Astron. Astrophys. 265, 106–114 (1992)

ASTRONOMY AND ASTROPHYSICS

The solar tachocline

E. A. Spiegel [1] and J.-P. Zahn [1,2]

[1] Astronomy Department, Columbia University, New York, NY 10027, USA
[2] Observatoire Midi-Pyrénées, 14 avenue E. Belin, F-31400 Toulouse, France

Received June 5, accepted July 20, 1992

Abstract. Acoustic sounding of the Sun reveals that the variation of angular velocity with latitude is independent of depth in the convection zone. By contrast, deep within the radiative zone, the rotation appears to be rigid. The transition between the two rotation laws occurs in a thin, unresolved layer that we here call the *tachocline*. This paper is an examination of the structure and previous evolution of this layer. We assume that the stress exerted by the convection zone is prescribed, much as oceanographers take the wind stress on the sea surface as given. We conclude that the helioseismic observations are best rationalized by a scenario in which, after an initial adjustment or spindown period, the subconvective rotation settles into a quasisteady state with a turbulent boundary layer. In the tachocline, the advection of angular momentum is controlled by horizontal turbulence. If this turbulence is intense enough, the tachocline is thin and is unresolved.

Key words: turbulence – Sun: interior, rotation

1. Introduction

The breaking of hydrostatic equilibrium by stellar rotation and the consequent internal circulation are still not well understood. The difficulties are particularly severe when there are extensive convection zones in the star, for then we have only crude means of describing the internal fluid stresses that shape these flows, and the internal rotation law cannot be reliably deduced. However, in the solar case, improved observations of solar surface oscillations, in combination with the study of normal modes, permit a sounding of the interior flows (Brown et al. 1989; Goode et al. 1991). It has been found that the latitudinal differential rotation of the solar surface prevails throughout the convection zone with little radius dependence.

Below the convection zone, the equatorial and polar rotation rates appear to approach a common value in a radial increment too small to be resolved as yet (Goode et al. 1991). This near discontinuity in rotation velocity may power dynamos and affect the large scale circulation and local mixing. It is therefore worthwhile to seek a theoretical rationalization of this velocity boundary layer, which we call the solar tachocline by analogy with the oceanic thermocline. The tachocline lies below the convection zone, just as the thermocline lies below the mixed layer in the ocean, although the relative depths of the corresponding layers differ in the two cases (Pedlosky 1990).

In this paper, we describe the flow within the tachocline.[1] After listing our simplifying assumptions in §2, we describe the initial adjustment processes under the influence of an imposed wind stress at the top of the radiative interior, first adiabatically in §3, then radiatively in §4. In §5, we seek a stationary state in which strong horizontal turbulence prevents the vertical spreading of the tachocline.

2. The formulation of the problem

2.1. Basic equations

We begin by writing the equations of motion with respect to a differentially rotating frame with angular velocity $\mathbf{\Omega}(r,t)$, where $r = |\mathbf{r}|$ is the radial coordinate and $\boldsymbol{r}$ is the absolute position. For the absolute velocity we have $\mathbf{\Omega} \times \boldsymbol{r} + \boldsymbol{V}$ where $\boldsymbol{V}$ is the local velocity in the differentially rotating frame. In the frame with this $\mathbf{\Omega}$ the equations governing the motions for the radiative interior of a star are the conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0, \tag{2.1}$$

the conservation of momentum

$$\rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} + 2\mathbf{\Omega} \times \boldsymbol{V} + \dot{\mathbf{\Omega}} \times \boldsymbol{r} \right] = -\nabla P - \rho \nabla \Phi + \nabla \cdot \| \tau \| , \tag{2.2}$$

and the conservation of heat (or entropy)

$$\rho T \frac{\partial S}{\partial t} + \rho T \boldsymbol{V} \cdot \nabla S = \nabla \cdot (\chi \nabla T). \tag{2.3}$$

Here $\dot{\mathbf{\Omega}} = \partial \mathbf{\Omega}/\partial t + u\, \partial \mathbf{\Omega}/\partial r$ is the substantial derivative of $\mathbf{\Omega}$, where u is the radial component of velocity. Conventional notation has been employed for the density, ρ, the pressure, P, the temperature, T and the specific entropy, S; $\| \tau \|$ represents the viscous (or turbulent) stress tensor, χ the thermal conductivity and Φ the gravitational potential. Since we shall deal with layers

[1] When the existence of such a layer was first adumbrated, the term *tachycline* was proposed (Spiegel 1972). Here we defer to the terminological sensibilities of D.O. Gough and modify that neologism.

Send offprint requests to: J.-P. Zahn (Observatoire Midi-Pyrénées)

that are far from the stellar core, the generation of nuclear energy has been ignored and Φ is prescribed. In this first exploration, we also ignore the effects of a magnetic field.

2.2. Assumptions

We seek solutions under the following enumerated conditions:

1) *The flow field is axisymmetric with respect to the rotation axis.* Thus, $\boldsymbol{V} = \boldsymbol{V}(r, \theta, t)$ where θ is $\pi/2$ minus the latitude.
2) *The oblateness due to the centrifugal force is negligible in the figure of the Sun, so that the level surfaces of the hydrostatic reference state can be treated as spheres.* This approximation filters out the Eddington-Sweet circulation. The omission may be compensated by the inclusion of a suitable thermal forcing term in the heat equation (Zahn 1992). Like the Euler force $\dot{\boldsymbol{\Omega}} \times \boldsymbol{r}$ in the equation of motion, this effect is responsible for producing weak meridional currents. Both of these secondary flows will be ignored in this work. Consequently, the reference state will be of uniform rotation.
3) *The flow occurs on such long time scales that the acoustic modes may be filtered out of the circulation problem.* This is the heart of the *anelastic* approximation in which we neglect the acoustic analogue of the displacement current, $\partial\rho/\partial t$. The density variation enters the problem, of course, but only through the stratification by way of the equation of state.
4) *Terms quadratic in fluctuations of thermodynamic quantities from their mean states may be safely neglected.* This too is normally considered as part of the anelastic approximation. Thus, we neglect the variations of the specific heat at constant pressure, C_P, of the radiative conductivity χ and of the turbulent viscosity, for which we assume different values, ν_V and ν_H, appropriate to the vertical and horizontal components of the viscous stress tensor. In the same way, except for a brief discussion in Appendix B, we neglect variations in the mean molecular weight of the medium. This leaves us with a perfect gas with a linear equation of state.
5) *The Rossby number, which measures the ratio of the Coriolis acceleration to the advective term, is small.* Quantitatively (Pedlosky 1979; Busse 1981) this means that

$$Ro = \left(\frac{r}{h}\right)\left(\frac{\Delta\Omega}{2\Omega}\right) \ll 1\,, \tag{2.4}$$

where $\Delta\Omega$ is the differential rotation imposed by the convection zone and h is the vertical scale of variation of the flow. This assumption, which can be verified *a posteriori*, implies that the dynamics is linear to good approximation. By the same token, a small Rossby number means that we can neglect the acceleration compared to the Coriolis force, at least, if we are interested in long time behavior.
6) *The tachocline is very thin: $h \ll r$.* Hence the scale height of any function describing the structure of the layer will be of order h. This allows the neglect of the vertical velocity, when compared with the horizontal velocity in the meridional plane. Failure of this approximation at the poles and the equator produces some slight problems, but they may be ignored in a first discussion.
7) *The flow is geostrophic.* This is a familiar condition of geophysical flows, that perhaps we should call heliostrophy. This condition requires that, below the convection zone, the turbulent viscous forces are much less important than the Coriolis forces. For this to be true the Ekman numbers

$$E_V = \frac{\nu_V}{2\Omega h^2}\quad , \quad E_H = \frac{\nu_H}{2\Omega R_\odot^2}\,, \tag{2.5}$$

where h is the thickness of the tachocline, must be small. In many geophysical and laboratory problems, this approximation has to be compensated by the introduction of Ekman boundary layers. In the solar case, the convection zone takes over the role played by Ekman layers of the oceanic analogue (Bretherton & Spiegel 1968; Sakurai 1970).

Because of condition 1), which rules out a zonal pressure gradient, geostrophic balance can hold only in the latitudinal direction. Hence, in the zonal (azimuthal) direction, the Coriolis force can be balanced only by viscous and inertial forces. This is consistent with condition 7) on account of condition 6).

2.3. Reduced equations

If we mix the equations of §2.1 with the assumptions listed in §2.2 and stir vigorously, we find that a number of terms evaporate. To view the results at their best, we express them in spherical coordinates (r, θ, ϕ). Each dependent variable is separated into a mean value on the sphere plus a perturbation. Thus, the temperature is $T(r,t) + \widehat{T}(r,\theta,t)$ with $\int_0^\pi \widehat{T}(\theta)\sin\theta\,\mathrm{d}\theta = 0$. Then the linearized version of the equation of state is

$$\frac{\widehat{P}}{P} = \frac{\widehat{\rho}}{\rho} + \frac{\widehat{T}}{T}\,. \tag{2.6}$$

We express the velocity field as $\boldsymbol{V} = (u, v, r\widehat{\Omega}\sin\theta)$ where $\widehat{\Omega}$ is the differential rotation with respect to the rotation Ω of the reference state. We can further simplify the look of the equations by introducing a stream function for the meridional flow:

$$r^2\rho u = \frac{\partial\Psi}{\partial x}\,, \qquad r\rho\sin\theta\, v = \frac{\partial\Psi}{\partial r} \tag{2.7}$$

with $x = \cos\theta$.

Equations (2.1)–(2.3) are then distilled down to

$$-\frac{1}{\rho}\frac{\partial\widehat{P}}{\partial r} + g\frac{\widehat{T}}{T} = 0 \tag{2.8}$$

$$-2\Omega r x\widehat{\Omega} = \frac{1}{\rho r}\frac{\partial\widehat{P}}{\partial x}\,, \tag{2.9}$$

$$\begin{aligned}&\rho r^2(1-x^2)\frac{\partial\widehat{\Omega}}{\partial t} + 2\Omega x\frac{\partial\Psi}{\partial r}\\ &= \frac{(1-x^2)}{r^2}\frac{\partial}{\partial r}\left[\rho\nu_V r^4\frac{\partial\widehat{\Omega}}{\partial r}\right] + \rho\frac{\partial}{\partial x}\left[\nu_H(1-x^2)^2\frac{\partial\widehat{\Omega}}{\partial x}\right],\end{aligned} \tag{2.10}$$

$$\frac{\partial\widehat{T}}{\partial t} + \frac{N^2}{g}\frac{T}{\rho r^2}\frac{\partial\Psi}{\partial x} = \frac{1}{\rho C_P r^2}\frac{\partial}{\partial r}\left(\chi r^2\frac{\partial\widehat{T}}{\partial r}\right). \tag{2.11}$$

The two first equations express the hydrostatic and geostrophic balances, widely adopted in geophysical fluid dynamics (Pedlosky 1990). The third and fourth equations describe the advection and diffusion of angular momentum and heat. The stability of the temperature stratification is measured by the buoyancy frequency $N^2 = (g/H_P)(\nabla_{ad} - \nabla)$ (with the usual notation for the pressure scale-height H_P and for the logarithmic temperature gradients $\nabla = \partial\ln T/\partial\ln P$). Since the turbulent stresses may be anisotropic we have both horizontal and vertical diffusion terms in (2.10) by contrast with (2.11), where the diffusion is radiative and isotropic.

2.4. Boundary conditions

To study the evolution of the differential rotation described by these equations, we shall assume that the dependence on latitude at $r = r_0$, is imposed from above by the convection zone. This means that we are not allowing for the feedback of the flows in the radiative zone on the overlying convective zone. In the solar case, this imposed latitude variation is determined from observations. The rotation law is generally expressed as a polynomial in $x = \cos\theta$, so the first boundary condition may be written as

$$\Omega_{zc} = \Omega + \widehat{\Omega}(r_0, x) = \Omega_0(1 - ax^2 - bx^4), \tag{2.12}$$

Ω_0 being the equatorial rate and Ω the reference rotation.

The helioseismic observations are still not precise enough to yield a firm rotation law. But from the data of Libbrecht (1989), Goode et al. (1991) have improved the spatial resolution to $\Delta r/R_\odot \approx 0.1$. Allowing for a vanishingly thin tachocline, they derived the following rotation law at the base of the convection zone:

$$\frac{\Omega_{zc}}{2\pi} = 462 - 64\,x^2 - 73\,x^4 \text{ nHz}, \tag{2.13}$$

which we shall adopt in our models.

A second boundary condition expresses the continuity of the temperature perturbation, $\widehat{T}$. In turn, this implies the continuity of $\partial\widehat{\Omega}/\partial r$, as can be seen by eliminating $\widehat{P}$ between (2.8) and (2.9). Since the observations indicate that the differential rotation varies little in the deep convection zone, we may take $\partial\widehat{\Omega}(r, x, t)/\partial r = 0$.

Furthermore, the solutions must vanish well outside the boundary layer. We formulate these conditions in this way:

$$\widehat{\Omega}(r, x, t) = \Omega_{zc}(x, t) - \Omega, \quad \frac{\partial\widehat{\Omega}(r, x, t)}{\partial r} = 0 \quad \text{at } r = r_0 \tag{2.14}$$
$$\text{and} \quad \widehat{\Omega}(r, x, t) \to 0 \quad \text{for } r \to 0.$$

The reference rotation Ω is thus identified as the rotation of the deep interior.

3. Adiabatic adjustment

The real initial conditions for our problem are not known and are probably too complicated to deal with in any case. We begin our calculations with a fully formed, differentially rotating convection zone at $t = 0$ exerting stress on the subjacent radiative interior across their mutual interface at $r = r_0$. After a quick adjustment which takes a dynamical time of order Ω^{-1}, the subconvective layers settle into a quasi-stationary stage, which then evolves much more slowly under the combined action of radiative diffusion and viscous torque as in the conventional solar spindown problem (Spiegel 1972).

The initial stage is rapid and adiabatic. It may be studied by neglecting the dissipative terms in (2.10) and (2.11). On eliminating Ψ between these equations, and integrating in time, we establish the following relation between the differential rotation and the vertical variation of the temperature perturbation (with $x = \cos\theta$):

$$\rho r^2 \frac{\partial}{\partial x}\left[\left(\frac{1-x^2}{x}\right)\widehat{\Omega}\right] = 2\Omega\,\frac{\partial}{\partial r}\left(\frac{g\rho r^2}{N^2}\,\frac{\widehat{T}}{T}\right). \tag{3.1}$$

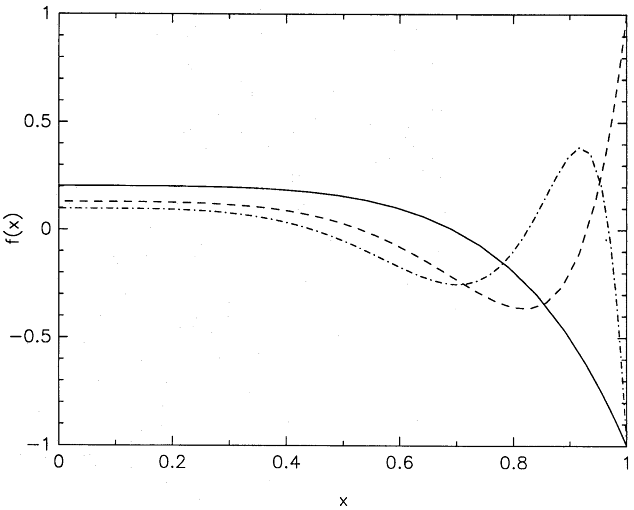

Fig. 1. Horizontal eigenfunctions, defined by (3.4), for the adiabatic and radiative regimes.

Elimination of $\widehat{\Omega}$ and $\widehat{T}/T$ using (2.8) and (2.9) yields

$$\frac{\partial}{\partial r}\left[\left(\frac{2\Omega}{N}\right)^2 r^2\,\frac{\partial\widehat{P}}{\partial r}\right] = -\frac{\partial}{\partial x}\left(\frac{1-x^2}{x^2}\,\frac{\partial\widehat{P}}{\partial x}\right). \tag{3.2}$$

This equation is separable in r and x, and we look for solutions of the form

$$\widehat{P}(r, x) = \sum_i \widetilde{P}_i(r)\,f_i(x)\,. \tag{3.3}$$

The horizontal eigenfunctions $f_i(x)$ obey

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^2}{x^2}\,\frac{\mathrm{d}f}{\mathrm{d}x}\right) + \lambda^2 f = 0\,, \tag{3.4}$$

and they are fully determined by the requirement that they be regular at the poles $x \pm 1$.

This equation is of the Sturm-Liouville type; it becomes the spheroidal wave equation when $\sqrt{1-x^2}\,\mathrm{d}f/\mathrm{d}x = x^2 S_n^1$, where the S_n^1 is the spheroidal wave function of the first kind of order n. Hence the eigenfunctions form a complete orthogonal set, when we include the constant function f_0 (with $\lambda_0^2 = 0$). The first even eigenfunctions are depicted in fig. 1. (See Appendix A for further discussion of the eigenfunctions.)

The modes of vertical structure satisfy

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{2\Omega}{N}\right)^2 r^2\,\frac{\mathrm{d}\widetilde{P}_i}{\mathrm{d}r}\right] = \lambda_i^2\,\widetilde{P}_i\,. \tag{3.5}$$

If there were no radiative or turbulent diffusion, the solution of (3.5) might serve to crudely describe a tachocline, with thickness or vertical scale height

$$h_0 = \frac{r_0}{\lambda_2}\left(\frac{2\Omega}{N}\right), \tag{3.6}$$

where $\lambda_2 = 3.542$ is the smallest eigenvalue of (3.4). This length scale appears in stratified spin-down theory (Holton 1965). In the present Sun, $h_0 \approx 500$ km; however, at earlier phases of its evolution, when the Sun was rotating much faster, the adiabatic adjustment layer penetrated well into the radiative interior, imposing a state of differential rotation similar to that in the convection zone. The thickness of this layer is the depth range over which the rotation overpowers stratification to locally establish a Taylor-Proudman regime (Greenspan 1968). Since the rotation of such an adiabatic tachocline cannot fulfill both boundary conditions (2.14) at $r = r_0$, we must either allow it to feed back onto the rotation of the convection zone or introduce a boundary layer. As those boundary conditions would involve dissipative mechanisms, such a study blends naturally into that of the following section.

4. Radiative spreading

Under the influence of radiative diffusion alone, the transition layer between the convection zone and the radiative interior thickens with time. We consider here the case where the horizontal diffusion of momentum is negligible, as it is when the flow is stable and the viscosity is due to microscopic interactions. Then, in § 5, we discuss what happens when the flow becomes unstable and the turbulent viscosity modifies the spreading of the transition layer.

The thin-layer approximation allows us to neglect the second term on the right of (2.10), provided that $\nu_H/\nu_V \ll (r_0/h)^2$, h being the thickness of the tachocline. If the turbulent stresses are not extremely anisotropic, this is a safe assumption. We use it for now and give it up in the next section.

The first consequence of ignoring possible strong anisotropy is that the diffusion terms that remain in (2.10) and (2.11) do not involve any derivatives with respect to $x = \cos\theta$. Therefore the horizontal eigenfunctions $f_i(x)$ introduced above can be used again to separate the variables r and x.

We may expand $\widehat{P}, \widehat{\rho}, \widehat{T}$ and u as

$$g(r,\theta,t) = \sum_{i>0} \widetilde{g}_i(r,t)\, f_i(x)\,, \tag{4.1}$$

where g stands for any of these dependent variables. Likewise, the stream function Ψ of the meridional flow becomes

$$\Psi = \sum_{i>0} \widetilde{\Psi}_i(r,t) \int_0^x f_i(x')\,\mathrm{d}x'\,. \tag{4.2}$$

For the differential rotation, similar manipulations lead to an expansion of the form

$$x\widehat{\Omega} = \sum_{i>0} \widetilde{\Omega}_i(r,t) \frac{\mathrm{d} f_i(x)}{\mathrm{d}x}\,. \tag{4.3}$$

Note that these eigenfunctions allow only a differential rotation $\widehat{\Omega}$ which vanishes at the equator, since $(1/x^2)\,\mathrm{d}f_i/\mathrm{d}x$ must be regular at $x = 0$ according to (3.4).

On dropping the index i, we transform (2.8)–(2.11) into the following system

$$0 = -\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial r} + g\frac{\widetilde{T}}{T} \tag{4.4}$$

$$-2\Omega\widetilde{\Omega} r^2 \rho = \widetilde{P} \tag{4.5}$$

$$r^2\rho\frac{\partial\widetilde{\Omega}}{\partial t} - \frac{2\Omega}{\lambda^2}\frac{\partial\widetilde{\Psi}}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\nu_V r^4 \frac{\partial\widetilde{\Omega}}{\partial r}\right) \tag{4.6}$$

$$\frac{\partial\widetilde{T}}{\partial t} + \frac{N^2}{g}\frac{T}{\rho r^2}\widetilde{\Psi} = \frac{1}{\rho C_P r^2}\frac{\partial}{\partial r}\left(\chi r^2 \frac{\partial\widetilde{T}}{\partial r}\right). \tag{4.7}$$

After a transient phase, which lasts about a thermal relaxation time, $h^2\rho C_P/\chi$, the evolution slows down and a balance is struck between advection and buoyancy in the heat equation. Then the time derivative in (4.7) may be neglected, as can be verified in the results of this section. With this simplification, the elimination of the perturbations of the thermodynamical variables yields an expression for the stream function in terms of the differential rotation:

$$\widetilde{\Psi} = -\frac{2\Omega g}{N^2 C_P T}\frac{\partial}{\partial r}\left[\chi\frac{\partial}{\partial r}\left(\frac{r^2 P T}{\rho g}\frac{\partial}{\partial r}\frac{\rho r^2\widetilde{\Omega}}{P}\right)\right]. \tag{4.8}$$

In using this formula we locate the base of the convection zone at the depth where the convective heat flux first becomes negligible. Because of penetrative convection, this occurs where N^2 is strictly positive (Zahn 1991), so no small denominators arise.

We then find an equation for the evolution of the differential rotation alone:

$$\begin{aligned}\frac{\partial\widetilde{\Omega}}{\partial t} &+ \frac{4\Omega^2}{\lambda^2}\frac{1}{\rho r^2}\frac{\partial}{\partial r}\left\{\frac{g}{N^2 C_P T}\frac{\partial}{\partial r}\left[\chi\frac{\partial}{\partial r}\left(\frac{r^2 P T}{\rho g}\frac{\partial}{\partial r}\frac{\rho r^2\widetilde{\Omega}}{P}\right)\right]\right\}\\ &- \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\nu_V r^4\frac{\partial\widetilde{\Omega}}{\partial r}\right) = 0\,.\end{aligned} \tag{4.9}$$

If the thickness of the transition layer is less than the scale heights of the background quantities, we may approximate this equation by

$$\frac{\partial\widetilde{\Omega}}{\partial t} + \kappa\left(\frac{2\Omega}{N}\right)^2\left(\frac{r_0}{\lambda}\right)^2\frac{\partial^4\widetilde{\Omega}}{\partial r^4} - \nu_V\frac{\partial^2\widetilde{\Omega}}{\partial r^2} = 0\,, \tag{4.10}$$

where $\kappa = \chi/\rho C_p$. The middle term is formally like the so-called hyperviscosity term of computational fluid dynamics and it operates in concert with the ordinary viscous term to spread the effects of the imposed shear radially.

We now apply the boundary conditions (2.14) to impose the value of $\widetilde{\Omega}(r_0)$. Thus, we integrate (4.3), expressed as in (2.13), and project the result onto the horizontal basis functions normalized so that $f_i(0) = 1$. The coefficents in (4.3), scaled by Ω_0, are then expressed by

$$\begin{aligned} Q_i &= \frac{\widetilde{\Omega}_i(r_0)}{\Omega_0} = -a\,I_i^4 - b\,I_i^6 \quad \text{with}\\ I_i^k &= \frac{1}{k}\,\frac{\int_{-1}^{1}\left(x^k - \frac{1}{k+1}\right) f_i\,\mathrm{d}x}{\int_{-1}^{1}(f_i)^2\,\mathrm{d}x}\,,\end{aligned} \tag{4.11}$$

where the integration constants have been chosen so that $I_0^k = 0$, in order to enforce $Q_0 = 0$. Note that the coefficients Q_i scale with the normalization adopted for the eigenfunctions f_i. On imposing the observed rotation law (2.13) as boundary condition, we calculate

$$Q_2 = 1.02\,10^{-2},\quad Q_4 = 2.2\,10^{-5},\quad Q_6 = -4.8\,10^{-5},\ \ldots\,;$$

and we see that the first mode ($i = 2$) dominates.

Now, folllowing an initial, rapid adiabatic response on the dynamical time-scale ($\approx \Omega^{-1}$) across a depth $h_0 = 2\Omega r_0/\lambda_2 N$ (§3), there is the subsequent, much slower, thermal relaxation when the flow evolves according to (4.9). Then, depending on the intensity of the turbulence, we shall have either (a) negligible turbulent viscosity, when the flow is essentially inviscid, or (b) significant turbulent diffusion of momentum controlling the further evolution of the flow. We consider case (a) here and a version of (b) in the next section.

If the spreading is laminar, we may ignore the viscous term in (4.9) or (4.10). We can look for a similarity solution of the resulting equation with the region of influence of the convection zone spreading like the 1/4 power of time:

$$h \approx r_0 \left(\frac{t}{t_{ES}}\right)^{1/4}, \tag{4.12}$$

where t_{ES} is a local Eddington-Sweet time given by

$$t_{ES} = \left(\frac{N}{2\Omega}\right)^2 \frac{r_0^2}{\kappa}.$$

Such a regime is familiar in the stratified spin-down problem (e.g. Sakurai 1970; Spiegel 1972). In the Sun, $t_{ES} = 2.2\,10^{11}$ years and $r_0 = 480,000$ km; thus $h \approx 200,000$ km after $4.6\,10^9$ yrs.

To be more precise, we define a scaled depth, measured from the top boundary, by $z = (r_0 - r)/r_0$, and a nondimensional time, $\tau = t/t_{ES}$ so that (4.10) is made non-dimensional:

$$\frac{\partial \widetilde{\Omega}_i}{\partial \tau} + \frac{1}{\lambda_i^2} \frac{\partial^4 \widetilde{\Omega}_i}{\partial z^4} = 0\,. \tag{4.13}$$

This equation has similarity solutions of the form

$$\widetilde{\Omega}_i(z,\tau) = Q_i\,\Omega_i\,Z(\xi), \quad \xi = \sqrt{\lambda}\frac{z}{\tau^{1/4}}, \tag{4.14}$$

where $y(\xi) = \mathrm{d}Z(\xi)/\mathrm{d}\xi$ satisfies

$$4\,y''' - \xi\,y = 0. \tag{4.15}$$

Of the three independent solutions of (4.15), only two, which are complex conjugate, remain finite for $\xi \to \infty$. We may combine them into a real function, which behaves as

$$y = y_0\,\phi' \exp\left(-\frac{1}{2}\phi\right) \cos\left(\frac{\sqrt{3}}{2}\phi\right) \text{ with } \phi' = \left(\frac{\xi}{4}\right)^{\frac{1}{3}}, \tag{4.16}$$

for large ξ, where prime denotes differentiation.

The solution depends on two constants, y_0 and the phase at the origin of ϕ. These are fixed by the boundary conditions, $Z(0) = 1$ and $Z'(0) = y(0) = 0$. We have integrated (4.15) numerically, and found that the solution has its first node at $\xi = 3.260$. In the present Sun this corresponds to $z = 0.536$, or about 270,000 km below the convection zone. This gives us some idea of how far the spreading has gone, but it also shows that it is too deep for (4.10) to be applicable.

A more accurate treatment requires the integration of the p.d.e. (4.9) with allowance for the variation of the mean rotation rate and for the evolution of the stellar structure. We have performed this integration for an unchanging mean Sun, to capture the effects of departures from the thin-layer approximation produced by density stratification and sphericity. We find, not surprisingly, that the penetration of the tachocline is less rapid than predicted above: $z = 0.375$ or 180,000 km below the convection zone. Since this estimate has not allowed for a higher rotation rate of the early Sun, it is too small because $\int \mathrm{d}t/t_{ES}$ is larger than t/t_{ES}, with the current value put in. We can make an estimate of this difference.

If the form of the spatial rotation law is assumed constant and the magnetic torque of the solar wind scales with a power of the surface rotation, Ω varies like $(t + t_*)^{-n}$ (Spiegel 1968). If t_* is neglected, the best fit to the data gives $n = 1/2$ (Skumanich 1972). With this value we get an increase in $\int_0^t \mathrm{d}t/t_{ES}$ by a factor 8 over the value obtained with contant Ω, if the equatorial rotation of the early Sun is taken as 100 km s^{-1} typical for very young stars of that mass. (For $n > 1/2$, the correction factor would be even larger.) The transition layer thickness determined above should therefore be scaled by approximately $8^{1/4}$, so that if the spreading began in the early Sun, it has now reached to 300,000 km below the convection zone. A more precise estimate would be hard to obtain without more knowledge of the initial internal motions.

The picture is that if the differential rotation in the convection zone was set up early, the effects will have spread quite deeply, in the case of a laminar adjustment process. As long as the slowing down rate of the solar rotation is not too great, this tracking of the surface motions by the interior flows works reasonably well. However, this simple theory does not explain why helioseismology reveals a thin tachocline. We thus have to seek a process which prevents the indefinite spreading of the influence of the convection zone. One possibility might be the inhibition of circulation by a gradient of molecular weight (Mestel 1953), which might arise if helium drifts down from the convective zone. We find that this does not work (see Appendix B). By contrast, a mechanism that at first glance seems to enhance the spreading rate, turbulent viscosity, is the one that we propose to explain the existence and the nature of the tachocline.

5. The turbulent tachocline

We have conluded that, without turbulent viscosity, the effects of the wind stress exerted on the radiative interior by the convection zone will have penetrated at least half way to the solar center. It might be thought that the inclusion of turbulent stresses in the interior flow will only enhance the spreading rate. That is the conclusion suggested by the equations of §4 but, in that section, we assumed at the outset that the horizontal turbulence was not much more intense than the vertical turbulence. The situation is drastically altered when we allow a suitably anisotropic turbulence; in particular, (4.6) no longer holds.

In this section, we explore the consequences of assuming that the horizontal turbulence transports angular momentum so effectively that in (2.10) the second term on the right is dominant. When

$$\frac{\nu_H}{\nu_V} \gg \left(\frac{r_0}{h}\right)^2, \tag{5.1}$$

where h is the thickness of the tachocline, we may neglect the first term in (2.10). This limit becomes relevant when the stable stratification in the subconvective flow produces the strong anisotropy familiar in geostrophic turbulence (Rhines 1979). As in many such studies, we shall not go into the details of the linear and nonlinear instability mechanisms that may produce the turbulence, except to remark that it is to be expected on account of the large Reynolds number of the horizontal shear (Zahn 1975).

For the purposes of understanding why there is a tachocline, we discuss the limiting case when (2.10) becomes

$$\rho r^2(1-x^2)\frac{\partial \widehat{\Omega}}{\partial t} + 2\Omega\,x\,\frac{\partial \Psi}{\partial r} = \rho\frac{\partial}{\partial x}\left[\nu_H(1-x^2)^2\frac{\partial \widehat{\Omega}}{\partial x}\right]. \tag{5.2}$$

In this limit, the horizontal transport of angular momentum by turbulence will be more important than the vertical transport. However, the horizontal transport of heat by turbulence remains smaller than the vertical radiative transport when we are below the region of penetrative convection. Though the effective Prandtl number ν_H/κ may well be larger than unity, it is appreciably less than $(r_0/h)^2$. Hence we neglect as before the net horizontal transport of heat in (2.11).

The important change that occurs when we adopt (5.2) is that, after a brief adjustment, the time derivative becomes unimportant and the remaining two terms can achieve a balance. This permits us to find a solution accounting for the observed tachocline, with suitable values of the parameters. Of course, the full solutions are now more difficult to come by since, with (2.10) replaced by (5.2), the new system (2.8)–(2.11) is not separable in r and x, but the sacrifice is well worthwhile. So we go straight to steady case. adopting the fiction that the solar rotation is independent of time now that the mean rotation rate is slowing down at a modest rate.

The stationary flow can be expanded in separable solutions, though a different set of horizontal eigenfunctions now appear. As before we write

$$g(r,\theta) = \sum_i \widetilde{g}_i(r)\, F_i(x)\,, \tag{5.3}$$

where $x = \cos\theta$ and g stands for any of the dependent variables $\widehat{P}, \widehat{\rho}, \widehat{T}$ and u. The stream function is again given by

$$\Psi = \sum_i \widetilde{\Psi}_i(r,t) \int_0^x F_i(x')\,\mathrm{d}x'\,, \tag{5.4}$$

and the rotation rate by

$$x\widehat{\Omega} = \sum_i \widetilde{\Omega}_i(r) \frac{\mathrm{d}\,F_i(x)}{\mathrm{d}x}\,. \tag{5.5}$$

Again, we have an equation for the new horizontal eigenfunctions, $F_i(x)$, of fourth order this time. When ν_H is independent of latitude, the eigenfunction equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}F_i}{\mathrm{d}x}\right)\right]\right\} - (\mu_i)^4 F_i = 0\,. \tag{5.6}$$

As with the function $f(x)$ introduced in § 3, we require that $F(x)$ be regular at the poles $x = \pm 1$. One easily verifies that these eigenfunctions form an orthogonal set, and that $(\mu_i)^4 \geq 0$. The functions of lowest order turn out to be the Legendre polynomials of degree $n \leq 2$; the first even eigenfunctions are displayed in fig. 2. (See Appendix A for more details.)

5.1. The interior rotation

A thin tachocline acts like a boundary layer in imposing the rotation (2.12) upon the radiative interior. We thus gain a certain flexiblity by comparison to the previous case since the interior rotation Ω is no longer required to be identical to the equatorial rate Ω_0. For $(1/x)\,\mathrm{d}F_i/\mathrm{d}x$ does not vanish at the equator ($x = 0$), for $i \geq 2$.

Once more, we derive the modal amplitudes $\widetilde{\Omega}_i(r_0)$ from the rotation law prevailing in the convection zone by integrating (5.5) and projecting onto the eigenfunctions, with $\widehat{\Omega}$ expressed as in (2.12). The result is

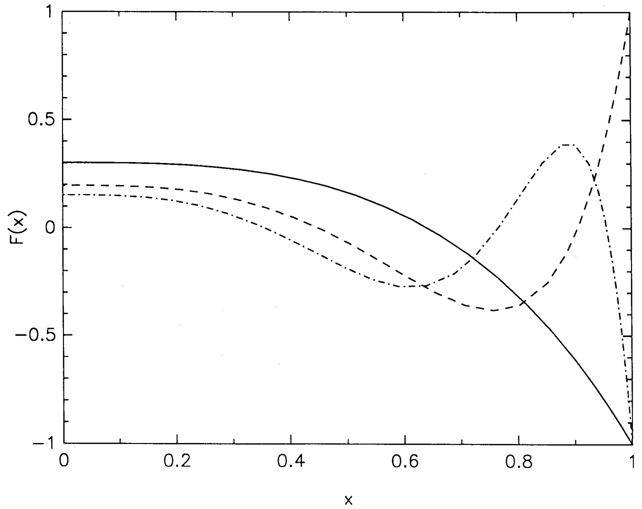

Fig. 2. Horizontal eigenfunctions, defined by (5.6), for the turbulent tachocline.

$$Q_i = \frac{\widetilde{\Omega}_i(r_0)}{\Omega_0} = \left(1 - \frac{\Omega}{\Omega_0}\right) I_i^2 - a\,I_i^4 - b\,I_i^6\,, \tag{5.7}$$

the I_i^k being defined as in (4.11b):

$$I_i^k = \frac{1}{k}\,\frac{\int_{-1}^{1}\left(x^k - \frac{1}{k+1}\right) F_i\,\mathrm{d}x}{\int_{-1}^{1}(F_i)^2\,\mathrm{d}x}\,. \tag{5.8}$$

In particular, for $F_2 = P_2 = (3x^2-1)/2$, we get

$$I_2^k = \frac{2}{(k+1)(k+3)}\,.$$

Note also that $I_i^2 = 0$ for $i > 2$, owing to the orthogonality property.

Since $(1/x)\,\mathrm{d}F_2/\mathrm{d}x$ is constant, $\widetilde{\Omega}_2(r)$ is a mode with no differential rotation, according to (5.5). In a sense, this is a mode associated with the global rotation, which we allowed at the outset to depend on r. Even though we have dropped that r-dependence, we must absorb this mode into the mean rotation, since otherwise it would be subject to an uncompensated viscous torque. On requiring that $Q_2 = 0$, we obtain for the value of the interior rotation

$$\left(1 - \frac{\Omega}{\Omega_0}\right) = \frac{a\,I_2^4 + b\,I_2^6}{I_2^2} = \frac{3}{7}a + \frac{5}{21}b\,. \tag{5.9}$$

When a and b are of the same sign, as they are in the Sun, the interior rotation rate is intermediate between the polar and equatorial rates at the base of the convection zone. Moreover, since $(3/7)^{1/2}$ is rather close to $(5/21)^{1/4}$, the interior rotation is approximately that of the convection zone at the latitude 42°, whose sine is the mean of those two values (cf. Adams 1979). The relation (5.9) still holds when ν_H is a function of latitude, because P_2 remains a solution of the o.d.e. which then replaces (5.6).

With the rotation law derived by Goode et al. (1991), one obtains the following value for the interior rotation:

$$\frac{\Omega}{2\pi} = 0.903\,\frac{\Omega_0}{2\pi} = 417 \text{ nHz}.$$

This value is compatible with those inferred from the helioseismic observations by various procedures.

The modal amplitudes $\widetilde{\Omega}_i/\Omega_0 = Q_i = -aI_i^4 - bI_i^6$ take the values (see 5.7)

$$Q_4 = -4.66\,10^{-3},\;\; Q_6 = -5.09\,10^{-4},\;\; Q_8 = 1.51\,10^{-4}.$$

Again, the gravest mode (here $i = 4$) dominates.

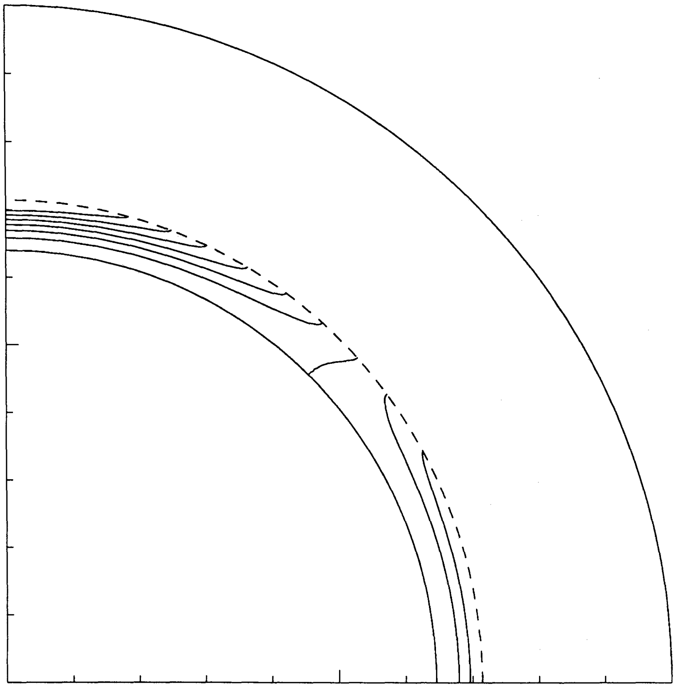

Fig. 3. The turbulent tachocline, whose thickness has been set here arbitrarily to 50,000 km (the actual value depends on the horizontal component ν_H of the turbulent viscosity). The continuous lines show the contours of the angular velocity. Below, the interior rotation is nearly uniform, and its angular velocity equals that of the base of the convection zone at the latitude of approximately 42°.

5.2. Tachocline thickness

As in writing (4.4)–(4.7), we drop the index i and have the analogous system

$$0 = -\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial r} + g\frac{\widetilde{T}}{T}, \tag{5.10}$$

$$-2\Omega\widetilde{\Omega} r^2 \rho = \widetilde{P}, \tag{5.11}$$

$$\frac{2\Omega}{\mu^4}\frac{\partial\widetilde{\Psi}}{\partial r} = \rho\nu_H\,\widetilde{\Omega}, \tag{5.12}$$

$$\frac{N^2}{g}\frac{T}{\rho r^2}\widetilde{\Psi} = \frac{1}{\rho C_P r^2}\frac{\partial}{\partial r}\left[\chi r^2\frac{\partial \widetilde{T}}{\partial r}\right]. \tag{5.13}$$

We then obtain the equation for the modal functions $\widetilde{\Omega}_i(r)$,

$$\frac{4\Omega^2}{\mu^4}\frac{1}{\rho\nu_H}\frac{\partial}{\partial r}\left\{\frac{g}{N^2 C_P T}\frac{\partial}{\partial r}\left[\chi r^2\frac{\partial}{\partial r}\left(\frac{T}{\rho g}\frac{\partial}{\partial r}\rho r^2\widetilde{\Omega}\right)\right]\right\} + \widetilde{\Omega} = 0, \tag{5.14}$$

whose fourth-order differential operator resembles that of (4.9). Again, we may simplify the problem by treating the tachocline as a boundary layer in which the rapidly varying quantity is $\widetilde{\Omega}$. Then, just below $r = r_0$,

$$\left(\frac{2\Omega}{N}\right)^2 \frac{\kappa}{\nu_H}\left(\frac{r_0}{\mu}\right)^4 \frac{\partial^4\widetilde{\Omega}}{\partial r^4} + \widetilde{\Omega} = 0\,. \tag{5.15}$$

We restore the modal index i ($i \geq 4$), and introduce the vertical coordinate

$$\zeta = \mu_i\left(\frac{r_0 - r}{d}\right) \quad \text{with} \quad d = r_0\left(\frac{2\Omega}{N}\right)^{1/2}\left(\frac{4\kappa}{\nu_H}\right)^{1/4}, \tag{5.16}$$

to cast (5.15) in non-dimensional form

$$\frac{\mathrm{d}^4\widetilde{\Omega}_i}{\mathrm{d}\zeta^4} + 4\widetilde{\Omega}_i = 0\,. \tag{5.17}$$

The solution that behaves well in the deep interior and satisfies the boundary conditions (2.14), $\widetilde{\Omega}_i = Q_i\Omega$ and $\mathrm{d}\widetilde{\Omega}_i/\mathrm{d}\zeta = 0$ at $\zeta = 0$, is

$$\widetilde{\Omega}_i(\zeta) = Q_i\,\Omega\,\sqrt{2}\,\exp(-\zeta)\,\cos(\zeta - \pi/4)\,. \tag{5.18}$$

We may thus conclude that if the horizontal eddy-diffusivity ν_H is large enough, the spread of the tachocline will be limited to a thickness fixed by the first zero of the lowest mode, which is located at $\zeta = 3\pi/4$, that is

$$h = \frac{3\pi}{4\mu_4}\,d = \frac{3\pi}{2\mu_4}\,r_0\left(\frac{\Omega}{N}\right)^{1/2}\left(\frac{\kappa}{\nu_H}\right)^{1/4}, \tag{5.19}$$

where $\mu_4 = 4.933$ is the smallest non-zero eigenvalue of (5.6). The horizontal turbulence then enforces a stationary state, in which the advection of angular momentum is balanced by the Reynolds stresses acting on the horizontal shear. Such a stationary solution is displayed in fig. 3, where we have arbitrarily set $h = 50,000$ km.

The thickness of the tachocline is approximately $20,000\,\sigma^{-\frac{1}{4}}$ km, in the present Sun, where σ is the effective Prandtl number $\sigma = \nu_H/\kappa$. In neglecting the net horizontal transport of heat, we have implicitly assumed that $\sigma < (r_0/h)^2$, a condition which is satisfied provided $h/r_0 \gtrsim \Omega/N$. As we come to the limit of validity for this approximation, the tachocline thickness reduces to about the scale height of the adiabatic adjustment layer (cf. 3.6) in the present Sun.

We should add that if the condition on σ were violated, we would need to include the turbulent diffusion of heat, which would then be competive with the radiative diffusivity. In that case, we should consider the region as part of the convection zone. Though a situation like this may have arisen briefly in the early evolution of the solar rotation, we may safely regard the tachocline as the present frontier of the radiative zone.

6. Discussion

By necessity, we have ignored the initial stages of solar rotation and turbulence and have assumed that the present interior rotation is not very sensitive to the initial conditions. So we have taken as our starting conditions a rigid interior subject to a "wind stress" imposed by the differential rotation in the overlying convection zone. The wind on the interface between the radiative core and the convective zone is taken from helioseismology. We have not allowed for feedback from the radiative flow onto motion in the convective zone. We have also assumed that this idealized initial state occurs after the era of rapid spindown so that the temporal variations of solar rotation could be left out of account.

The first effect seen when the surface differential rotation law is imposed on the convective-radiative interface is a spreading of the differential rotation into the solar interior by radiatively controlled circulation. In the regime where the behavior is controlled by radiative diffusion, the lowest order mode, $f_2(\cos\theta)$, dominates the latitude variation. This horizontal eigenfunction has the property that the rotation rate does not vary with depth on the equator; in other words, the interior rotation is equal to the equatorial rotation of the convection zone. At higher latitude, the spreading in time, with $h \approx r_0(t/t_{ES})^{1/4}$, would by now extend over half way into the center of the Sun if we take proper account of its rotational history.

The radiative spreading does not produce the sharp transition layer between the convective and radiative zones detected in the recent inversions of helioseismic data (Brown et al. 1989; Goode et al. 1991). Their spatial resolution has improved to 1/10 of the solar radius, and the observed tachocline appears more shallow than that. The implication is that a stationary layer has been formed that changes in step with the continuing spin down, with the large scale advection of angular momentum balanced by horizontal turbulent stresses.

Support for this picture comes from its prediction of an interior rotation that is intermediate between the polar and equatorial rates within the convection zone, in agreement with the present interpretations of helioseismic observations. The turbulent tachocline is the seat of a permanent circulation whose stream function is dominated by the lowest mode, $\Psi_4 \propto \exp(-\zeta)\cos(-\zeta)\,F_4(x)$, with $\zeta = \mu_4(r_0 - r)/r_0$. This means that the vertical velocity decreases monotonically from $\zeta = 0$ to its first zero at $\zeta = \pi/2$, and that the circulation has an octopolar configuration, with two cells emerging from the convection zone in each hemisphere. When future observations allow the resolution of the tachocline, we can use its observed thickness to estimate the horizontal component of the eddy diffusivity.

Through such considerations, the tachocline may play a pivotal role in the problem of the mixing of lithium. The horizontal turbulence, which prevents the spread of the layer, should also significantly affect the lithium transport (Chaboyer & Zahn 1992). This and related matters are discussed elsewhere (Zahn 1992).

Similarly, we note the possibility that the tachocline is implicated in solar magnetic activity. This is in line with the currently favored idea that the seat of solar activity is in a layer near the bottom of the tachocline (Stenflo 1990, and other papers in the same volume). In this picture, motions overshooting from above would carry magnetic fields from the convection zone into the tachocline, to be stretched there into toroidal fields whence they would erupt again at the surface as sunspots (e.g. Spiegel & Weiss 1980). As the observational resolution improves, we should be able to improve our understanding of these issues.

Acknowledgements. We are grateful to the Air Force Office of Scientific Research for supporting this work under grant AFOSR 89-0012 and to the GFD course at the Woods Hole Oceanographic Institution in which we participated fruitfully in 1990. J.-P. Z. was also supported by the Centre National de la Recherche Scientifique (GDR 132). We thank N.J. Balmforth, A. Brandenburg, M. Rieutord and N.O. Weiss for their comments on earlier versions of this work. Some of the calculations in this paper relate to an ongoing study of solar spindown with F.P. Bretherton, and we acknowledge some beneficial spillover.

Appendix A: Properties of the tachocline functions

A.1. Horizontal eigenfunctions in the laminar regimes

In the tachocline, all fields are expanded in horizontal eigenfunctions which satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^2}{x^2}\frac{\mathrm{d}f_i}{\mathrm{d}x}\right) + (\lambda_i)^2 f_i = 0\,, \tag{3.4}$$

with $x = \cos\theta$. That equation is similar to the Legendre equation; here also the solutions must be regular at $x = \pm 1$, and therefore

$$\int_{-1}^{1} f_i(x)\,\mathrm{d}x = 0.$$

This means that the constant function is an eigenfunction, for $\lambda_0 = 0$, as required both by the linearization we have performed and by the continuity equation.

Equation (3.4) has been integrated numerically by a shooting method to yield the even eigenfunctions. We took the boundary condition $f_{2k}(1) = (-1)^k$, and the eigenvalue λ was iterated until $f'(0) = 0$. The first three even solutions are displayed in fig. 1; note their very flat profile around $x = 0$, due to the vanishing f''. (To calculate the integrals appearing in (4.11), the eigenfunctions were normalized at the equator: $f_{2k}(0) = 1$.)

The eigenvalues and the nodes were found at the following locations:

$\lambda_0 = 0.000$;
$\lambda_2 = 3.542$; $x = \pm 0.697$;
$\lambda_4 = 6.688$; $x = \pm 0.533,\ \pm 0.932$;
$\lambda_6 = 9.830$; $x = \pm 0.447,\ \pm 0.825,\ \pm 0.970$;
$\lambda_8 = 12.1071$; $x = \pm 0.393,\ \pm 0.742,\ \pm 0.904,\ \pm 0.983$.

There is excellent agreement with the asymptotic theory, which for large eigenvalue predicts

$$\lambda_{2k} = \frac{\pi}{8} + k\pi, \quad k = 1, 2, 3, \ldots$$

A.2. Horizontal eigenfunctions in the turbulent regime

It is the horizontal viscosity which dominates in this regime, and the horizontal eigenfunctions $F_i(x)$ obey the fourth-order equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}F_i}{\mathrm{d}x}\right)\right]\right\}-(\mu_i)^4F_i=0\,. \qquad (5.6)$$

With the classical method of multiplying by F_j and integrating by parts, one checks that the eigenfunctions are orthogonal, provided they have different eigenvalues.

A curious property of this equation is that it is obeyed by any second degree polynomial in x, with $\mu_i = 0$. Among these degenerate solutions, only the Legendre polynomials form an orthogonal sequence, and therefore we conclude that $F_i(x) \equiv P_i(x)$ for $i = 0$, 1 and 2.

The other eigenvalues μ^4 are strictly positive, since they verify

$$\int_{-1}^{1}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}F_i}{\mathrm{d}x}\right)\right]^2 dx=(\mu_i)^4\int_{-1}^{1}(F_i)^2\mathrm{d}x\,.$$

Equation (5.6) has been integrated likewise by a shooting method to yield the even eigenfunctions. We took the boundary condition $F_{2k}(1) = (-1)^k$, and the trial values of $F'(1)$ and μ^4 were interated until $F'(0) = F'''(0) = 0$. The first three even solutions are displayed in fig. 2; they are less steep near the poles $x = \pm 1$ than the functions $f(x)$.

The eigenvalues and the nodes were found at the following locations:

$\mu_0 = 0.000$;
$\mu_2 = 0.000$; $x = \pm 0.577$;
$\mu_4 = 4.933$; $x = \pm 0.374$, ± 0.877;
$\mu_6 = 7.680$; $x = \pm 0.296$, ± 0.728, ± 0.949;
$\mu_8 = 10.356$; $x = \pm 0.249$, ± 0.624, ± 0.853, ± 0.972.

Appendix B: Effect of a molecular weight gradient

The laminar spreading of the adjustment process proceeds through the radiative diffusion of buoyancy in the stable layers. At first sight, it looks as if a strongly stable molecular weight gradient would slow down this process, as we mentioned at the end of §4. Such a gradient of molecular weight might occur below the solar convection zone, because helium spreads downward through gravitational settling and thermal diffusion (Profitt & Michaud 1991).

With the gradient $\mathrm{d}\ln\mu/\mathrm{d}\ln r$ in the problem, we need to consider the perturbation of molecular weight. This quantity may be developed in the eigenfunctions $f(x)$, and its expansion coefficient obeys an equation similar to that of the temperature perturbation:

$$\frac{\partial\widetilde{\mu}}{\partial t}+\frac{\mathrm{d}\mu}{\mathrm{d}r}\frac{\widetilde{\Psi}}{\rho r^2}=0\,. \qquad (B.1)$$

We have omitted the diffusion term, since the molecular diffusivity is of the same order as the viscosity, which we have neglected.

This molecular weight perturbation reacts on the buoyancy, and thus the hydrostatic equation becomes

$$0=-\frac{1}{\rho}\frac{\partial\widetilde{P}}{\partial r}+g\left(\frac{\widetilde{T}}{T}-\frac{\widetilde{\mu}}{\mu}\right), \qquad (B.2)$$

instead of (4.4).

In the thin layer limit, the modified system reduces to

$$\frac{\partial\widetilde{\Psi}}{\partial t}+\kappa\left(\frac{2\Omega}{N}\right)^2\left(\frac{r_0}{\lambda}\right)^2\frac{\partial^4\widetilde{\Psi}}{\partial r^4}+\kappa\frac{g}{rN^2}\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln r}\frac{\partial^2\widetilde{\Psi}}{\partial r^2}=0\,. \qquad (B.3)$$

This equation is similar to (4.10); since the helium abundance increases with depth, $\mathrm{d}\ln\mu/\mathrm{d}\ln r$ is negative, and the additional term is of diffusive nature.

References

Adams D., 1979, The Hitch-Hiker's Guide to the Galaxy (Pan Books, London)
Bretherton F.P., Spiegel E.A., 1968, ApJ 153, 277
Brown T.M., Christensen-Dalsgaard J., Dziembowski W.A., Goode P., Gough D.O., Morrow C.A., 1989, ApJ 343, 526
Busse F.H., 1981, Geophys. Astrophys. Fluid Dynamics 17, 215
Chaboyer B., Zahn J.-P., 1992, A&A 253, 173
Goode P.R., Dziembowski W.A., Korzennik S.G., Rhodes E.J., 1991, ApJ 367, 649
Greenspan, H.: 1968, The Theory of Rotating Fluids (Cambridge Univ. Press)
Holton, J.R. 1965, J. Atmos. Sci. **22**, 402
Libbrecht K.G., 1989, ApJ 336, 1092
Mestel L., 1953, MNRAS 113, 716
Pedlosky J., 1979, Science 248, 316
Pedlosky J., 1990, Geophysical Fluid Dynamics (Springer)
Profitt C.R., Michaud G., 1991, ApJ 380, 238
Randers G., 1941, ApJ 94, 109
Rhines P.B., 1979, Ann. Rev. Fluid Mech. 11, 401
Sakurai T., 1970, Stellar Rotation (ed. A. Slettebak; Reidel), 329
Skumanich A., 1972, ApJ 200, 747
Spiegel E.A., 1968, Highlights of Astronomy (ed. L. Perek; D. Reidel Pub. Co.) p. 261
Spiegel E.A., 1972, Physics of the Solar System (ed. S.I. Rasool; NASA, Washington)
Spiegel E.A., Weiss N.O., 1980, Nat 287, 616
Stenflo J.O., 1991, The Sun and Cool Stars: activity, magnetism, dynamos (eds. I. Tuominen, D. Moss, G. Rüdiger; Springer), 193
Zahn J.-P., 1975, Mém. Soc. Roy. Sci. Liège, 6e série 8, 31
Zahn J.-P., 1991, A&A 252, 179
Zahn J.-P., 1992, A&A (this volume)

This article was processed by the author using Springer-Verlag TEX A&A macro package 1991.