$q_0 = \sum_s \eta_s \mathscr{Q}_s , \qquad q_{\pm 1} = \eta_1 + \eta_2(\mathscr{Q}_{1,l} + \mathscr{Q}_{1,l\pm 2}) , \qquad q_{\pm 2} = \eta_2 \tag{51}$$

and

$$s_0 = \eta_1(1 - 5\mathscr{Q}_1) + \eta_2(3\mathscr{Q}_1 - 7\mathscr{Q}_2) \tag{52}$$

and

$$s_{\pm 1} = -5\eta_1 + 2\eta_2(3 - 7\mathscr{Q}_{1,l} - 7\mathscr{Q}_{1,l\pm 2}) , \qquad s_{\pm 2} = -14\eta_2 . \tag{53}$$

Note that equation (31) implies that $\mathscr{Q}_{k,l}$ is an m^{2k}-polynomial. The unitless frequency is given by

$$\sigma = \frac{\omega_0}{\sqrt{4\pi G\langle\rho\rangle}} \tag{54}$$

and V_g and C are properties of the unperturbed model, where

$$V_g = \frac{gr\rho}{p\Gamma} \tag{55}$$

and

$$C = 3\,\frac{r^3 M}{R^3 M_r} . \tag{56}$$

Equations (45) and (46) have to be supplemented with two corresponding differential equations for $\tilde{v}_j$ and $\tilde{w}_j$ which will not be reproduced here because they are the same as they would be in the absence of rotation. We do not give explicit forms for the boundary conditions because they are easy to derive following the way in which they are derived for a nonrotating star.

This problem must be solved numerically. This is the price for having a finite system. The case of $j = 0$, uniform rotation, has been considered by Saio (1981). In this case a nontrivial homogeneous solution exists; therefore we must consider $\tilde{\sigma}_1$ as an unknown instead of using its value from equation (47). The orthogonality condition is employed to ensure the uniqueness of the solution. For $j \neq 0$, we do not encounter any problem in solving the inhomogeneous problem unless there are accidental degeneracies. We treat the latter problem in the next section. Still another approach is useful in the case of dense spectra such as one encounters in evolved δ Scuti stars; see § 6.

3.3. *Accidental Degeneracies*

Both formulations for the poloidal mode given in the previous section presume no degeneracy in frequency between the mode of interest and modes differing in degree by 2, 4, ... 2*S*. θ-dependent rotation couples such modes within linear theory. The coupling would occur at higher order in Ω for spherical rotation. We note that in real solar oscillation data, there are accidental degeneracies. We formally treat the problem of accidental degeneracies assuming there are two nearly degenerate modes, *a* and *b*. Their frequency difference is

$$\Delta\omega = \omega_{0,b} - \omega_{0,a} , \tag{57}$$

where $\Delta\omega$ is a small quantity of order Ω and the true mode ξ is a linear combination of *a* and *b*,

$$\xi = B_a \xi_{0,a} + B_b \xi_{0,b} . \tag{58}$$

We determine the constant of proportionality by solving the equations arising after inserting ξ into equation (8) which is extended to include the $-2\boldsymbol{K}\xi_0$ term of equation (9),

$$(\omega^2 - 2\omega\omega_{1,a} - \omega_{0,a}^2)B_a - 2\omega\omega_{1,a,b} B_b = 0 \tag{59}$$

and

$$-2\omega\omega_{1,a,b} B_a + (\omega^2 - 2\omega\omega_{1,b} - \omega_{0,b}^2)B_b = 0 , \tag{60}$$

where we define

$$\omega_{1,a,b} = -\langle \xi_{0,a} | \boldsymbol{K} | \xi_{0,b} \rangle . \tag{61}$$

We denote the perturbed frequency by

$$\omega = \omega_{0,a} + \tilde{\omega}_1 , \tag{62}$$

where here the tilde serves as a reminder that in degenerate perturbation theory the general result includes terms beyond first order. We obtain for the frequency shift

$$\tilde{\omega}_1 = \frac{\omega_{1,a,a} + \omega_{1,b,b} + \Delta\omega \pm \sqrt{(\omega_{1,a,a} - \omega_{1,b,b} - \Delta\omega)^2 + 4\omega_{1,a,b}^2}}{2} . \tag{63}$$

We remark that in the far-resonance limit, $\Delta\omega \gg |\omega_{1,a,a} - \omega_{1,b,b}|$, the coupling term contributes only to the second-order frequency correction, specifically to the part due to the poloidal eigenvector perturbation,

$$\omega_{2,a}^P = -\omega_{2,b}^P = \frac{\omega_{1,a,b}^2}{\Delta\omega} . \tag{64}$$

4. ROTATIONAL PERTURBATION OF EQUILIBRIUM

The rotation law in equation (20) implies that Ω^2 may be written as

$$\Omega^2 = \bar{\Omega}^2 \left[1 + \sum_{j=0}^{2S} \tilde{\eta}_j(r) \mu^{2j} \right] , \tag{65}$$

where there is an obvious relation between $\tilde{\eta}_s$ and η_s. The centrifugal force is given by

$$\Omega^2 r \sin\theta \boldsymbol{e}_s = -\nabla\Phi^c + F\boldsymbol{e}_r , \tag{66}$$

where we have arbitrarily separated it into a part arising from a potential and a nonpotential part. The potential part is determined, up to a θ-independent function $f(r)$, where

$$\Phi^c = \frac{r\bar{\Omega}^2}{2} \left[\mu^2 + \sum_{j=1} \tilde{\eta}_{j-1} \mu^{2j} \right] + f(r) , \tag{67}$$

and the nonpotential part is given by

$$F = r\bar{\Omega}^2 \left[\sum_{j=1} d_j \mu^{2j} + 1 + \tilde{\eta}_0 \right] + \frac{df}{dr} , \tag{68}$$

where

$$d_j = \tilde{\eta}_j - \frac{j-1}{j} \tilde{\eta}_{j-1} + \frac{r}{2j} \frac{d\tilde{\eta}_{j-1}}{dr} . \tag{69}$$

As usual we use a Legendre expansion, and therefore, we need to change our expansion in μ's to one in P's, whereby

$$\mu^{2j} = \sum_{k=0} T_{j,k} P_{2k}(\mu) \tag{70}$$

and

$$T_{j,k} = \frac{2^{2k}(4k+1)(j+k)!\,(2j)!}{(2j+2k+1)!\,(j-k)!}\,, \qquad j \geq k\,, \qquad \text{and 0 otherwise}\,. \tag{71}$$

We choose $f(r)$ so that our expansion of the potential begins with P_2 rather than P_0,

$$\Phi^c = r^2\bar{\Omega}^2 \sum_{k=1} c_k P_{2k}(\mu)\,. \tag{72}$$

Then the force, F, is given by

$$F = r\bar{\Omega}^2\left[1 + \tilde{\eta}_0 + \sum_j T_{j,0}\, d_j + \sum_{k=1} b_k P_{2k}(\mu)\right]. \tag{73}$$

We ignore the first three terms in equation (73). This does not have any effect on the multiplet structure. Rather, it causes an overall frequency shift of the multiplet. We remark that a purely spherical perturbation must be considered in conjunction with the complete set of stellar structure equations (Gough & Thompson 1990).

The expansion coefficients c_k and b_k are given by

$$c_k = \frac{1}{3}\,\delta_{k,0} + \sum_{j\geq k} T_{j,k}\,\frac{\tilde{\eta}_{j-1}}{2j} \tag{74}$$

$$b_k = \sum_{j\geq k} T_{j,k}\, d_j\,. \tag{75}$$

The Legendre expansion of the gravitational potential is written in the form

$$\Phi_2 = \sum_{k=1} \Phi_{2,k} P_{2k}\,. \tag{76}$$

From the condition for mechanical equilibrium, we obtain the corresponding expansion coefficients for pressure and density,

$$p_{2,k} = -\rho_0 \bar{\Omega}^2 r^2 u_k \tag{77}$$

and

$$\rho_{2,k} = \frac{r\bar{\Omega}^2}{g}\left(r\,\frac{d\rho_0}{dr}\,u_k + \rho_0 b_k\right) \tag{78}$$

where

$$u_k = \frac{\Phi_{2,k}}{\bar{\Omega}^2 r^2} + c_k\,. \tag{79}$$

Poisson's equation for the perturbed potential becomes

$$\frac{d}{dr}\left(r^2\,\frac{d\Phi_{2,k}}{dr}\right) - 2k(2k+1)\Phi_{2,k} = r^2\bar{\Omega}^2 U\left(\frac{d\ln\rho_0}{d\ln r}\,u_k + b_k\right), \tag{80}$$

where

$$U = \frac{4\pi r^3 \rho}{M_r}\,. \tag{81}$$

Our boundary conditions for the potential are

$$\Phi_{2,k} \propto r^{2k}\,, \qquad r \to 0 \tag{82}$$

and

$$\Phi_{2,k} \propto r^{-(2k+1)}\,, \qquad r \to R\,. \tag{83}$$

These equations completely specify the distortion of the stellar structure to the extent it is needed in the equation for stellar oscillation. Distortion of the temperature profile is not specifically needed, but we do need the perturbation of the adiabatic exponent Γ_1. Fortunately, this perturbation may be significant only in the outer layers. In these chemically homogeneous layers, the perturbation of Γ_1 may be expressed in terms of $p_{2,k}$ and $\rho_{2,k}$ alone.

We could have chosen a distorted geometry such that all $u_k \equiv 0$ eliminating the term in equation (78) involving the derivative of the density which is relatively large near the surface. However, we choose to work in spherical geometry since the results are the same as for the distorted geometry and the equations are easier to handle.

5. THE SECOND-ORDER EFFECT OF ROTATION

We evaluate ω_2^D from equation (16) using $\boldsymbol{L}_2$ obtained from perturbing equation (3) in which we replace p, ρ, and ϕ by their perturbed expansions given in equations (76)–(79), The eigenfunctions are given in equations (21) and (23) and used in the integral for ω_2^D. With this, we perform the angular integrals and some manipulations to obtain

$$\omega_2^D = \frac{\bar{\Omega}^2}{\omega_0} \sum_{k=1} \tilde{\mathcal{Q}}_k \int_0^R \left\{ \frac{\lambda^2}{V_g} \left[\frac{d \ln p\Gamma}{d \ln r} u_k + \left. \frac{\partial \ln \Gamma}{\partial \ln \rho} \right|_p b_k \right] + [-y^2(4 + C\sigma^2 - U) - z^2 C\sigma^2 \tilde{\Lambda}_k + yz(\Lambda + \tilde{\Lambda}_k) + 2yw + 2zv\tilde{\Lambda}_k] \right.$$
$$\left. \times \left(\frac{d \ln \rho}{d \ln r} u_k + b_k \right) + \left(y^2 \frac{d \ln \rho}{d \ln r} + 2\lambda y \right) \left(b_k - 2u_k - r \frac{du_k}{dr} \right) - yzu_k \kappa - 2\lambda z u_k \right\} \rho r^4 \, dr , \quad (84)$$

where

$$\lambda = V_g(y - C\sigma^2 z + v) \quad (85)$$

and

$$\tilde{\Lambda}_k = \Lambda - \kappa \quad (86)$$

and

$$\kappa = k(2k + 1) . \quad (87)$$

The angular integrals, $\tilde{\mathcal{Q}}_k$, are related to the $\mathcal{Q}_j$ through the following relation

$$\tilde{\mathcal{Q}}_k = \sum_{j=1} P_{k,j} \mathcal{Q}_j , \quad (88)$$

where the $P_{k,j}$ are coefficients in the Legendre expansion

$$P_{2k}(\mu) = \sum_{s=0}^{k} P_{k,s} \mu^{2s} . \quad (89)$$

An alternative expression for ω_2^D can be obtained from equation (84) by a series of integrations by parts in which the oscillation equation and equation (80) are used and the contributions from the boundary terms are ignored. These terms are proportional to ρ and p; for trapped modes, they can be made arbitrarily small by moving the outer boundary far away. The same applies to the boundary condition perturbation mentioned in the previous section. For further details, see the appendix in Gough & Thompson (1990). This alternative expression is

$$\omega_2^D = \frac{\bar{\Omega}^2}{2\omega_0} \sum_{k=1} \tilde{\mathcal{Q}}_k \int_0^R \left[C\sigma^2 \left\{ (2y^2 + \kappa z^2) \left[r \frac{du_k}{dr} + (4 - U)u_k \right] + 2z^2 \Lambda u_k + yz(\Psi_k + 2\kappa u_k) \right\} \right.$$
$$+ 2y\left(\lambda + y \frac{d \ln \rho}{d \ln r} \right) U u_k - yw\left[3r \frac{du_k}{dr} + u_k(10 - 3U) \right] - yv(\Psi_k + 2\Lambda u_k) - 2zw\Lambda u_k$$
$$\left. - \left[y^2(C\sigma^2 + 3 - U) + z^2 C\sigma^2 \tilde{\Lambda}_k + yz\kappa - \frac{\lambda^2}{V_g} \left. \frac{\partial \ln \Gamma}{\partial \ln \rho} \right|_p - 2yw - 2zv\tilde{\Lambda}_k \right] b_k - y^2 r \frac{db_k}{dr} \right] \rho r^4 \, dr , \quad (90)$$

where

$$\Psi_k = r \frac{du_k}{dr} (4 - 2U) + u_k[(4 - 2U)(4 - U) - 4 + 2\kappa] + r^2 \frac{d^2 c_k}{dr^2} + 6r \frac{dc_k}{dr} + (6 - 2\kappa)c_k + U b_k . \quad (91)$$

This expression for the frequency change is what one would obtain working directly in a distorted geometry. Equation (90) reduces to that of Saio (1981) in the limit of rigid rotation ($\tilde{\eta}_s = 0$).

The frequency perturbation due to explicit second-order inertial effects follows from inserting expressions for Ω from equation (20) and equation (21) for ξ_0 into equation (17),

$$\omega_2^I = \frac{\bar{\Omega}^2}{2\omega_0} \left[m^2 \left\{ 2C_L - 1 - \sum_s \int \tilde{\eta}_s \mathcal{Q}_s [y^2 - 4yz + (\Lambda - 2)z^2] \rho r^4 \, dr \right\} + \sum_s \int \left[s\tilde{\eta}_s (z^2 D_s + yzE_s) + r \frac{d\tilde{\eta}_s}{dr} (y^2 G_s + yzH_s) \right] \rho r^4 \, dr \right] , \quad (92)$$

where

$$D_s = (2s - 1)(m^2 - s)\mathcal{Q}_{s-1} + [m^2(2s + 7) + (2s + 1)^2 - \Lambda]\mathcal{Q}_s + [\Lambda - (s + 1)(2s + 3)]\mathcal{Q}_{s+1} \quad (93)$$

and

$$E_s = -(2s - 1)\mathcal{Q}_{s-1} + (4s + 2)\mathcal{Q}_s - (2s + 3)\mathcal{Q}_{s+1} \quad (94)$$

and

$$G_s = \mathcal{Q}_s - \mathcal{Q}_{s+1} \quad (95)$$

and

$$H_s = \frac{2s+1}{2}\mathscr{Q}_s - \frac{2s+3}{2}\mathscr{Q}_{s+1} . \tag{96}$$

The dependence on m, except where explicit, comes only through the $\mathscr{Q}$'s like in the expressions for ω_2^D.

Making use of equation (9) and

$$\langle \xi_1^T | \xi_1^P \rangle = 0 \tag{97}$$

in equation (18) to obtain $\langle \xi_1^T | \xi_1^T \rangle$ and then explicitly using equations (21) and (35), we obtain

$$\omega_2^T = \frac{\omega_0}{2}\langle \xi_1^T | \xi_1^T \rangle = \frac{2\bar{\Omega}^2}{\omega_0}\sum_{k=-S}^{k=S+1}\frac{1}{(l+2k+1)(l+2k)}\int \tilde{\tau}_{2k-1}^2 \rho r^4 dr . \tag{98}$$

Here the dependence on m occurs through $\tilde{\tau}^2$, and we may verify with the use of equations (33), (36), and (37) that it is a polynomial in m^2.

We determine ξ_P from equation (43) and use it to obtain an explicit expression for ω_2^P,

$$\omega_2^P = \frac{2m^2\bar{\Omega}^2}{\omega_0}\int\left\{\tilde{y}_0 z + y\tilde{z}_0 + \tilde{z}_0 z - \sum_{j=-S}^{j=+S}\mathscr{A}_j[q_j(\tilde{y}_j y + \tilde{\Lambda}_j \tilde{z}_j z - \tilde{z}_j y - \tilde{y}_j z) + s_j \tilde{z}_j z]\right\}\rho r^4 dr . \tag{99}$$

Let us recall that the perturbed eigenfunctions, like $\tilde{y}$, are proportional to $\mathscr{Q}$'s, and the $\mathscr{A}_j$'s are products of the $\mathscr{I}$'s. Thus, the ω_2^P is a polynomial in m^2.

In sum, the total second-order effect of rotation is given by equation (15). The first term appearing on the right-hand side may be calculated from equation (25), the second from equation (84) or (90), and the third, fourth, and fifth from equations (92), (99), and (98), respectively. For any rotation law presented in the form of equation (20), any stellar model represented by certain radial functions, and eigenfunctions calculated from that model, the fine structure in any multiplet can be determined from equation (15). The second-order part of the fine structure is described as a polynomial in m^2.

6. *g*-MODE ASYMPTOTICS

Here we introduce g-mode asymptotics because they are relevant for white dwarfs and δ Scuti stars and the resultant formulae are exceptionally simple for spherical rotation and model-independent for rigid rotation. In the formulation of the g-mode asymptotics, we include the Coriolis term in zeroth order. Such a development can also be done for p-modes, but it does not lead to model-independent formulae. For simplicity, we limit ourselves to the case of θ-independent rotation. The treatment, however, may be easily generalized to the rotation law of equation (20) in § 3.

One approach to the g-mode asymptotics would be to use equations (45) and (46) with $j = 0$ and then let $\tilde{y}_0 + y \to y$, etc., and drop the $\tilde{\sigma}_1$-terms because they are now already included in σ. However, it is advantageous in the asymptotics to include the effect of trivial modes because the resulting formulae become valid through third order in Ω. We shall see that this validity is also a consequence of the fact that distortion does not contribute. We start with equation (32) which expresses a toroidal component in terms of poloidal ones. Then, taking the radial component and horizontal divergence of equation (2), we see that the equations for eigenfunctions are separable in terms of a single Y_l^m through third order in Ω. Ignoring terms of order Ω^4 and higher, we thus get

$$ry' = \left(V_g - 3 + \frac{h_1}{h_2}\right)y + \left(\frac{\Lambda}{C\tilde{\sigma}^2} - V_g\right)s + V_g v \tag{100}$$

and

$$rs' = (C\tilde{\sigma}^2 h_3 - A)y - \left(1 + A - U - \frac{h_1}{h_2}\right)s - Av . \tag{101}$$

We used here the radial eigenfunction, $s \propto (r/g)[(p'/\rho) + \Phi']$. This eigenfunction is related to y and z by

$$s = C\sigma^2\left(zh_2 - \frac{h_1}{\Lambda}y\right) \tag{102}$$

where

$$h_1 = \chi - l\delta_l + (l+1)\gamma_l , \tag{103}$$

$$h_2 = 1 - \frac{\chi}{\Lambda} - \frac{l}{l+1}\delta_l - \frac{l+1}{l}\gamma_l , \tag{104}$$

$$h_3 = 1 - \delta_l - \gamma_l - \frac{h_1^2}{h_2\Lambda} , \tag{105}$$

$$\chi = \frac{2m\Omega}{\tilde{\omega}} , \tag{106}$$

$$\tilde{\omega} = \omega + m\Omega , \tag{107}$$

$$\tilde{\sigma} = \frac{\tilde{\omega}}{\sqrt{4\pi G\langle\rho\rangle}} , \tag{108}$$

$$\delta_l = \frac{4\Omega^2}{\tilde{\omega}^2} \frac{(l+2)^2 J_{l+1}}{(l+2)(l+1) - \chi} , \tag{109}$$

and

$$\gamma_l = \frac{4\Omega^2}{\tilde{\omega}^2} \frac{(l-1)^2 J_l}{l(l-1) - \chi} . \tag{110}$$

For uniform rotation, $\tilde{\omega}$ may be regarded as the eigenvalue and it is equal to the frequency of oscillation in the corotating system. We remark that δ_l and γ_l would be zero in the absence of trivial modes.

We consider the g-mode limit of ω much less than the Brunt-Väisälä and Lamb frequencies while still being much larger than Ω. With this, we are able to reduce our equations to a single, second-order equation in normal form,

$$\frac{d^2q}{dr^2} + \frac{q}{r^2}\left[\frac{N^2\Lambda}{\tilde{\omega}^2 h_2} + O(1)\right] = 0 , \tag{111}$$

where the transformed variable, q, is related to y by

$$y = \frac{q}{r^3\sqrt{\rho}} . \tag{112}$$

The Brunt-Väisälä frequency, N, is given by

$$N^2 = 4\pi G\langle\rho\rangle \frac{A}{C} . \tag{113}$$

Equation (111) implies (see, e.g., Unno et al. 1989) the following implicit relation:

$$(n + \alpha_g)\pi \approx \sqrt{\Lambda} \int_{r_1}^{r_2} \frac{N}{\tilde{\omega}} \frac{1}{\sqrt{h_2}} \frac{dr}{r} \tag{114}$$

where r_1 and r_2 are the boundaries of the g-mode propagation region and the phase constant α_g would be obtained from being explicit about $O(1)$ in equation (111). Equation (113) for uniform rotation in the inertial system implies that

$$\tilde{\omega}\sqrt{h_2} = \frac{\sqrt{\Lambda}}{n + \alpha_g} \int_{r_1}^{r_2} N \frac{dr}{r} . \tag{115}$$

If we ignore the rotational pertubation of the boundary conditions and of α, we obtain

$$\tilde{\omega}\sqrt{h_2} = \omega_0 . \tag{116}$$

Expanding the square root in equation (116), we determine that

$$\omega = \omega_0 - m\Omega\left(1 - \frac{1}{\Lambda}\right) - \frac{m^2\Omega^2}{\omega_0} \frac{4\Lambda(2\Lambda - 3) - 9}{2\Lambda^2(4\Lambda - 3)} . \tag{117}$$

which is equivalent to the formula obtained by Chlebowski (1978). However, the derivation here is more satisfactory because Chlebowski, without justification, ignored the contribution of ω_2^p in obtaining his expression.

Applying g-mode asymptotics to equation (90) for distortion, we find that

$$\omega_2^p \sim \frac{m^2\Omega^2}{\omega} \sigma^2 \bar{C} , \tag{118}$$

where $\bar{C}$ means the average value of C for the g-mode propagation zone. This contribution is higher order in the g-mode asymptotics than the corresponding term in Ω^2 in equation (117).

Even though equation (117) is appealing, it may be difficult in practice for second order in Ω. In particular, for $l = 1$ the second-order coefficient is only 0.025 so that higher order terms easily come into play. This was illustrated by the realistic calculations for white dwarfs made by Brassard, Wesemael, & Fontaine (1989)–see their Table 1.

We note that equation (117) remains valid even if

$$\Omega > \omega_{n-1,l,0} - \omega_{n,l,0} \approx \frac{\sqrt{\Lambda}}{n^2} \int_{r_1}^{r_2} N \frac{dr}{r} . \tag{119}$$

This fact is contrary to what one of us wrote in the context of δ Scuti stars (Dziembowski 1990). In particular, in model calculations of these stars, we may have that $\Omega > \omega_{n-1,l,0} - \omega_{n,l,0}$ while $\omega_{n,l,} > \Omega$; even under these two conditions, equation (117) holds implying that a calculated spectral multiplet would be nearly uniformly spaced while overlapping with its nearly uniformly spaced neighboring multiplets.

7. THE SECOND-ORDER EFFECT OF ROTATION IN THE SUN

The observational data describing the fine structure in (nl)-multiplets is usually defined by

$$\nu_{n,l,m} - \nu_{n,l,0} = L \sum_{i=1}^{N} a_{i,n,l} P_i\left(\frac{m}{L}\right) = \sum_{i=1}^{N} \alpha_{i,n,l} P_i\left(\frac{m}{L}\right), \tag{120}$$

where $\nu_{n,l,m}$ is the frequency of the oscillation, P_i is a Legendre polynomial, and the $a_{i,n,l}$'s are the splitting coefficients. The symmetric splitting coefficients are sometimes defined as $\alpha_{i,n,l}$'s because they tend to be dominated by near-surface effects. N is 5 or 6 and $L = l$ or $\Lambda^{1/2}$ depending on the choice of the observer. In the existing helioseismic data, the $a_{i,n,l}$'s are usually presented in nanohertz (nHz) with any effect on them of less than 1 nHz being negligible (not detectable). The antisymmetric part of the data arises from the linear effect of rotation discussed in § 3.1. The symmetric part of the data must contain a contribution from the second-order effect of rotation which was discussed in § 5. The fact that this expansion for the fine structure works tells us that to the limit we know it, the Sun rotates on a single axis and any perturbation which is visible in the data has the rotation axis as its axis of symmetry.

Using the antisymmetric part of the 1986 data of Libbrecht & Woodard (1990 and private communication) and inverting it with the method used by Dziembowski, Goode, & Libbrecht (1989), we calculated the solar rotation law shown in Figure 1. The ordinate parameters are defined in equation (20). Each component, $\eta_s(r)$, is assumed to be constant beneath 0.31 of the solar radius reflecting our lack of seismic information about the rotation at those depths. We use the rotation law of Figure 1 in all subsequent calculations calling for the Sun's internal differential rotation.

The theory for the calculation of the second-order effect of rotation was developed in § 5. There all terms in ω_2 were found to be polynomials in m^2. It is straightforward to convert them to a series in P_{2k} using the $T_{j,k}$-coefficients defined in equation (71).

To obtain a crude sense of the scale of the second-order effect of rotation, we note that the coefficient of equation (90) or (92) is $\bar{\Omega}^2/\omega_0 \sim 0.1$ nHz. This small number is further reduced when we realize we need to divide it by l to compare it to the observed a_i-coefficients. However, ω_2^D contains the large factor σ^2 (order 100–1000 for the observed mode frequencies) while ω_1^2/ω_0, ω_2^I, and ω_2^P have the factor m^2 which could make this term measurable for high-l. The ω_2^T term is the only one in equation (15) which does not contain terms in σ^2 or m^2 and is everywhere negligible.

Gough & Thompson (1990) calculated the second-order effect of rotation assuming a specific spherical rotation law for the Sun, and they found that the effect of distortion dominated. Also, they wondered whether the aforementioned terms containing an m^2 factor could be important for higher l values than they considered.

7.1. *Distortion*

For high-frequency p-modes that are the 5-minute solar oscillations, equation (90) may be simplified. We may ignore quantities like yz and z^2 compared to y^2, unless they are multiplied by Λ or $V_g C\sigma^2 = \omega^2 r^2/c^2$ because for low l-values, the motion is nearly radial, and for large l-values, we have $\Lambda \gg 1$. Further, we ignore the effect of perturbing the gravitational potential (v and $w = 0$). We

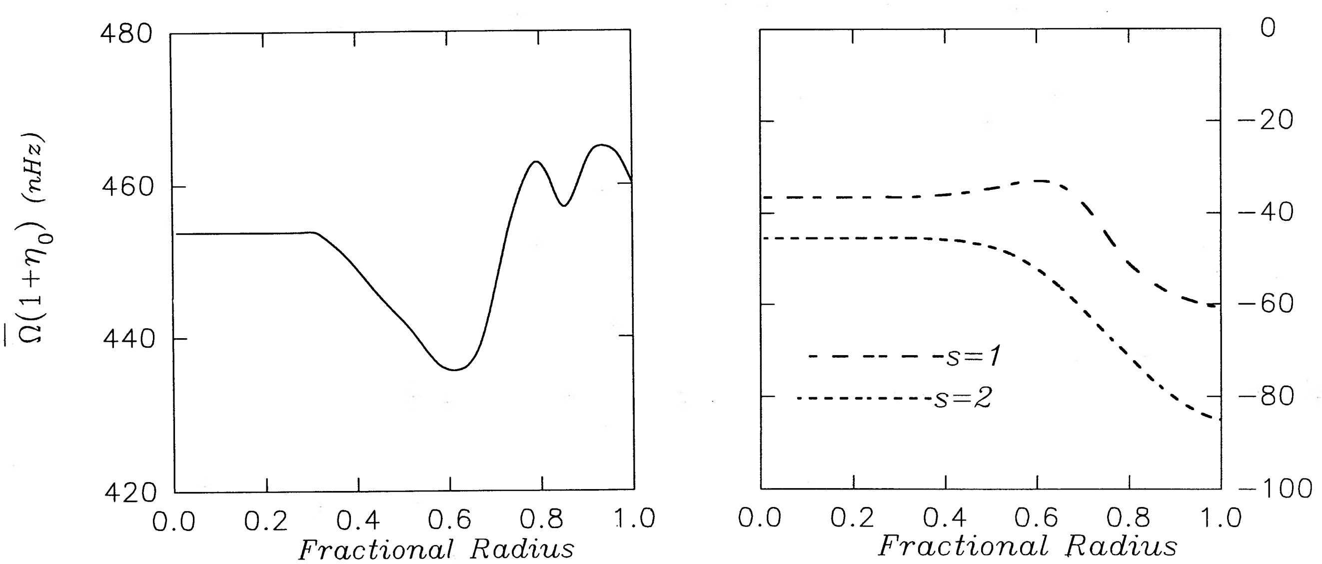

FIG. 1*a* FIG. 1*b*

FIG. 1.—The Sun's internal rotation law calculated from the data of Libbrecht & Woodard (1990, and private communication) and employing the method from Dziembowski et al. (1989). The rotation law is expressed in terms of eq. (20).

then obtain

$$\omega_2^D \approx \frac{\bar{\Omega}^2}{\omega_0}\sigma^2 \sum \tilde{\mathscr{Q}}_k \int C\left\{y^2\left[r\frac{du_k}{dr} + (4-U)u_k - \frac{1}{2}b_k - \frac{1}{2C\sigma^2}r\frac{db_k}{dr}\right] + z^2\Lambda u_k + yzr^2\frac{d^2c_k}{dr^2} + \frac{1}{2}V_g C\sigma^2 \left.\frac{\partial \ln \Gamma}{\partial \ln \rho}\right|_p b_k\right\}\rho r^4\, dr\,. \quad (121)$$

We note that the integral is of the order of unity. In addition to the aforementioned leading terms, we retain terms involving the second derivative of $\tilde{\eta}_k$ for reasons that will become clear in § 7.2. Most of the contribution to ω_2^D comes from the outer layers where one may additionally assume that $C \approx 3(r/R)^3$, $u_k \approx c_k$, and $U \approx 0$.

In Figure 2 we show α_2 and α_4 as a function of l arising from the effect of distortion as given in equation (90). In these calculations, we used the rotation law in Figure 1. The l ranges shown are between 1 and 50 and between 50 and 500. Not surprisingly, distortion causes a much larger quadrupole than hexadecupole distortion. Higher order terms, like a_6's, are negligible like the a_4's. The apparent discontinuities in Figure 2 arise because we include no mode below 1.5 mHz and no mode above 4.5 mHz. The α's are fairly independent of degree with a slight tendency to increase with l. The tendency arises because the higher l modes are more confined to the outer layers where the effect is larger. Similarly, the slight departure from an almost linear increase of the α's with σ occurs because the higher frequency modes are also more confined to the surface region.

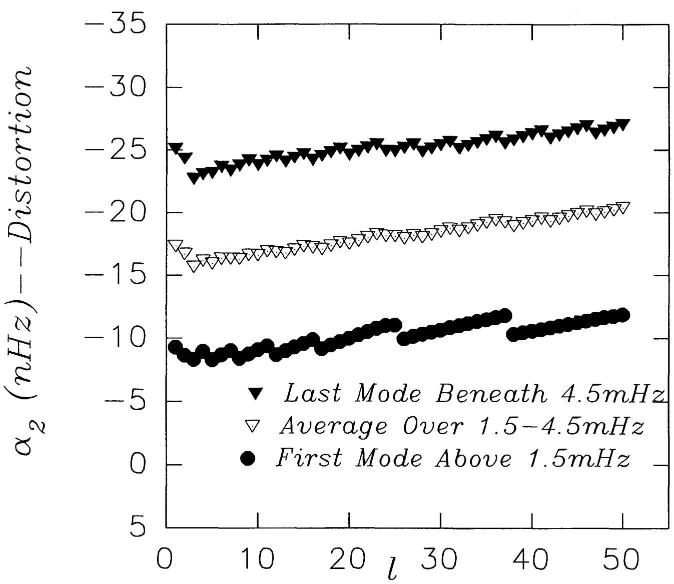

FIG. 2a

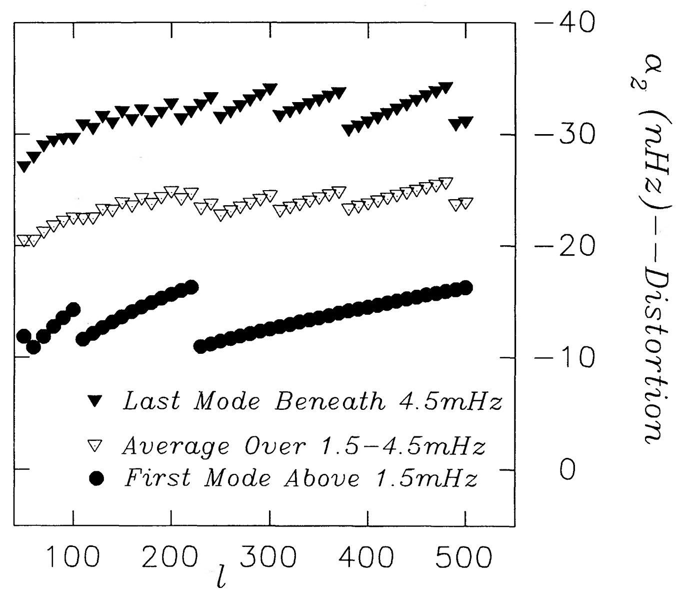

FIG. 2b

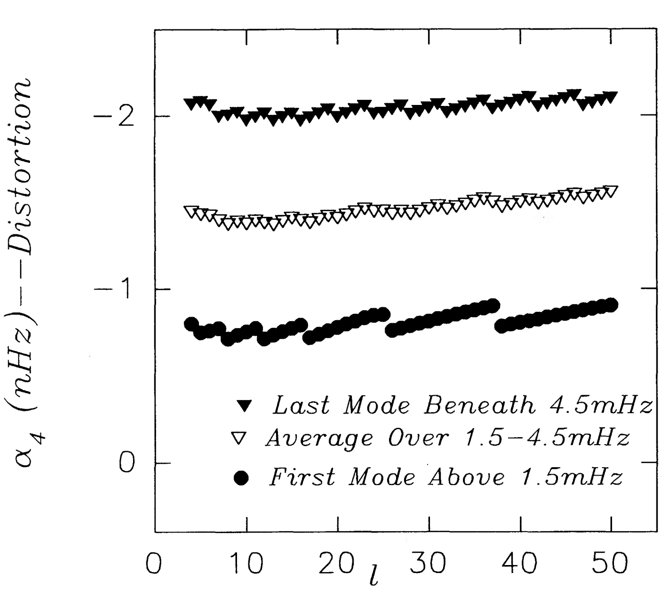

FIG. 2c

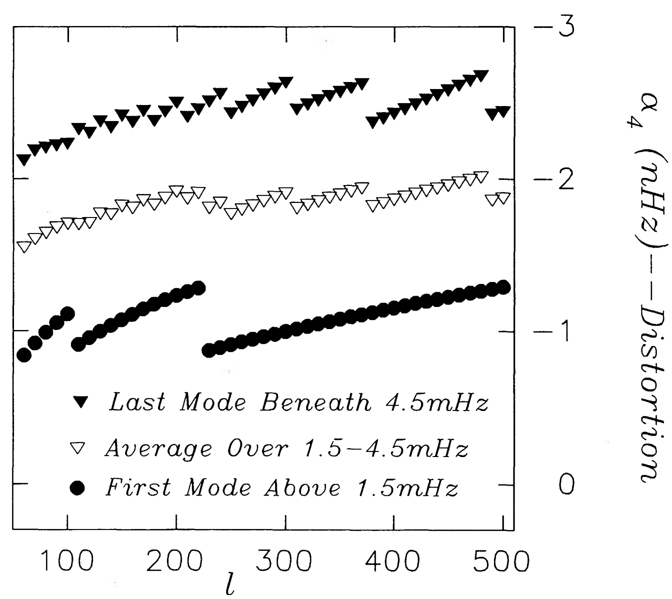

FIG. 2d

FIG. 2.—The calculated effect of distortion on (a) α_2 for l from 1 to 50 and (b) l from 50 to 500 for every tenth l (c) and (d) α_4 for the same ranges as in (a) and (b), respectively.

7.2. *Effect of Steep Gradient in Rotation in the Outer Layers*

We cannot completely trust the $\eta_s(r)$ in the outermost layers where the disproportionate part of the contribution to the second-order effect of rotation arises. That is, the available helioseismic data do not contain sufficiently high l-values. This implies that a steep change in rotation there could have a significant effect (Goode & Dziembowski 1983). We note that Libbrecht & Woodard (1990) report large symmetric α-coefficients for the summer of 1988. Could large coefficients arise from a sharp change in rotation rate near the surface? We check this by considering α_6 so that we need only consider $\tilde{\eta}_2 = 2\eta_2$, and we write $\Omega_2 \equiv \bar{\Omega}\eta_2$. We use here the expression for ω_2 given in equation (121) to evaluate

$$\frac{d\Omega_2}{dr} = -\frac{\Delta\Omega_2}{RD} d(1-d) \text{ for } r_{\text{phot}} - D \le r \le r_{\text{phot}} \text{ and 0 outside this range}, \tag{122}$$

where $d = (r_{\text{phot}} - r)/D$. Then we express $u_k \approx c_k$ and b_k in terms of $\tilde{\eta}$ employing equations (74) and (75). Finally, equations (31) and (88) are used to evaluate $\tilde{\mathscr{Q}}_k$, and equation (74) is used to convert m^6 to $P_6(m/L)$. We then have

$$\frac{\alpha_6}{\Delta\Omega} \approx \frac{-15}{1386}\frac{R}{D}\frac{\bar{\Omega}}{\omega_0}\sigma^2 \int_{r_{\text{phot}}-D}^{r_{\text{phot}}} \left[d(1-d)\left(y^2 - z^2 V_g C\sigma^2 \left.\frac{\partial \ln \Gamma}{\partial \ln \rho}\right|_p \right) + \frac{R}{D}(1-2d)y\left(z - \frac{y}{C\sigma^2} \right) \right] \rho r^4\, dr\,. \tag{123}$$

We see from Figure 3 that an abrupt change in the rate near the surface makes a negligible contribution to α_6 even generously assuming that $\Delta\Omega_2$ is 100 nHz which is above the maximum the splitting data will allow. This miniscule result is a bit of a surprise because the order-of-magnitude estimate of the coefficient on the right-hand side of equation (123) is unity, rather than the actual value of 15/1386.

7.3. *Inertia*

Equation (92) for ω_2^I contains the factor m^2, and therefore one might anticipate that this term could be large for high l-values. However, for the case of uniform rotation and high-frequency p-modes, there is a nearly exact cancellation in the calculation of ω_2 between the first and third terms on the right-hand side of equation (15). This can be seen by inspecting equations (25) and (92). In this regard, we emphasize that $C_L \sim 10^{-2}$ to 10^{-3} for solar oscillations. The Sun exhibits a very roughly uniform rotation so we must consider together the aforementioned two terms in equation (15). Still, before calculation the degree of cancellation between the two terms is not clear. For the solar case, their sum is well approximated, to leading order, by

$$\omega_2^I + \frac{\omega_1^2}{2\omega_0} \approx \frac{m^2\bar{\Omega}^2}{2\omega_0}\left[\left(\sum_s \mathscr{Q}_s \langle \eta_s \rangle \right)^2 - \sum_s \mathscr{Q}_s \langle \tilde{\eta}_s \rangle \right], \tag{124}$$

where, for instance,

$$\langle \tilde{\eta}_s \rangle = \int \tilde{\eta}_s (y^2 + \Lambda z^2) d^3\boldsymbol{x}\,. \tag{125}$$

In Figure 4, the α_2's from $\omega_2^I + (\omega_1^2/2\omega_0)$ are plotted. The ω_2^I's used in that calculation follow from equation (92). The α_2's grow with

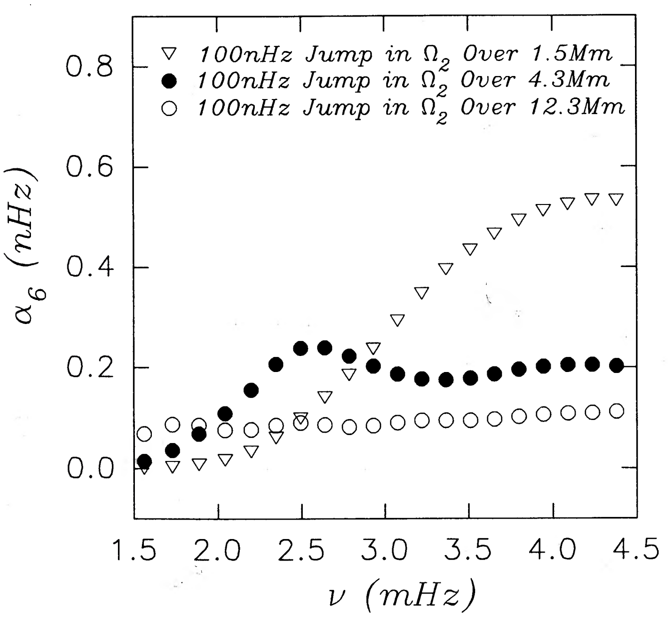

FIG. 3

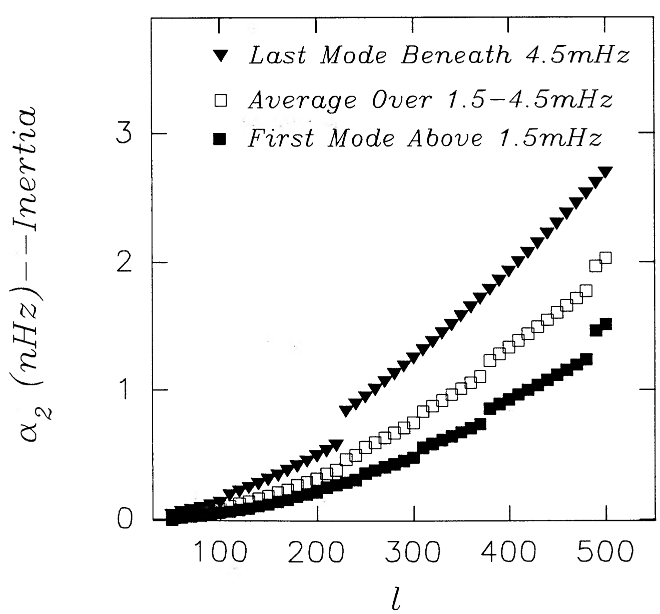

FIG. 4

FIG. 3.—The calculated values of α_6 through the 5 minute band following from a 100 nHz jump in Ω_2 spread over three ranges in depth

FIG. 4.—The calculated effect of inertial terms, $\omega_2^I + (\omega_1^2/2\omega_0)$, from eqs. (92) and (25) shown as α_2 vs. l

l, as could have been anticipated from equation (104), but, perhaps surprisingly, are still considerably smaller than expected from order-of-magnitude estimates. This is largely due to the cancellation between terms in the brackets of equation (124). The corresponding α_4's and α_6's behave very much like the α_2's.

7.4. *Poloidal Mode Effects and Accidental Degeneracies*

From examining equation (64), one may anticipate that ω_2^P is exceptionally large. Near-degeneracies occur in the spectrum of 5-minute solar oscillations for the pairs (n, l) and $(n-1, l+4)$ if $n = l+1$ or $l+2$. In Table 1, we list the higher l-value member of each pair present in the 1986 Big Bear data, Libbrecht & Woodard (1990 and private communication), which have an observed frequency difference of less than 1 mHz. All pairs, except the last one in the table, are of the first type ($n = l + 1$).

The a-coefficients in the table were obtained numerically—using equation (64) and assuming the observed frequency difference, $\nu_{l+4} - \nu_l$. We quote only values of the a's for the higher l-value of each pair. The other member of each pair has the opposite sign and is somewhat smaller in magnitude. From the table, we see that the observed frequency difference is always smaller than the calculated one. The far resonance limit of equation (64) is justified even for the closest resonance pair. It is clear from the table that the effect on the a_k is not significant even in the case of the closest near resonance.

We cannot exclude the possibility that in reality the near resonance is closer than implied by the table. With this in mind, we show in Figure 5 the frequency difference, $\Delta\nu$, between the modes of azimuthal order m and 0 for $l = 20$ and $n = 21$. For this multiplet, the observed frequency difference between its centroid and the $l = 24$ partner is only -0.11 mHz—the smallest difference in the table. The frequency differences, $\Delta\nu$ and the B's, are those of equation (58). The calculations were performed for $\nu_{l+4} - \nu_l = 0$ and -0.11 mHz. To roughly compare $\Delta\nu$ with the a's, we need to divide $\Delta\nu$ by the l-value. From this we see that even in the case of an exact resonance, the largest frequency shifts are about at our 1 nHz limit of observability. On the other hand, Figure 5 makes it clear that the mixing of modes is quite different depending on whether there is a near resonance or an exact resonance. In the long run, measuring this mixing would provide an interesting diagnostic.

One might also expect a significant contribution to the even a's from ω_2^P for very large l-values. This is due not only to the aforementioned m^2 factor, but also because $\omega_{n,l} - \omega_{n,l\pm2}$ and $\omega_{n,l} - \omega_{n,l\pm4}$ are decreasing with l. Woodard (1989) studied the observational consequences of the eigenfunction perturbation in such cases. He used the eigenfunction expansion given here in equation (38). In our calculation of ω_2^P, we used equation (99) and obtained the $\tilde{y}_j$ and $\tilde{z}_j$ by solving equations (45) and (46). The result is that even for $l = 500$, the largest effect is for a_4 and is less than 0.2 nHz. Thus, the effect is negligible.

7.5. *Comparison with Data*

Distortion causes the only detectable second-order effect of rotation in existing solar oscillation data. Further, only the a_2-coefficients are large enough to be detected. This signature is clearly visible as the only trend in the 1986 data of Libbrecht & Woodard taken at solar minimum. In Figure 6, we show the full second-order effect of rotation compared to the 1986 and 1988 Big Bear datasets. The nearly constant offsets between the 1986 and 1988 data sets are presumably due to magnetic field effects associated with activity (Libbrecht & Woodard 1990; Woodard et al. 1991).

8. RESULTS FOR δ SCUTI STARS

In the interpretation of the spectra of δ Scuti stars, the formula,

$$\omega = (C_L - 1)m\Omega + D_L \frac{m^2\Omega^2}{\omega_0}, \tag{126}$$

is commonly used to describe effects of rotation through second order. The C_L- and D_L-coefficients were calculated by Saio (1981) who employed a polytrope of index 3. This formula is valid for rigid rotation. We used equation (90), assuming rigid rotation, in our calculations of D_L. We shall consider the validity of equation (126) for realistic models, examining the frequency range which may be

TABLE 1

POLOIDAL MODE SPLITTING COEFFICIENTS FROM ACCIDENTAL NEAR DEGENERACIES IN FREQUENCY (nHz)

$\Delta\nu_{obs}$	$\Delta\nu_{cal}$	l	n	a_2	a_4	a_6
0.47.........	0.52	11	7	−0.01	−0.00	0.01
0.43.........	0.91	12	8	−0.01	−0.00	0.02
0.46.........	1.09	13	9	−0.01	−0.00	0.02
0.45.........	1.22	14	10	−0.01	−0.01	0.02
0.60.........	1.25	15	11	−0.01	−0.01	0.02
0.65.........	1.24	16	12	−0.01	−0.01	0.02
0.66.........	1.23	17	13	−0.01	−0.01	0.02
0.52.........	1.22	18	14	−0.01	−0.01	0.02
0.57.........	1.21	19	15	−0.01	−0.01	0.02
0.56.........	1.18	20	16	−0.01	−0.01	0.03
0.58.........	1.12	21	17	−0.01	−0.01	0.03
0.45.........	1.09	22	18	−0.02	−0.02	0.04
0.28.........	1.10	23	19	−0.03	−0.03	0.06
−0.11.........	1.09	24	20	0.08	0.08	−0.16
−0.59.........	−0.96	24	21	0.02	0.01	−0.03

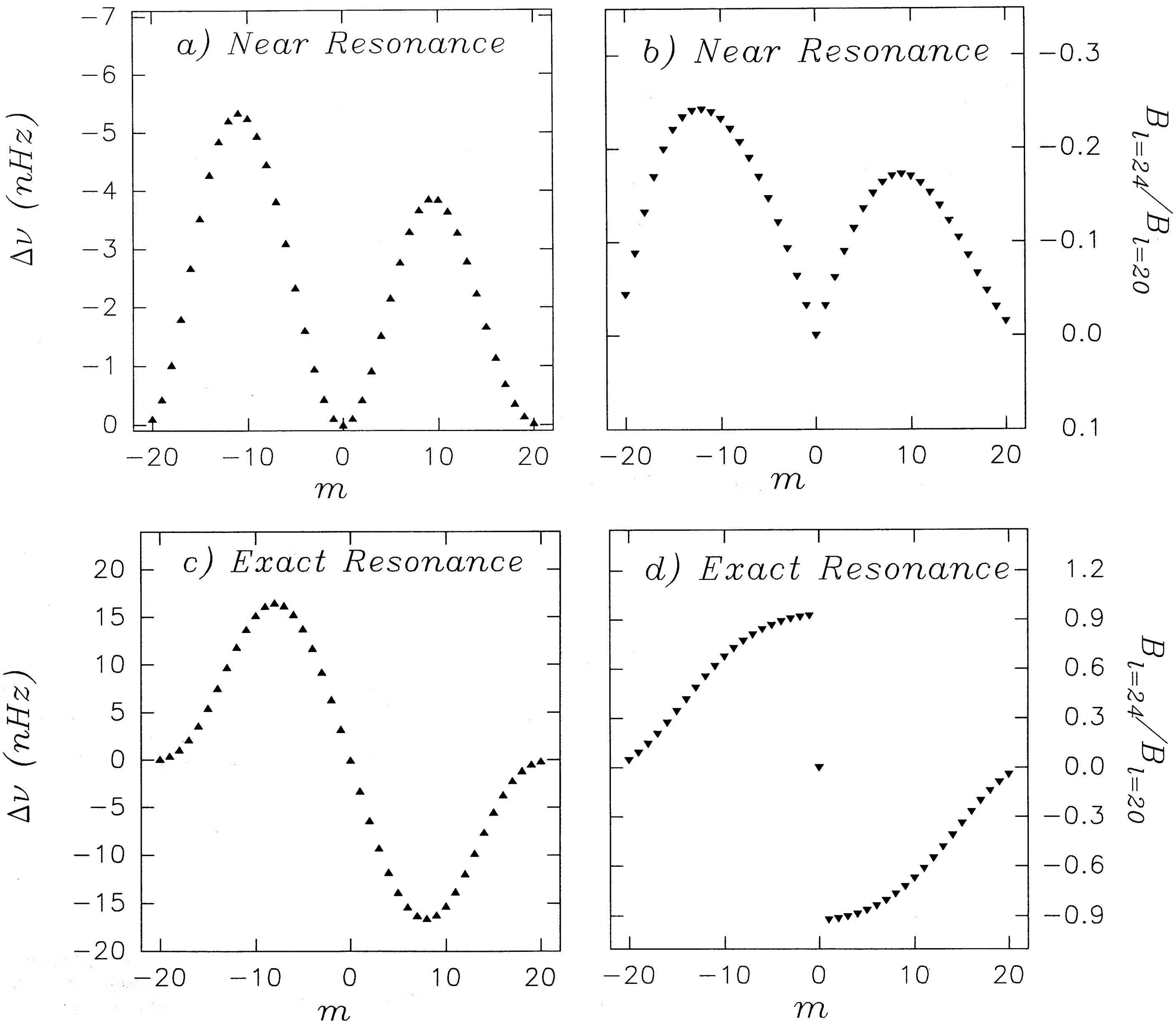

FIG. 5.—A comparison of mode mixing for the most nearly degenerate pair in Table 1. For the near resonance (*a*) the frequency change within the $l = 20$, $n = 21$ multiplet as a function of *m* and (*b*) the amount of mixing of the $l = 24$ partner. Assuming an exact resonance, (*c*) and (*d*) correspond to (*a*) and (*b*), respectively.

excited in δ Scuti stars. In Table 2, we compare the C_L- and D_L-coefficients from the polytrope with those from two realistic models. Our polytropic model coefficients, C_L and D_L, agree sufficiently well with those of Saio (1981). The solar models are for $M = 2M_\odot$. The first of the two models is essentially ZAMS having $X_c = 0.699$ and $\log T_{\rm eff} = 3.965$. X_c is the central stellar hydrogen abundance. We can see that the polytropic values agree quite well with those from this model. The second model ($X_c = 0.313$ and $\log T_{\rm eff} = 3.925$) is an evolved one from which we have two additional modes in the frequency range under consideration. These two modes have a very different character. The two modes are trapped in the vicinity of the convective core boundary, and their importance for stellar evolution theory has been recently discussed by Dziembowski & Pamyatnykh (1991). We note here that because they have such different fine-structure coefficients (see Table 2), the modes would be detectable through observed splittings.

Most well-studied δ Scuti variables are more evolved stars which have exhausted their core hydrogen. In the models of such objects, the Brunt-Väisälä frequency has a large maximum close to the center which results in a dual character for all non-radial modes in the relevant frequency range. They propagate as high-order *g*-modes in the deep interior, while remaining low-order *p*-modes in the outer layers.

In Table 3, we give C_L- and D_L-coefficients for a chosen model, in a selected range of the spectrum which is close to that of the radial p_2-mode (second overtone). For each mode, we also give the fraction of the total energy which belongs to the *g*-mode region. All these modes but one ($\sigma = 3.454$) should be regarded as being predominantly *g*-modes. We can see that the C_L-values for all these modes are close to the asymptotic ones for *g*-modes: $C_L = \frac{1}{2}$ for $l = 1$ and $C_L = \frac{1}{6}$ for $l = 2$. On the other hand, the D_L-values are far

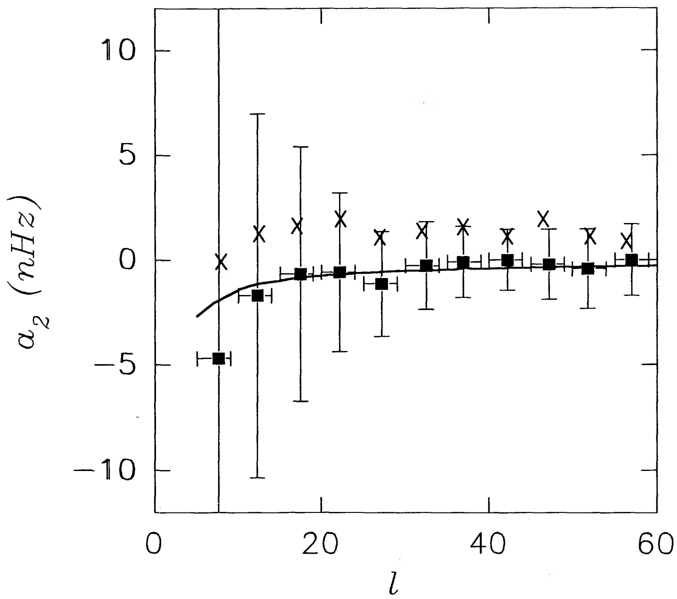

FIG. 6.—The solid boxes represent the a_2's following from binning the 1986 Big Bear data. The crosses represent their 1988 data. The solid line is the calculated a_2, including all second-order effects, binned for each l-value. This latter binning is done by weighting each mode using the errors in the 1986 Big Bear modes. The solid line would not change if the 1988 modes were used instead.

TABLE 2

FINE STRUCTURE SPLITTING COEFFICIENTS FOR δ SCUTI STARS

Mode	l	σ	C_L	D_L	Model
		1.949	0.030	−3.351	Polytrope
p_1	1	2.087	0.012	−4.005	$X_c = 0.699$
		2.083	0.015	−3.906	$X_c = 0.313$
		2.680	0.034	−7.720	Polytrope
p_2	1	2.799	0.012	−8.647	$X_c = 0.699$
		2.833	0.159	−5.737	$X_c = 0.313$
		3.411	0.034	−13.489	Polytrope
p_3	1	3.477	0.017	−14.055	$X_c = 0.699$
		3.478	0.010	−14.235	$X_c = 0.313$
		4.143	0.031	−20.624	Polytrope
p_4	1	4.148	0.021	−20.737	$X_c = 0.699$
		4.147	0.007	−20.844	$X_c = 0.313$
		1.651	0.253	−0.319	Polytrope
f.............	2	2.053	0.040	−0.682	$X_c = 0.699$
		2.252	0.054	−0.772	$X_c = 0.313$
		2.255	0.154	−1.082	Polytrope
p_1	2	2.505	0.216	−1.150	$X_c = 0.699$
		2.699	0.060	−1.319	$X_c = 0.313$
		2.984	0.082	−2.313	Polytrope
p_2	2	3.076	0.119	−2.300	$X_c = 0.699$
		3.185	0.145	−2.331	$X_c = 0.313$
		3.718	0.054	−3.842	Polytrope
p_3	2	3.738	0.064	−3.856	$X_c = 0.699$
		3.788	0.096	−3.810	$X_c = 0.313$
		4.450	0.040	−5.679	Polytrope
p_4	2	4.413	0.043	−5.564	$X_c = 0.699$
		4.477	0.090	−4.805	$X_c = 0.313$
g_c	1	2.651	0.428	−2.827	$X_c = 0.313$
g_c	2	4.325	0.139	−1.063	$X_c = 0.313$

TABLE 3

C_L AND D_L FOR MODES NEAR $p_3(l = 0)$ FROM MODES WITH $X_c = 0$ AND $\log T_{\rm eff} = 3.88$

l	σ	C_L	D_L	$\frac{I_g}{I}$
1.........	3.199	0.488	−0.110	0.958
	3.367	0.447	−1.984	0.881
	3.454	0.092	−10.834	0.205
	3.580	0.482	−0.138	0.946
	3.802	0.495	0.055	0.967
2.........	3.250	0.157	−0.424	0.888
	3.344	0.154	−0.338	0.922
	3.442	0.143	−0.428	0.902
	3.535	0.129	−0.580	0.856
	3.629	0.138	−0.545	0.862
	3.738	0.151	−0.593	0.865

from the asymptotic values. This is related to the problem of small asymptotic values for $l = 1$ as discussed in § 6. The mode which is predominantly acoustic is trapped in the envelope and this mode is of special interest. Dziembowski & Królikowska (1990) speculated that such a trapped mode, among a sea of other unstable modes, may be preferentially excited to a visible level. For this trapped mode, C_L- and D_L-coefficients are very much like the one for modes of similar frequency in Table 2 ($l = 1$, p_3).

In a few δ Scuti–type objects, notably in 4 CVn (Breger et al. 1990) and θ^2 Tau (Breger et al. 1989), closely spaced peaks are found in the periodograms. The peaks have been interpreted as a manifestation of rotational splitting. Here we would like to point out problems one may encounter with this picture, considering that the objects are relatively rapidly rotating and evolved. In particular, an overlapping of multiplets may take place, and there is a significant departure from uniform spacing. These may lead to considerable confusion in mode identification.

With this in mind, we calculated the complete spectra for $l = 1$ in a selected frequency range using the model of Table 3. The equatorial rotation velocity is 100 km s^{-1}. The model parameters are not far from the ones describing θ^2 Tau. In this calculation, equations (100) and (101) were solved. These equations are accurate to Ω^3 except for the effect of distortion. The correction due to this effect was calculated separately and added in for each m-component of a multiplet using equation (90) for rigid rotation. The results are shown in Figure 7. The frequencies here are given as would be reported by an observer in an inertial system. The ordering of the modes in each multiplet is such that the leftmost one is the prograde mode. Significant departure from equal spacing occurs only in the trapped mode multiplet. This is a consequence of the effect of distortion being large only in that multiplet (see Table 3). Each member of this multiplet remains trapped, and the trapping spreads to the closest members of adjacent multiplets. In this

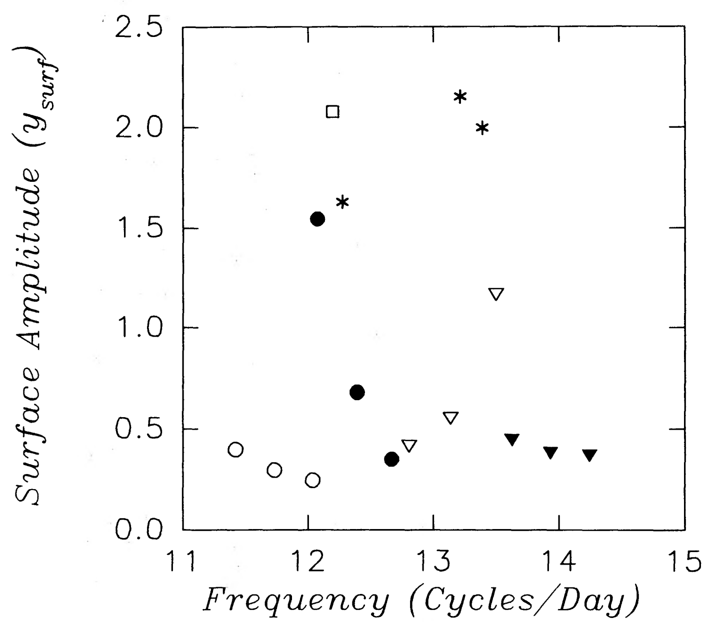

FIG. 7.—The surface amplitude in terms of the radial eigenfunction corresponding to the radial component at the surface vs. mode frequency. The modes shown are the $l = 0$ and $l = 1$ multiplets in the frequency range used in Table 3. Members of a multiplet are identified with the same symbol. The circle represents the $l = 0$ mode, and the asterisks represent the members of the trapped multiplet. The assumed surface rotational velocity is 100 km s^{-1}. In the calculation of $y_{\rm surf}$, we assume all modes have the same inertia, eq. (22). The amplitudes shown may be suggestive but should not be regarded as a prediction. However, the larger the $y_{\rm surf}$ value, the greater the trapping, and a larger value may reflect preferential excitation.

regard, the reader may be misled by the use of an inertial system in Figure 7. The transformation to the inertial frame reverses the mode ordering from that in the intrinsic frame. We emphasize that with the assumed velocity of rotation, there are significant departures from the splitting law of equation (126).

Suppose that the amplitudes in Figure 7 are real and, say, modes exceeding 1 in amplitude are detectable. Then one might well interpret the spectrum as two rotationally split triplets. If we do not have observationally determined m and l, model calculations like the ones reported here are necessary for proper mode identification.

Just as the g_c-mode in the model with $x_c = 0.313$ was distinguishable from its p-mode neighbors on the basis of its splitting characteristics, here the trapped mode is distinguishable from its g-mode neighbors. This is important because it will allow us to test the hypothesis of preferential excitation of the trapped mode.

P. R. G. is partially supported by NSF-Astronomy and SERC. P. R. G. is on leave from the New Jersey Institute of Technology. Part of this work was done when W. A. D. was a visitor in the Institute of Astronomy, Cambridge University. He gratefully acknowledges their hospitality and support. We thank R. Sienkiewicz for the solar model and A. A. Pamyatnykh for the δ Scuti star models.

REFERENCES

Brassard, P., Wesemael, F., & Fontaine, G. 1989, in IAU Colloq. 114, White Dwarfs, ed. G. Wegner (Berlin: Springer), 258
Breger, M., Garrido, R., Huang, L., Jiang, S.-Y., Guo, Z.-H., Fruch, M., & Paparo, M. 1989, A&A, 214, 209
Breger, M., McNamara, B. J., Kershbaum, F., Huang, L., Jiang, S.-Y., Guo, Z.-H., & Poretti, E. 1990, A&A, 231, 56
Chlebowski, T. 1978, Acta Astron., 28, 441
Dziembowski, W. A. 1990, in Progress of Seismology of the Sun and Stars, ed. Y. Osaki & H. Shibahashi (Berlin: Springer), 359
Dziembowski, W. A., & Goode, P. R. 1984, Mem. Soc. Astron. Ital., 55, 185
———. 1989, ApJ, 347, 540
———. 1991, ApJ, 376, 782
Dziembowski, W. A., Goode, P. R., & Libbrecht, K. G. 1989, ApJ, 337, L53
Dziembowski, W., & Królikowska, M. 1990, Acta Astron., 40, 19
Dziembowski, W. A., & Pamyatnykh, A. A. 1991, A&A, 248, L11
Goode, P. R., & Dziembowski, W. A. 1984, in Solar Seismology from Space, ed. R. K. Ulrich (JPL Publ. 84-84) (Pasadena: JPL), 351
Gough, D. O., & Taylor, P. P. 1984, Mem. Soc. Astron. Ital., 55, 215
Gough, D. O., & Thompson, M. J. 1990, MNRAS, 242, 25
Hansen, C. J., Cox, J. P., & Carroll, B. W. 1978, ApJ, 226, 210
Libbrecht, K. G., & Woodward, M. F. 1990, Nature, 345, 779
McGraw, J. T. 1977, ApJ, 214, L123
McGraw, J. T., & Robinson, E. L. 1975, ApJ, 200, L89
Saio, H. 1981, ApJ, 244, 299
Unno, W., Osaki, Y., Ando, H., Saio, H., & Shibahashi, H. 1989, Nonradial Oscillations of Stars (Tokyo: Univ. of Tokyo Press)
Winget, D., et al. 1991, ApJ, 378, 326
Woodward, M. F. 1989, ApJ, 347, 1176
Woodward, M. F., Kuhn, J. R., Murray, N., & Libbrecht, K. G. 1991, ApJ, 373, L81