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GRAVITATIONAL LENSING, TIME DELAY, AND GAMMA-RAY BURSTS

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ABSTRACT

The probability distributions of time delay in gravitational lensing by point masses and isolated galaxies (modeled as singular isothermal spheres) are studied. We find that for point lenses (all with the same mass M_*), the probability distribution is broad, and with a peak at $\Delta t \approx 50$ s $(M_*/10^6\ M_\odot)$; for singular isothermal spheres, the probability distribution is a rapidly decreasing function with increasing time delay, with a median $\Delta t \approx 1\ h^{-1}$ month, and its behavior depends sensitively on the luminosity function of galaxies. The present simplified calculation is particularly relevant to the gamma-ray bursts if they are of cosmological origin. The frequency of "recurrent" bursts due to gravitational lensing by galaxies is probably between 0.05% and 0.4%. BATSE on *Gamma Ray Observatory* may observe several such lensing events. BATSE may also provide the most stringent limit on the mass density of cosmological massive black holes. Gravitational lensing therefore can be used as a test of the cosmological origin of gamma-ray bursts.

Subject headings: gamma rays: bursts — gravitational lensing

1. INTRODUCTION

Turner, Ostriker, & Gott (1984, hereafter TOG) developed a formalism to compute analytically the gravitational lensing probability and the angular image separation distribution for different mass distributions. Since TOG, much progress has been made (e.g., Fukugita et al. 1992, and references therein), but so far most studies have calculated only dimensionless quantities such as image angular separation, lens redshift, and differential (total) lensing probabilities. Time delay between different images, a dimensional quantity which was first proposed by Refsdal (1964, 1966) to determine the Hubble constant H_0, has not been studied statistically.

Gamma-ray bursts (GRBs) remain enigmatic almost 20 yr after the discovery paper by Klebesadel, Strong, & Olson (1973); even their distances are still highly controversial (e.g., Higdon & Lingenfelter 1990; Harding 1991; Paczyński 1992). Recent results from BATSE on *Gamma Ray Observatory* (*GRO*) provided new insights into these problems (Meegan et al. 1992). The BATSE results are very difficult to explain using the galactic disk and halo models (Mao & Pacyński 1992a), while a simple cosmological model with no adjustable parameters explains the BATSE results surprisingly well (Mao & Pacyński 1992b). Cosmological origin of GRBs has been put forward by many authors (Paczyński 1986, 1988; Goodman 1986; Babul, Paczyński, & Spergel 1987, and references therein). If the GRBs are cosmological, then they should be gravitationally lensed just as quasars are (Paczyński 1986, 1987). The gamma-ray detectors on BATSE have poor angular resolution, but they have excellent time resolutions, so the lensed bursts can be verified only by studying their temporal behaviors. The following question naturally rises: what is the time interval (time delay) between different images caused by gravitational lensing? As mentioned above, the answer to the question is not yet provided in the literature; this *Letter* is the first attempt to study quantitatively the statistics of time delays.

2. MODELS

We follow the treatment and notations of TOG. We study two types of mass distributions: point masses and singular isothermal galaxies. Point masses are an appropriate model for lensing by stars or any cosmological distributed compact sources such as primordial black holes. Singular isothermal spheres reasonably approximate the mass distributions of isolated galaxies. Treatments will be given in turn.

2.1. *Point Masses*

For lensing by a point mass, M_*, the time delay is given by

$$t = \frac{1}{c}(1 + z_L)\left[\frac{(\mathbf{r} - \mathbf{r}_s)^2}{2D} - \frac{4GM_*}{c^2}\ln r\right], \qquad (1)$$

where z_L is the lens (point mass) redshift, $\mathbf{r}$ is the position where a virtual light ray intercepts the image plane, $\mathbf{r}_s$ is the position of the source projected to the image plane, and $D = D_{\rm OL} D_{\rm LS}/D_{\rm OS}$ is the effective distance with $D_{\rm OL}$, $D_{\rm LS}$, and $D_{\rm OS}$ being, respectively, the lens, lens-source, and source angular diameter distances.

Using Fermat's principle (Schneider 1985; Blandford & Narayan 1986), the actual image positions $\mathbf{r}_i$ are the virtual light rays which extremize the time delay ($\nabla_{\mathbf{r}} t = 0$). For a point mass lens, Fermat's principle always gives two images; their positions and brightness ratio are

$$\mathbf{r}_{i,\pm} = \frac{r_s \pm \sqrt{r_s^2 + 4r_E^2}}{2}\,\frac{\mathbf{r}_s}{r_s}, \quad \frac{I_+}{I_-} = \left(\frac{\sqrt{f^2 + 4} + f}{\sqrt{f^2 + 4} - f}\right)^2, \qquad (2)$$

where $f = r_s/r_E$, $r_E = (4GM_* D/c^2)^{1/2}$ is the Einstein radius. The time delay between these two images can be readily obtained using equation (1):

$$\Delta t = t(\mathbf{r}_{i,-}) - t(\mathbf{r}_{i,+})$$
$$= \Delta t_0(1 + z_L)\left[\frac{1}{2} f\sqrt{f^2 + 4} + \ln\frac{\sqrt{f^2 + 4} + f}{\sqrt{f^2 + 4} - f}\right], \qquad (3)$$

where $\Delta t_0 = 4GM_*/c^3 \approx 2 \times 10^{-5} M_*/M_\odot$ (s).

The differential probability $d\tau(r_s, z_L)$ of a beam having a lensing event in transversing a shell $r_s \rightarrow r_s + dr_s$ at redshift interval $z_L \rightarrow z_L + dz_L$ is (cf. TOG, eq. [2.9])

$$d\tau(r_s, z_L) = n_L(z_L)(2\pi r_s\, dr_s)\frac{c\, dt}{dz_L}\, dz_L\,. \qquad (4)$$

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We choose the effective cross section for lensing to be $\pi r_{\rm cr}^2$, where the "critical" radius is $r_{\rm cr} = r_E$ ($f = 1$), inside of which significant magnification occurs and the brightness ratio of the two images is not too large (cf. eq. [2]; $I_+/I_- \approx 6.85$ for $f = 1$). This choice of "critical" radius is in acordance with TOG. If the comoving density of gravitational deflectors is a constant [i.e., $n_L(z_L) = n_L(0)(1 + z_L)^3$], then after substituting in all the cosmological terms (TOG, Table 1), we obtain (cf. TOG, eq. [2.13])

$$d\tau(f, x) = \frac{3}{4}\,\Omega_L\,\frac{(y^2 - x^2)(x^2 - 1)}{y^2 - 1}\,\frac{dx}{x^2}\,(2f df)\,,\, f \leq 1\,, \quad \text{for} \quad \Omega_0 = 0\,, \tag{5.1}$$

and

$$d\tau(f, x) = \frac{3}{5}\,\Omega_L\,\frac{(y^{5/2} - x^{5/2})(x^{5/2} - 1)}{y^{5/2} - 1}\,\frac{dx}{x^{7/2}}\,(2f df)\,,\, f \leq 1\,, \quad \text{for} \quad \Omega_0 = 1\,, \tag{5.2}$$

where

$$x \equiv 1 + z_L\,, \quad y \equiv 1 + z_s \tag{5.3}$$

and

$$\Omega_L \equiv \left(\frac{\rho_L}{\rho_{\rm crit}}\right)_0 = \frac{8\pi G M_* n_L(0)}{3H_0^2}\,, \tag{5.4}$$

is the ratio of the local density in point masses to the critical density. The probability distribution of time delay can be obtained by using equations (3) and (5.1)–(5.3). The differential probability distributions are shown in Figure 1 for four representative source redshifts, $z_s = 3, 2, 1, 0.5$. All the distributions peak around $\Delta t \approx 2.5\,\Delta t_0$, and approach 0 at $\Delta t = 0$ and $\Delta t = \Delta t_{\rm max} \approx 2.08(1 + z_s)\Delta t_0$. $\Delta t_{\rm max}$ is determined by equation (3) and our choice of $r_{\rm cr}(= r_E)$.

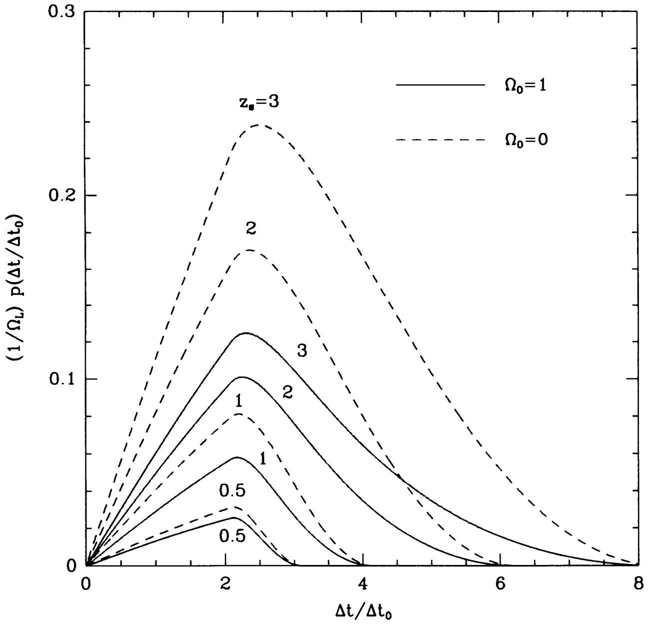

FIG. 1.—The differential probability distribution of time delay due to lensing by point masses (all with mass M_*, and having density Ω_L in units of the critical density) in open ($\Omega_0 = 0$, *dashed lines*) and flat ($\Omega_0 = 1$, *solid lines*) cosmologies for four point source redshifts, $z_s = 3, 2, 1, 0.5$ (highest to lowest curves, respectively, for each cosmology). The time delay is measured in units of $\Delta t_0 = 4GM_*/c^3 \approx 2 \times 10^{-5} M/M_\odot$ (s).

2.2. *Singular Isothermal Galaxies*

The time delay for a singular isothermal galaxy is given by

$$t = \frac{1}{c}\,(1 + z_L)\left[\frac{(\boldsymbol{r} - \boldsymbol{r}_s)^2}{2D} - \alpha r\right], \quad \alpha \equiv 4\pi\left(\frac{\sigma}{c}\right)^2\,, \tag{6}$$

where z_L is the redshift of the lensing galaxy, α is the deflection angle, σ is the velocity dispersion of the galaxy, and all other symbols are as defined in equation (1). Again using Fermat's principle, we find that there are two images when $r_s \leq r_{\rm cr} \equiv D\alpha$, the image positions and the brightness ratio are

$$\boldsymbol{r}_{i,\pm} = (r_s \pm D\alpha)\,\frac{\boldsymbol{r}_s}{r_s}\,, \quad \frac{I_+}{I_-} = \frac{D\alpha + r_s}{D\alpha - r_s}\,. \tag{7}$$

The time delay between the images is

$$\Delta t = t(\boldsymbol{r}_{i,-}) - t(\boldsymbol{r}_{i,+}) = 2(1 + z_L)\alpha\,\frac{r_s}{c}\,. \tag{8}$$

Clearly r_s scales as h^{-1}; therefore, the time delay also scales as h^{-1}, here h is the Hubble constant H_0 in units of 100 km s^{-1} Mpc^{-1}. The brightness ratio of the two images can also be expressed as a function of time delay Δt:

$$\frac{I_+}{I_-} = \frac{\Delta t_{\rm max} + \Delta t}{\Delta t_{\rm max} - \Delta t}\,, \tag{9}$$

where $\Delta t_{\rm max} = 2(1 + z_L)\alpha r_{\rm cr}/c$.

The differential probability $d\tau(r_s, z_L, L)$ of a beam having a lensing event in transversing a shell $r_s \to r_s + dr_s$ at redshift interval $z_L \to z_L + dz_L$ by a galaxy with luminosity between $L \to L + dL$ is (cf. TOG, eq. [2.26])

$$d\tau(r_s, z_L, L) = \phi(L)dL\,(2\pi r_s\,dr_s)\,\frac{c\,dt}{dz_L}\,dz_L\,, \tag{10}$$

where $\phi(L)dL$ is the luminosity function of galaxies. We adopt a Schechter luminosity function

$$\phi(L)dL = \phi^* e^{-L/L^*}\left(\frac{L}{L^*}\right)^\beta \frac{dL}{L^*} \tag{11}$$

and relate galaxy luminosity to velocity dispersion with the Faber-Jackson or the Tully-Fisher relation:

$$\frac{L}{L^*} = \left(\frac{\sigma}{\sigma_*}\right)^n\,, \tag{12}$$

where σ_* is the velocity dispersion associated with an L^* galaxy. Combining equations (10)–(12) and substituting in all the cosmological terms then lead to

$$d\tau(f, x, l) = \frac{1}{4}\,F\,\frac{(x^2 - 1)^2}{x^5}\left(\frac{y^2 - x^2}{y^2 - 1}\right)^2 dx\,(2f df)\,(e^{-l} l^\beta\,dl)\,, \quad f \leq 1\,, \tag{13.1}$$

for $\Omega_0 = 0$, and

$$d\tau(f, x, l) = 4F\,\frac{(x^{1/2} - 1)^2}{x^{7/2}}\left(\frac{y^{1/2} - x^{1/2}}{y^{1/2} - 1}\right)^2 dx\,(2f df)\,(e^{-l} l^\beta\,dl)\,, \quad f \leq 1\,, \tag{13.2}$$

for $\Omega_0 = 1$ (filled beam), where $F = 16\pi^3\phi^* R_0^3(\sigma/c)^4$, $f = r_s/r_{\rm cr}$, $l = L/L^*$, $R_0 = c/H_0$, and x, y are as defined in equation (5.3). The parameter values are adopted from Fukugita & Turner (1991): $n = 4$, $\sigma_* = 276$ km s^{-1}, $\beta = -1.1$ for elliptical gal-

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axies; $n = 4$, $\sigma_* = 252$ km s^{-1}, $\beta = -1.1$ for S0's; and $n = 2.6$, $\sigma_* = 134$ km s^{-1}, $\beta = -1.1$ for spiral galaxies. $\phi^* = 1.56 \times 10^{-2} h^3$ Mpc^{-3}, and the morphological composition is taken to be E:S0:S = 12:19:69.

Integrating equations (13.1) and (13.2) gives the total lensing probabilities (TOG; Fukugita & Turner 1991)

$$\tau(y) = \frac{F_l}{4(y^2-1)^2}\left[(y^4+4y^2+1)\ln y - \frac{3}{2}(y^4-1)\right], \quad \Omega_0 = 0 , \quad (14.1)$$

and

$$\tau(y) = \frac{4F_l}{15}(1-y^{-1/2})^3 , \quad \Omega_0 = 1 \text{ (filled beam)} , \quad (14.2)$$

where $y \equiv 1 + z_s$ and $F_l = 0.019$, 0.021, 0.0048 for elliptical galaxies, S0's, and spiral galaxies respectively, each with approximately 40% uncertainty. Notice that $d\tau$, τ are dimensionless; therefore, they are independent of the Hubble constant H_0.

Equations (8) and (13.1)–(13.2) can be used to obtain the probability distribution of time delay. Differential probability distributions, which combine all the galaxy types, are shown in Figure 2 for four source redshifts, $z_s = 3, 2, 1, 0.5$. It is apparent that all the distributions decrease rapidly with increasing time delay; the median time delay is $\Delta t \approx 1\ h^{-1}$ month. The small time delays are produced mostly by a diverging number of faint (small velocity dispersion) galaxies (but with cross section remaining finite). However, small time delays can also be produced by galaxies close to the line of sight (cf. eq. [8]). If we introduce a cutoff at the low-luminosity end of the Schechter luminosity function, then the amplitude for small time delay decreases. Such a luminosity cutoff also suppresses some small separation lenses.

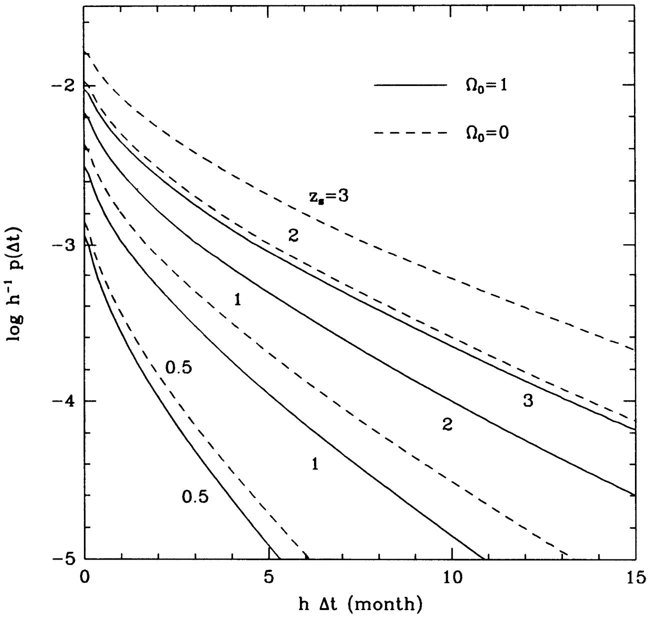

FIG. 2.—The differential probability distribution of time delay (logarithmic scale) due to lensing by isolated galaxies in open ($\Omega_0 = 0$, *dashed lines*) and flat ($\Omega_0 = 1$, filled beam, *solid lines*) cosmologies for four point source redshifts, $z_s = 3, 2, 1, 0.5$ (highest to lowest curves respectively for each cosmology). Note the time delay scales as h^{-1}, where h is the Hubble constant in units of 100 km s^{-1} Mpc^{-1}.

3. DISCUSSION

It is clear from many previous studies that there are a variety of uncertainties and selection effects which affect the lensing statistics (TOG; Blandford & Kochanek 1987; Kochanek & Blandford 1987; Bartelmann & Schneider 1990; Kochanek 1991; Fukugita & Turner 1991; Mao 1991; Fukugita et al. 1992). Aside from improving observations of galaxy properties including luminosity function and core properties, there is a need for future theoretical elaboration which should include magnification bias; angular resolution bias; finite core radius of galaxies; effects of clusters of galaxies and galaxy clustering; ellipticities of galactic potentials; different cosmological models; effects of microlensing; and self-consistent angular diameter distances. We qualitatively discuss how some of these complications could modify our results.

For a flux-limited quasar sample, the lensing statistics is dominated by magnification bias when the flux limit is brighter than the turnover magnitude of the quasar luminosity function. Due to the magnification bias, coupled with the finite angular resolution bias and the influence of clusters of galaxies and galaxy clustering, the distribution of time delay of observed quasar lensing systems could be significantly biased toward larger time delay, as the only time delay measurement available, $\Delta t \approx 1.5$ yr, for Q0957+561 (Lehár et al. 1992; Press, Rybicki, & Hewitt 1992a, b) seems to indicate. Thus the present calculation which models the galaxies as spherical lens is clearly insufficient for a magnitude limited quasar sample. However, as we will show below, the magnification bias for cosmological GRBs is moderate and there is no "angular" resolution bias due to the excellent time resolution of gamma-ray detectors; therefore, the time-delay distribution obtained here is more applicable to GRBs of cosmological origin. From now on, we confine ourselves to lensing of GRBs, although the discussion is also qualitatively correct for a quasar sample.

Of particular interest is using lensing to put stringent limit on postulated primeval compact objects like black holes (Carr 1990, and references therein). Compact objects with mass of order $\sim 10^6\ M_\odot$ will produce time delays ~ 50 s, and therefore can be readily detected. Given a mass density Ω_L in these massive compact objects, one can integrate equations (5.1)–(5.2) to obtain the total probabilities (cf. TOG, eqs. [2.14a]–[2.14c]). A null detection thus will put a strong limit on the mass density of cosmologically distributed massive black holes. The limit may be much stronger than any limit so far from null results of (optical and radio) searches for lensing of quasars (Nemiroff 1991).

Having discussed an application of the point mass model, we will focus on lensing by galaxies. Magnification bias for the cosmological GRBs can be calculated in the same way as that for a quasar sample (TOG; Fukugita & Turner 1991). We adopt the simple cosmological model of GRBs studied by Mao & Paczyński (1992b). For a canonical spectral index 1 ($\alpha = 1$ in the notation of Mao & Paczyński 1992b), we obtain the lensing frequency and the magnification bias:

$$f = 1.42 \times 10^{-3} , \quad B = 5.37 . \quad (15)$$

The lensing frequency is rather small. The magnification bias is moderate, which can also be seen from the fact that the slope of the integrated source counts with flux is rather shallow (~ -0.8) at the faint end (Meegan et al. 1992).

Both models described here have spherical potentials; the caustics produced are degenerate points. However, even a slight asymmetry in the potential will induce nontrivial caustic structures. For an elliptical potential with a finite core radius, the caustics consist of an inner diamond and an outer ellipse. A source located inside the diamond produces five images, while a source located outside the diamond caustic but inside the ellipse three images are formed. However, the central image is usually strongly deamplified, so one mostly observes doubly or quadruply imaged systems. Equation (8) makes a statistical prediction: quadrupole "recurrent" bursts have smaller time intervals than double "recurrent" bursts. The relative frequency of doubly or quadruply imaged systems is hard to estimate, due to the unknown galactic potential ellipticity and the complication of magnification bias (Narayan & Wallington 1992). From the fact that about one-half of the lensed quasars are doubles, we conclude that our results are probably valid within a factor of 2 if there were no additional selection bias present. However, BATSE could have detected ~ 800 GRBs per year (Meegan et al. 1992), while in reality it is only detecting ~ 300 GRBS per year due to the Earth eclipse and other selection effects, so the requirement that at least two images will be observed (so that they can be identified as lensed) lowers the lensing frequency of doubles by roughly a factor of 3 while the frequency of quadrupoles is essentially unaffected. Hence, the quadruply imaged systems might be significantly overrepresented. A detailed calculation is needed to predict the distribution of time delays for quadrupole systems. Qualitatively they are likely to have smaller time delays than the doubles, and sources close to the nontrivial caustics may induce double bursts with very short time interval and similar peak counts. Note this selection effect do not affect the lensing by massive black holes, due to the time delay ~ 50 s is much shorter than a typical noninterrupted observing by BATSE.

The core radii observed in early-type galaxies are very small (some of them are not even resolved); thus the lensing probability is not likely to be reduced by more than a factor of 2 (Fukugita & Turner 1991; Kochanek 1991; Mao 1991). A core radius in a spherical potential also produces a third (mostly faint) image if lensed. The time delay between the two outer images is smaller from that produced by a singular isothermal sphere.

The effects of clusters of galaxies and galaxy clustering are hard to evaluate. If we model a cluster as a constant density sheet, some elegant results are known (Falco, Gorenstein, & Shapiro 1985; Gorenstein, Falco, & Shapiro 1988; Narayan 1991). A constant density sheet increases the time delay by providing an extra deflection (thus an extra geometrical time delay), and it also introduces some magnification bias. Both effects are difficult to analyze.

Microlensing by stars in the lensing galaxy (or by galaxies in a rich cluster of galaxies; see Paczyński 1987) introduces time delays of order 2×10^{-5} s $(M/M_\odot)$ producing complicated temporal variations on a microsecond time scale, which may potentially be detected. This effect in principle can be used to determine the mass of the lensing star (Krauss & Small 1991). Massive black holes in galactic halos (Lacey & Ostriker 1985; Fuller, Woosley, & Weaver 1986) with mass of order $\sim 10^6$ $M_\odot$ will produce time delays ~ 50 s, and therefore can be identified because they typically lie outside the critical radius (a black hole inside the critical radius is likely to be confused with lensing by the galaxy as a whole). Hence BATSE may be the first instrument to (dis)prove the existence of massive black holes in the halos.

To summarize, if the GRBs are cosmological, then gravitational lensing predicts that a GRB may appear several times ("recurrent"), each with the same angular position in the sky, identical spectra, and identical "light curves" except for a scale factor. The frequency of "recurrent" bursts produced by lensing is very uncertain: if we allow roughly a factor of 3 uncertainty in the estimate of equation (15), we obtain a guess estimate of the "recurrent" frequency 0.05%–0.4%. With a probable *GRO* life of 6–10 yr and a detection rate of 300 yr^{-1} (Meegan et al. 1991), BATSE may observe several "recurrent" bursts due to gravitational lensing. These "recurrent" bursts should have high-redshift galaxies in the foreground and the time intervals between the bursts are likely to be days–months. BATSE may also provide the strongest limit on the mass density of massive black holes.

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