

Research Note

Radial oscillations of neutron stars and strange stars

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Abstract. We have determined the two lowest frequency radial oscillation modes constructed for several equations of state. The results for neutron stars differ from those of Glass & Lindblom (1983). We find that the central density where $\omega_0^2=0$ agrees in our calculations with that where the star reaches its maximal mass, as expected, but contrary to their calculations. The oscillation frequencies of strange stars have qualitatively a different dependence on the central density compared with the case of neutron stars, as the periods of all modes go to zero when the central density of the strange star approaches its smallest possible value.

Key words: neutron stars – oscillations of stars

1. Introduction

The possibility that radial oscillations of neutron stars give rise to the oscillations observed within radio subpulses of pulsars was proposed by Boriakoff (1976). While this interpretation has been doubted by some (Cordes 1976), others have not excluded the possibility that radial oscillations might be detectable (van Horn 1980). More recently, periodicities have been observed in X-ray bursts (Sadeh et al. 1982), which has raised considerable interest in radial oscillations (Glass & Lindblom 1983, hereafter GL; Muslimov & Tsygan 1986; Martí et al. 1988) and other types of oscillations of neutron stars (Lindblom & Detweiler 1983; McDermott et al. 1988). Then Kristian et al. (1989) reported the discovery of the neutron star remnant of SN 1987 A (Arnett et al. 1989), which showed optical oscillations with a period of 0.5 ms. This was interpreted by Wang et al. (1989) as radial oscillations of the neutron star. But the observations were found to be wrong and the remnant of this supernova remains undiscovered. Nevertheless, it is possible that if the neutron star in SN 1987 A is discovered, it could show pulsations which may be due to radial oscillations. Hence studies of radial oscillations of neutron stars may be valuable to understanding future observations.

Studies of radial oscillations of neutron stars have been performed by a number of authors with equations of state (EOS) at zero temperature (e.g. Harrison et al. 1965; Chanmugam 1977; Glass & Lindblom 1983, GL) and more recently at finite temperature (Martí et al. 1988). Here, we have calculated radial oscillations using new EOS at zero temperature, which include the effects of heavy baryons, hyperons, and pions (Glendenning

1985). But we also computed the oscillation frequencies using EOS similar to those of GL in order to check our results.

We also calculated the radial oscillation frequencies of strange stars, whose existence was proposed by Witten (1984). Further interest in the study of the properties of strange stars has been stimulated by Haensel et al. (1991), who proposed that gamma ray bursts originate as a result of collisions between strange stars at cosmological distances. Even though radial oscillations of strange stars seem to be damped rapidly (Haensel et al. 1989), one cannot completely exclude the possibility of oscillations of strange stars during short time scales. Furthermore, fully relativistic calculations of the oscillation frequencies of strange stars have not been performed previously.

2. Equations of state

The characteristics of the EOS at densities higher than nuclear densities together with their abbreviations as used in this study are shown in Table 1. All are for catalyzed matter at zero temperature. In the case of neutron stars we used the EOS described by Canuto (1974) and composed of those due to Feynman et al. (1949), Baym et al. (1971a, b) for densities below nuclear densities. The fitting points to the EOS above nuclear densities are given in Table 1. The EOS of model V of Bethe & Johnson (1974; hereafter BJ) and Serot (1979; hereafter S) were already used by GL and the same fitting points were used here to facilitate comparison. Two EOS of Glendenning (1985) were also used, one with pions (G1, his case 1) and the other without (G2, his case 2).

The idealized EOS of strange matter is given by (Witten 1984)

$$P = (\rho c^2 - 4B)/3, \quad (1)$$

where P is the pressure, ρ the density, c the speed of light, and B is the bag constant. From this EOS it follows that the density of strange matter cannot be smaller than $\rho_{\min} = 4B/c^2$. The relativistic adiabatic index is defined as

$$\gamma \equiv (1 + \rho c^2/P)(dP/d\rho)/c^2 \quad (2)$$

so that for strange matter one gets the analytic expression

$$\gamma = (4 + 4B/P)/3. \quad (3)$$

Two different bag constants were used here, one with $B = 60 \text{ MeV fm}^{-3}$ (S1) and the other with $B = 70 \text{ MeV fm}^{-3}$ (S2). Even though the value of the bag constant is not well known, these two are reasonable estimates (Alcock & Olinto 1988). The minimal densities $\rho_{\min}$ for these values are given in Table 1.

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Table 1. The EOS at densities higher than nuclear densities as used in this paper

Reference	Abb.	Constituents	Density range [$10^{14} \text{ g cm}^{-3}$]
Bethe & Johnson (1974)	BJ	Neutrons, Δ Hyperons (model V)	> 1.7
Serot (1979)	S	Neutrons	> 1.926
Glendenning (1985)	G1	p, n, e^- , μ^- , Δ , Hyperons, π^- (case 1)	> 1.0101
Glendenning (1985)	G2	p, n, e^- , μ^- , Δ , Hyperons (case 2)	> 1.0101
Witten (1984)	S1	u, d, s quarks $B = 60 \text{ MeV fm}^{-3}$	$> \rho_{\min} \approx 4.3$
Witten (1984)	S2	u, d, s quarks $B = 70 \text{ MeV fm}^{-3}$	$> \rho_{\min} \approx 4.7$

3. Equations for the radial oscillations

A spherically symmetric system is described by the Schwarzschild metric

$$ds^2 = e^v c^2 dt^2 - e^\lambda dr^2 - r^2(d\Theta^2 + \sin^2 \Theta d\Phi^2), \quad (4)$$

where r is the radial coordinate, t the time, Θ and Φ are the angles, and λ, v are metric functions. The function λ may be replaced by

$$e^\lambda = (1 - 2Gm/rc^2)^{-1}, \quad (5)$$

where G is the gravitational constant and m the mass within the radius r . The structure of the star when in hydrostatic equilibrium is described by the Tolman–Oppenheimer–Volkoff equations (Tolman 1939; Oppenheimer & Volkoff 1939)

$$\frac{dm}{dr} = \frac{4\pi r^2}{c^2} \varepsilon, \quad (6)$$

$$\frac{dP}{dr} = -\frac{(P+\varepsilon)}{c^2} \left(\frac{Gm}{r^2} + \frac{4\pi G}{c^2} Pr \right) \left(1 - \frac{2Gm}{rc^2} \right)^{-1}, \quad (7)$$

$$\frac{dv}{dr} = -\frac{2}{(\varepsilon + P)} \frac{dP}{dr}, \quad (8)$$

where $\varepsilon = \rho c^2$ is the energy density. The equations governing radial adiabatic oscillations were first derived by Chandrasekhar (1964). The time dependence of the harmonic oscillations is written as $\exp(i\omega t)$. If Δr is the radial displacement, ΔP the corresponding Lagrangian perturbation of the pressure, $\xi = \Delta r/r$ and $\eta = \Delta P/P$, one gets (Chanmugam 1977)

$$\frac{d\xi}{dr} = -\frac{1}{r} \left(3\xi + \frac{\eta}{\gamma} \right) - \frac{dP}{dr} \frac{\xi}{(P+\varepsilon)}, \quad (9)$$

$$\begin{aligned} \frac{d\eta}{dr} = & \xi \left\{ \frac{\omega^2}{c^2} e^{\lambda-v} \left(\frac{P+\varepsilon}{P} \right) r - \frac{4}{P} \frac{dP}{dr} \right. \\ & \left. - \frac{8\pi G}{c^4} e^\lambda (P+\varepsilon) r + \left(\frac{dP}{dr} \right)^2 \frac{r}{P(P+\varepsilon)} \right\} \\ & + \eta \left[-\frac{dP}{dr} \frac{\varepsilon}{P(P+\varepsilon)} - \frac{4\pi G}{c^4} (P+\varepsilon) r e^\lambda \right], \end{aligned} \quad (10)$$

where ω is the eigenfrequency of the vibration and γ is the relativistic adiabatic index given by Eq. (2). The advantage of

using this system of equations in contrast to others (see GL) is that one does not need any derivatives of γ . In order for $d\xi/dr$ to be finite everywhere the first term on the r.h.s. of Eq. (9) must be finite as $r \rightarrow 0$, and hence one requires at the center that

$$\eta = -3\gamma\xi. \quad (11)$$

Similarly, for $d\eta/dr$ to be finite one requires since $P \rightarrow 0$ and $P/\varepsilon \rightarrow 0$ at the surface that

$$\eta = \xi \left[\left(1 - \frac{2GM}{Rc^2} \right)^{-1} \left(-\frac{\omega^2 R^3}{GM} - \frac{GM}{Rc^2} \right) - 4 \right]. \quad (12)$$

Here M and R are mass and radius of the star, respectively. As Eqs. (9) and (10) form a system of linear differential equations, one is free to choose the scaling factor. Here the eigenfunctions are normalized so that $\xi = 1$ at the center. Equations (9) and (10) with the boundary conditions (11) and (12) form a two point boundary value problem of the Sturm–Liouville type with ω^2 being the eigenvalue. For a given star there are therefore different eigenvalues $\omega_0^2 < \omega_1^2 < \dots < \omega_n^2 < \dots$ with n being the number of nodes (Cox 1980). The $n=0$ mode is also called the fundamental mode.

4. Numerical methods

Equations (6)–(8) for the equilibrium structure of the star were integrated for a given EOS and a given central density ρ_c (see Arnett & Bowers 1977) by using Runge–Kutta integration (Press et al. 1989). Thus ρ , P and v were obtained as functions of the radius r .

Equations (9) and (10) for the radial oscillations with the boundary conditions given by Eqs. (11) and (12) were solved by a shooting method using Runge–Kutta integration (Press et al. 1989). One needs in addition the adiabatic index γ and the metric function λ , the latter being given by Eq. (5). We calculated γ directly from the EOS for densities above nuclear densities as in GL by using a simple difference formula for the derivative in Eq. (2). In the regions below nuclear densities we used the tabulated values of γ whenever given and obtained the values for intermediate densities by interpolation. In the case of strange matter, the adiabatic index was given analytically by Eq. (3). The singularity at the center was overcome by a Taylor expansion.

The eigenvalue ω^2 was determined to an accuracy of one part in 10^6 for a given input of data points for the adiabatic index. However, as γ has to be calculated by a difference equation, except in the case of strange matter, the accuracy of ω^2 is somewhat less. By changing the number of input points, it was estimated that ω^2 is accurate to one part in 10^4 .

5. Results and discussion

The mass versus central density relations for all EOS used in this work are shown in Fig. 1. The relations differ from those of GL, Haensel et al. (1986), and Alcock et al. (1986) by less than 1%. Glendenning (1985) used a different EOS for densities below nuclear densities, but the difference in our calculations is less than 1% as well, since the mass and radius of a neutron star are mainly determined by the high density region.

The period $T=2\pi/\omega$ versus the central density relations for stable stars for the fundamental ($n=0$) and first excited ($n=1$) mode are shown in the Figs. 2 and 3. The periods, masses, and radii for the BJ and S EOS are given in Table 2 together with those due to GL. Table 3 shows these values for strange matter with the S1 EOS. The e-folding time $T=1/|\omega|$ is given if the star is unstable in that mode (negative ω^2).

One can see from Table 2 that the periods for the BJ and S EOS calculated by us differ from those calculated by GL. The difference increases with increasing central density. While the difference is less than 1% at $\rho_c=6 \cdot 10^{14} \text{ g cm}^{-3}$ for the BJ EOS and at $\rho_c=4 \cdot 10^{14} \text{ g cm}^{-3}$ for the S EOS, it is 23% at $\rho_c=1 \cdot 10^{15} \text{ g cm}^{-3}$ for the S EOS. Moreover, we find $\omega_0^2 < 0$, for $\rho_c \geq 3.37 \cdot 10^{15} \text{ g cm}^{-3}$ for the BJ EOS. But GL have a stable star at $\rho_c=3.981 \cdot 10^{15} \text{ g cm}^{-3}$ for this EOS. In both our and GL's calculations the equilibrium adiabatic index is used. Hence the central density ρ_{cmax} , at which the star reaches its maximal mass, must be identical with the critical central density ρ_{ccrit} , above which ω_0^2 is negative (Harrison et al. 1965). For the BJ EOS $\rho_{\text{cmax}}=3.35 \cdot 10^{15} \text{ g cm}^{-3}$, and ρ_{ccrit} as calculated by our oscillation program agrees with ρ_{cmax} to better than about 1%. In the case of strange matter, the EOS and the adiabatic index are given analytically. Here, $\rho_{\text{cmax}}=2.0590 \cdot 10^{15} \text{ g cm}^{-3}$, and its agreement with ρ_{ccrit} is better than 0.4%.

A possible reason for the discrepancy between the calculations of GL and ours can be seen in the systematic difference in the calculated periods. As we pointed out above, the periods do not differ much at low central densities, where the redshift of the star $z=(1-2GM/Rc^2)^{-1/2}-1$ is below $z \approx 0.1$ and general relativistic effects are not very important. But there are large differences at high central densities where $z \approx 0.5$. It is therefore possible that an error in some general relativistic term of the pulsation equation of GL as implemented in their numerical code caused the discrepancy.

At $\rho_c \approx 4 \cdot 10^{14} \text{ g cm}^{-3}$ the period for stars with the G1 EOS increases by about 13% in the fundamental mode compared to the period of the G2 EOS. This is because pion condensation occurs in the G1 EOS for $\rho > 3 \cdot 10^{14} \text{ g cm}^{-3}$ (Glendenning 1985) and causes a drop of the adiabatic index (Shapiro & Teukolsky 1983) with respect to the G2 EOS, which does not contain pions but is otherwise similar. At $\rho > 7 \cdot 10^{14} \text{ g cm}^{-3}$ the pions disappear again due to the condensation of other particles (Glendenning 1985). Now the eigenvalue ω_0^2 for the fundamental mode becomes in the case of a homogeneous, nonrelativistic star (Shapiro &

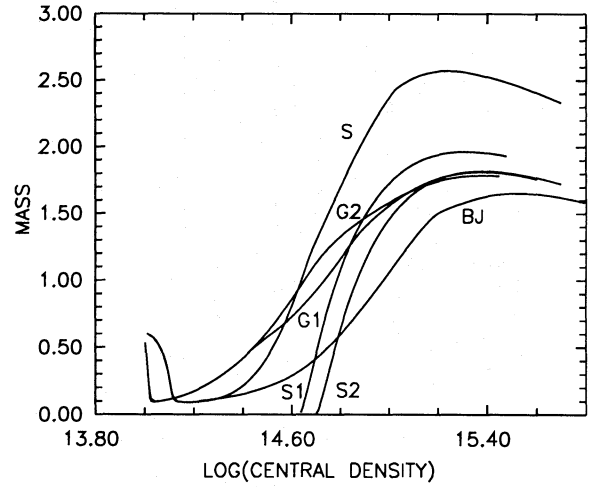


Fig. 1. Mass in $M_{\odot}$ vs. central density in g cm^{-3} for stars constructed from different EOS

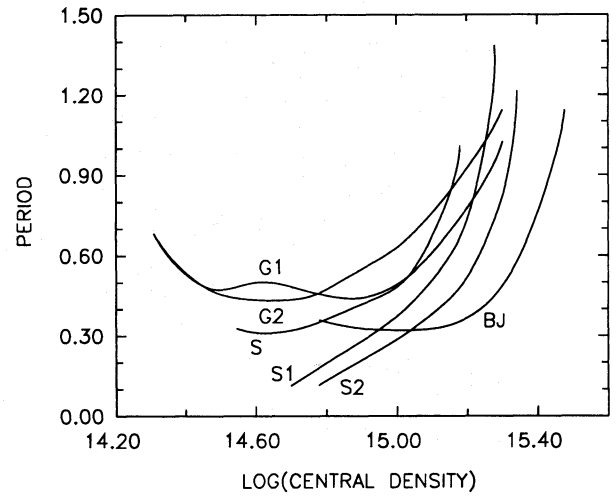


Fig. 2. Period in milliseconds of the fundamental mode vs. central density in g cm^{-3} for stars constructed from different EOS

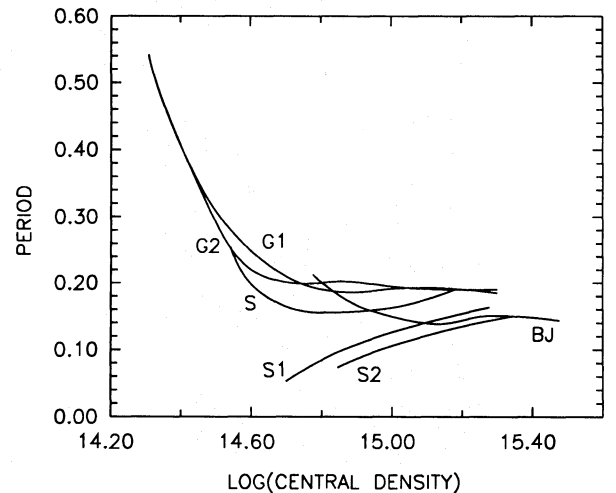


Fig. 3. Period in milliseconds of the $n=1$ mode vs. central density in g cm^{-3} for stars constructed from different EOS

Table 2. Periods of radial oscillations. Values in parenthesis are from Lindblom & Detweiler (1983) for the masses and radii and from Glass & Lindblom (1983) for the periods. For unstable modes (marked by *) the e-folding time is given

EOS	Central density ρ_c [$10^{15} \text{ g cm}^{-3}$]	Mass [$M_\odot$]	Radius [km]	Period	
				$n=0$ [10^{-3} s]	$n=1$ [10^{-3} s]
BJ	3.9810	1.646	9.041	0.1399*	0.1462
		(1.645)	(9.040)	(0.6335)	(0.1530)
	3.5480	1.652	9.264	0.2450*	0.1447
		(1.650)	(9.261)	(0.5817)	(0.1509)
	3.3700	1.65290	9.363	0.8355*	0.1441
	3.3300	1.65290	9.386	4.5880	0.1439
	3.0000	1.650	9.584	1.1436	0.1435
		(1.648)	(9.598)	(0.5412)	(0.1483)
	2.2390	1.610	10.135	0.6004	0.1498
		(1.610)	(10.154)	(0.4922)	(0.1470)
	1.7780	1.546	10.440	0.4121	0.1492
		(1.547)	(10.450)	(0.4305)	(0.1455)
	1.4130	1.422	10.671	0.3388	0.1386
		(1.424)	(10.678)	(0.3556)	(0.1420)
	1.0000	1.059	11.070	0.3205	0.1494
		(1.060)	(11.088)	(0.3181)	(0.1511)
	0.8000	0.818	11.280	0.3267	0.1659
		(0.820)	(11.301)	(0.3245)	(0.1667)
S	0.6000	0.548	11.660	0.3590	0.2130
		(0.549)	(11.697)	(0.3617)	(0.2042)
	2.0000	2.563	12.271	0.1789*	0.1988
		(2.563)	(12.270)		
	1.5000	2.561	12.825	1.0134	0.1914
		(2.561)	(12.825)		
	1.0000	2.370	13.470	0.4825	0.1622
		(2.370)	(13.472)	(0.6123)	(0.1704)
	0.8000	2.080	13.793	0.4195	0.1569
		(2.080)	(13.800)	(0.4197)	(0.1596)
	0.4000	0.801	13.335	0.3113	0.1984
		(0.801)	(13.338)	(0.3082)	(0.1887)
	0.3500	0.567	13.242	0.3277	0.2545

Teukolsky 1983)

$$\omega_0^2 = 4\pi G \rho (3\gamma - 4)/3. \tag{13}$$

Most of the matter of a neutron star is at high densities, and the star can therefore be approximated as being homogeneous (Arnett & Bowers 1977). From Eq. (13) it follows that the value of ω_0^2 scales with γ . Correspondingly, the period of a neutron star constructed with the G1 EOS increases with respect to a star constructed with the G2 EOS, when pions start to condense.

From Figs. 2, 3, and Table 3, one notices that the periods for all modes of the strange stars go to zero when the central density approaches its smallest possible value $\rho_{\min}$. This can be explained as follows. When the central density of a strange star is close to $\rho_{\min}$, the star becomes approximately homogeneous, and furthermore nonrelativistic because of its small mass (Alcock & Olinto 1988), and ω_0^2 is therefore approximately given by Eq. (13). But the adiabatic index of strange matter given by Eq. (3) goes to infinity, because $P \rightarrow 0$ everywhere and not only at the surface.

Table 3. Periods of radial oscillations: S1 EOS. For unstable modes (marked by *) the e-folding time is given

Central density ρ_c [$10^{15} \text{ g cm}^{-3}$]	Mass [$M_\odot$]	Radius [km]	Period	
			$n=0$ [10^{-3} s]	$n=1$ [10^{-3} s]
2.1000	1.963	10.690	0.4499*	0.1667
2.0626	1.9631	10.713	1.4981*	0.1659
2.0551	1.9637	10.707	9.1167	0.1658
1.9000	1.962	10.802	1.3853	0.1631
1.5000	1.929	11.027	0.6570	0.1507
1.0000	1.563	11.143	0.3722	0.1257
0.9000	1.629	11.071	0.3275	0.1178
0.7520	1.399	10.762	0.2599	0.1026
0.7000	1.278	10.537	0.2345	0.09574
0.5000	0.427	7.639	0.1118	0.05253

Therefore $\omega_0^2 \rightarrow \infty$ and the period $T \rightarrow 0$ for the fundamental mode. As the period of a higher mode is always smaller than that of a lower stable mode, the periods of all modes of strange stars have to go to zero, when $\rho_c \rightarrow \rho_{\min}$.

The periods in all modes of the strange star with a bag constant $B = 70 \text{ MeV fm}^{-3}$ are lower than those of the strange star with a bag constant of $B = 60 \text{ MeV fm}^{-3}$. This is essentially a shift to the right of the graph of the period versus central density with increasing bag constant, as $\rho_{\min}$ increases with increasing bag constant.

6. Conclusions

The periods and e-folding times of radial oscillations for neutron stars and strange stars were calculated. These calculations were compared to those made by GL for the BJ and S EOS. It was found that differences occur which increase with increasing central density. If the equilibrium adiabatic index is used, then the critical densities above which the star becomes unstable to radial perturbations, as calculated by the oscillation equations and as derived from the mass to central density relation, must be the same (Harrison et al. 1965). This is consistent with our results, but not with those of GL.

The effect of pion condensation on the period of radial oscillations was seen in the calculations for the two EOS due to Glendenning (1985). One contains pion condensation for $\rho > 3 \cdot 10^{14} \text{ g cm}^{-3}$, which causes a drop of the adiabatic index and a change in the structure, and ultimately causes a significant increase of the period in the fundamental mode compared to the one without pions.

We also show that the periods of radial oscillations of strange stars behave very differently from those of neutron stars. Instead of having a smallest possible period for a given mode, the periods of all modes for strange stars go to zero when the central density of the strange star approaches its smallest possible value. This behavior arises because $\gamma \rightarrow \infty$ when $P \rightarrow 0$, which occurs when the central density of the strange star is close to its smallest possible value.

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