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3C 345: IS THE VARIABILITY OF QUASARS NONLINEAR?[1]

R. VIO,[2] S. CRISTIANI,[2] O. LESSI,[3] AND L. SALVADORI[2]
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ABSTRACT

We analyze the optical light curve of the OVV quasar 3C 345, showing that it is generated by a nonlinear and nonstationary stochastic process. The observed characteristics of the 3C 345 variability, such as the varying power spectrum and the strong luminosity bursts, are easily accounted for in this mathematical framework. Conventional algorithms suitable for linear analysis (power spectrum, structure function, covariance function, ...) are not useful in this case and different approaches have to be devised. The physics acting in the central regions of QSOs are probably nonlinear and dominated by a quantity subject to random variations.

Subject heading: quasars

1. INTRODUCTION

In recent years the optical variability of QSOs has been extensively studied by several authors. The reason for such a widespread interest is that variability can provide direct information about the central "engine" of quasars. However, in spite of the huge quantity of data collected in more than two decades of observations, variability has provided only few indications for the understanding of the QSO scenario. The studies have followed two directions. On the one hand, correlations between the variability and physical parameters such as redshift, absolute luminosity, and radio properties have been investigated using large samples of objects (Bonoli et al. 1979; Barbieri et al. 1983; Netzer & Sheffer 1983; Cristiani 1986; Pica et al. 1987; Trevese et al. 1989; Cristiani, Vio, & Andreani 1990). On the other hand, by investigating detailed light curves of individual objects, mainly OVVs, investigators have tried to characterize statistically the QSO variability phenomenon (e.g., Bregman et al. 1988; Webb et al. 1988; Kidger 1989; Barbieri et al. 1990). This latter task turned out to be particularly difficult due to the erratic nature of the OVV light curves.

In this paper we tackle the QSO variability problem by studying 3C 345, one of the quasars with the longest and most uniformly sampled optical record. Its light curve shows short periods of intense activity within long interludes of quiescence, a behavior lacking up to now any convincing statistical description.

2. THE DATA

Photometric data about 3C 345 have been found in the literature from the sources listed in Kidger (1989) (exactly the same references have been used). In addition to these measurements, we have analysed a number of plates taken at the Asiago Observatory with the 67/92/215 cm Schmidt Telescope. The 103a-O Eastman emulsion and a Schott GG 13 filter were used to match the *B* system.

The images of 3C 345 and 14 reference stars, calibrated by Goldsmith & Kinman (1965), Kinman et al. (1968), and Angione (1971), have been digitized on each plate with the PDS of the Padova Observatory. Small areas of 64 × 64 pixels have been scanned around each object, adopting an aperture of 25 × 25 μm^2. Magnitudes have then been derived with the MIDAS package (Banse et al. 1983) running on the VAX8530 of the Department of Astronomy in Padova.

Table 1 lists the magnitudes obtained at the various epochs. A comparison with the values given by Barbieri et al. (1977) on the basis of visual estimates provides a ΔB (this paper − Barbieri et al.) = 0.02 ± 0.15.

The resulting light curve obtained merging the present data with those found in the literature is shown in Figure 1*a*. Note that it is given in flux units (the transformation from magnitudes to fluxes has been carried out according to Allen 1963). Fluxes are the only meaningful physical quantities, and the use of magnitudes would make the light curve artificially compressed, distorting the subsequent statistical analysis.

3. DATA ANALYSIS

3.1. *Problematics of Data Analysis*

An important finding about the quasar variability has been the recognition of its stochastic nature. In 1968 Manwell & Simon realized that the time series of 3C 273 showed a "*noise component*" not associated with the measurement errors. This means that the dynamical evolution of the system cannot be described by deterministic differential equations. Manwell & Simon suggested also that the temporal behavior of 3C 273 could be due to a *shot noise* process (see also Terrel & Olsen 1970). This model, however, is not acceptable: *shot noise* processes have a power spectrum which does not change (within the statistical fluctuations) with time and are not able to produce time series characterized by sudden bursts of very large amplitude, contrary to what has been observed for the QSO time series (e.g., Kidger 1989; Barbieri et al. 1990). In general this limitation is common to all linear stochastic processes (to which *shot noise* belongs) (Tong 1983). This implies that the dynamical evolution of quasars is nonlinear (i.e., described by nonlinear stochastic differential equations).

We have first tried to quantify the nonlinear hypothesis for the variability of 3C 345, applying a test suggested by Keenan (1985), that evaluates the probability for the Volterra expansion of the time series to have terms beyond the linear one. Since this test requires equally spaced data, a regularization of the observed time series has been necessary. Techniques for the reconstruction of uneven time series such those suggested by Scargle (1989) or Roberts, Lehar, & Dreher (1987) cannot be

[1] Based on material collected at the Asiago Astrophysical Observatory.

[2] Dipartimento di Astronomia della Università di Padova, Vicolo dell' Osservatorio 5, I-35122 Padova, Italy.

[3] Dipartimento di Statistica della Università di Padova, Via S. Francesco 33, I-35122 Padova, Italy.

TABLE 1

LIST OF THE MAGNITUDES OF 3C 345

Plate Number	B (mag)	σ_B (mag)	Date (yy-mm-dd)	Julian Day (2,400,000+)
732	15.86	0.09	1967 Jul 1	39,673
810	16.34	0.04	1967 Aug 9	39,712
1684	17.20	0.07	1969 Jun 25	40,033
2397	16.51	0.10	1969 May 23	40,365
2561	17.56	0.02	1969 Oct 5	40,500
2615	17.53	0.08	1969 Oct 11	40,506
2652	17.33	0.06	1969 Oct 14	40,509
3279	16.93	0.02	1970 Apr 26	40,703
3281	16.90	0.07	1970 Apr 29	40,706
3290	16.76	0.05	1970 Apr 30	40,707
3303	16.96	0.06	1970 May 1	40,708
3322	16.83	0.04	1970 May 2	40,709
3330	17.11	0.11	1970 May 10	40,718
3332	17.20	0.07	1970 May 11	40,718
3362	17.05	0.07	1970 May 26	40,733
3370	17.06	0.08	1970 May 27	40,734
3379	16.99	0.09	1970 May 31	40,738
3387	16.97	0.07	1970 Jun 1	40,739
3401	17.09	0.05	1970 Jun 12	40,750
3415	17.11	0.12	1970 Jun 23	40,761
3424	16.69	0.07	1970 Jun 27	40,765
3453	16.87	0.05	1970 Jul 8	40,776
4367	16.05	0.05	1971 May 15	41,087
4375	16.23	0.06	1971 May 16	41,088
4380	16.21	0.05	1971 May 17	41,089
4399	15.65	0.07	1971 May 27	41,099
4423	15.99	0.07	1971 Jun 16	41,119
4424	15.98	0.09	1971 Jun 17	41,120
4428	16.13	0.05	1971 Jun 19	41,122
4433	16.06	0.06	1971 Jun 20	41,123
4441	15.94	0.11	1971 Jun 26	41,129
4449	15.77	0.14	1971 Jun 28	41,131
4458	15.62	0.03	1971 Jun 30	41,133
7255	16.77	0.14	1974 Jul 10	42,245
7459	16.48	0.15	1974 Sep 16	42,307
7934	16.70	0.06	1975 May 2	42,535
7970	16.69	0.07	1975 Jun 4	42,568
8552	16.12	0.05	1976 May 2	42,901
8977	16.40	0.11	1977 Apr 12	43,246
9043	16.43	0.06	1977 Jun 14	43,309
9047	16.57	0.11	1977 Jun 17	43,312
12063	15.57	0.07	1983 Apr 15	45,440
12076	15.69	0.07	1983 May 4	45,459
12153	15.86	0.10	1983 Aug 4	45,551
12165	15.90	0.08	1983 Aug 9	45,556
12491	16.41	0.08	1984 Apr 25	45,816
12850	17.03	0.08	1985 Jun 11	46,228
12856	16.44	0.08	1985 Jun 20	46,237
12903	16.54	0.12	1985 Aug 9	46,287
12911	16.53	0.06	1985 Aug 12	46,290
12946	16.31	0.10	1985 Aug 19	46,297
12975	16.55	0.08	1985 Sep 8	46,317
12982	16.51	0.09	1985 Sep 10	46,319
13000	16.69	0.06	1985 Sep 14	46,323
13361	16.86	0.04	1986 Jul 10	46,622
13369	16.89	0.08	1986 Aug 2	46,645
13384	16.62	0.11	1986 Aug 7	46,650
13385	16.83	0.04	1986 Aug 7	46,650
13818	17.21	0.07	1987 Jun 1	46,948
13819	17.06	0.08	1987 Jun 22	46,969
13847	17.11	0.09	1987 Jul 23	47,000
13866	17.37	0.06	1987 Aug 18	47,026
14149	17.45	0.07	1988 Jun 1	47,322
14166	17.22	0.07	1988 Jul 7	47,350
14179	17.60	0.08	1988 Aug 9	47,383
14185	17.51	0.13	1988 Aug 10	47,384
15541	17.71	0.07	1989 May 5	47,652
14586	17.47	0.06	1989 Aug 3	47,742
14599	17.82	0.08	1980 Aug 9	47,748

applied here because they require a certain "regularity" of the signal, and clearly this is not the case for 3C 345. Therefore we have limited ourselves to average the data in temporal intervals of about 11 days (800 bins) and fill the empty bins with a linear interpolation between adjacent observations, "dirtied" by a white-noise process with the same variance of the signal during quiescence. Figure 1*b* shows the time series of 3C 345 obtained in this way. The bin size is the compromise between two contrasting conditions:

1. The bins cannot be too long to avoid a signal sampling with a time scale larger than that characteristic of the process. If this condition is not verified the possibility to recover the "physics" of the system is lost.

2. The bins cannot be too short to avoid the necessity of filling too many gaps.

With a bin size of 11 days, the number of empty bins is already 53%. On the other hand, with a larger bin size, a number of "secondary" bursts in the light curve disappear.

After the regularization of the sampling, the Keenan test shows that the light curve of 3C 345 is nonlinear at a confidence level of 95%. We stress that the resulting nonlinearity can not be an artifice of the technique used for regularizing the sampling of the time series, since the gaps have been filled by means of a linear process. Numerical simulations carried out with a set of linear stochastic models and a sampling similar to that of 3C 345 confirm this result.

We have then tackled the problem of a statistical description of the nonlinearity of the light curve of 3C 345. Such an operation has to be carried out with great care since we are studying a continuous signal by means of a discrete model. In time series data analysis, this is a rather common situation; the practical unavoidability of this approach and the consequent problems are delineated for example by Pandit & Wu (1975). In the following we will deal with the discrete light curve of 3C 345 with all the consequent limitations. Even with this simplification the task is very difficult: the "explosive" evolution of the time series suggests that certain moments of the process do not exist. Moreover, although the last decade registered a growing interest of statisticians concerning nonlinear time series (Subba Rao & Gabr 1984; Tong 1983; Ozaki 1985), most of the work has been limited to such specific problems as limit cycle, and amplitude-dependent frequency and jump phenomena. The only nonparametric approach available for the nonlinear time series analysis and applicable to our case is represented by the "state-dependent models" (SDM) introduced by Priestley (1980, 1982). This general class of models may give an indication of the specific type of nonlinearity which is appropriate to a particular situation, and whether a linear model might prove equally satisfactory (see below). These models have not yet been fully developed, and many problems remain to be solved; it is, however, interesting to apply them to the data of 3C 345.

3.2. *State-dependent Models* (*SDM*)

Since a comprehensive and detailed introduction to the SDM is given in the excellent book of Priestley (1988), only a general description is provided here.

The most general form of a time series model can be expressed as (Priestley 1988)

$$X_t = h(X_{t-1}, X_{t-2}, \ldots, e_t, e_{t-1}, \ldots) \tag{1}$$

where $h(\cdot)$ is a general function, and X_t and e_t denote, respec-

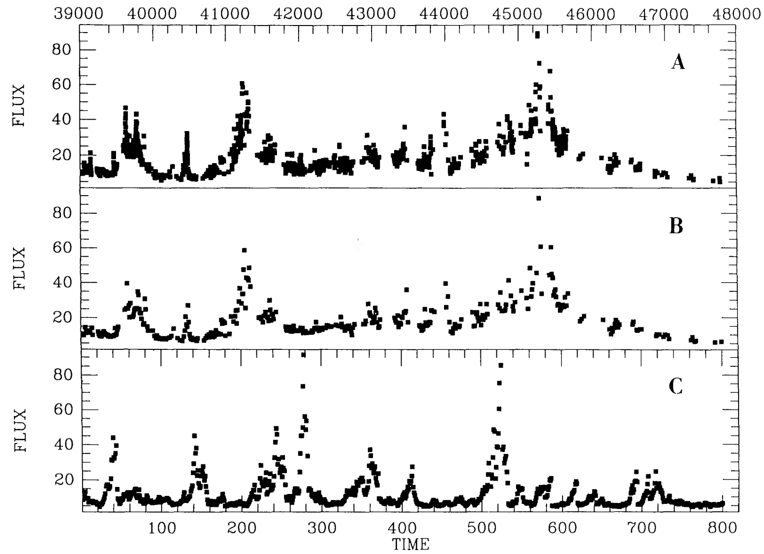

FIG. 1.—(*a*) Original optical light curve of 3C 345. Time in units of 11 days; flux in units of 10^{-12} ergs s^{-1} cm^{-2}. The top axis units are in Julian days (2,400,000+). (*b*) Optical light curve of 3C 345 binned in temporal intervals of 11 days. Time in units of 11 days; flux in units of 10^{-12} ergs s^{-1} cm^{-2}, (*c*) Simulation of a bilinear model of order (1, 0, 1, 1) with $a^* = 0.90$, $b = 0.79$, $\mu_e = 0.07$, $\mu_0 = 5.50$, $\sigma_e^2 = 0.15$. Time in units of 11 days; flux in units of 10^{-12} ergs s^{-1} cm^{-2}.

tively, the values of the series and of a strictly white-noise process at the time t. As it stands, equation (1) is "infinite dimensional" in the sense that it involves a relationship between infinitely many variables. This form is very difficult to treat, and therefore we have to limit ourselves to a "finite dimensional" one. In other words we assume that the relationship between X_t and the "past history" of the series can be described, or approximated, in terms of finitely many values of past $\{X_t\}$ and $\{e_t\}$, so that we may write, say,

$$X_t = h(X_{t-1}, \ldots, X_{t-p}, e_{t-1}, \ldots, e_{t-q}) + e_t \quad (2)$$

or

$$X_t = h(x_{t-1}) + e_t \ ,$$

where x_t denotes the vector:

$$x_{t-1} = (X_{t-p}, \ldots, X_{t-1}, e_{t-q}, \ldots, e_{t-1}) \ .$$

With this formulation, e_t plays the role of the innovation process for X_t, and the function h describes the information on X_t contained within its past history. Note that at the time $(t-1)$, the evolution of the process described by equation (2) is completely determined by the vector x_{t-1} (together with the future values of e_t), and thus x_{t-1} may be interpreted as the "state vector" at time $(t-1)$.

In the classical statistical literature $h(\cdot)$ is always a linear function, and in this case the Wold's theorem guarantees that for a given time series always exists an "unique" MA (moving average) representation (Scargle 1981; Wei 1990). If $h(\cdot)$ is a nonlinear function, a theorem corresponding to that of Wold does not exist. Therefore, without "a priori" information, in the nonlinear case we have the problem to determine the functional form of $h(\cdot)$. Expanding equation (2) in a Taylor series about an arbitrary but fixed time point t_0 and using only the linear terms, we obtain

$$X_t + \sum_{u=1}^{p} \phi_u(x_{t-1})X_{t-u} = \mu(x_{t-1}) + e_t + \sum_{u=1}^{q} \psi_u(x_{t-1})e_{t-u} \ , \quad (3)$$

where $\phi_u \sim -\partial h(\cdot)/\partial X_{t-u}$, $\psi_u \sim \partial h(\cdot)/\partial e_{t-u}$, and $\mu(\cdot)$ is a sort of mean level.

This is the basic model of Priestley, and it is called state-dependent model of order (p, q). Formally equation (3) is identical to a linear ARMA (AutoRegressive-Moving Average) model. Now, however, the coefficients ϕ_u, ψ_u and the mean μ at time $(t-1)$ are not constant but depend on the "state vector" x_{t-1}. In other words with equation (3) we assume that a general time series may be considered the realization of a *locally* linear ARMA model whose evolution at time $(t-1)$ is governed by the "state" of the process at the same instant. Of course the fitting problem for SDM consists in the determination of the coefficients μ, ϕ_u, and ψ_u. However, these coefficients depend on the state vector x_t, and the problem thus becomes the estimation of the functional forms of this dependency. Provided that the coefficients are smooth functions of the state vector, such a dependency may be determined by using an optimal recursive processing algorithm similar to the Kalman filter (Priestley 1980).

The useful characteristic of the SDM is that these models can be used without any "a priori" assumption on the possible nonlinearity present in the data. In effect the functional forms of $\phi_u(x_{t-1})$ and $\psi_u(x_{t-1})$ strictly depend on the "type" of nonlinearity inherent in the time series. In spite of the attractive theoretical capabilities of SDM, in practice some fundamental problems remain to be solved:

1. A general method is not yet available to determine the order (p, q) of the SDM necessary to identify a nonlinear model from a given time series.

2. It is difficult, expecially with few data, to estimate the parameter "surfaces" $\phi_u(x_{t-1})$ and $\psi_u(x_{t-1})$ when the dimension of the state vector x_{t-1} is greater than 1 or 2 (however, see Priestley 1988 and Priestley & Chao 1972).

3. The estimate of the parameters of the nonlinear models provided by the SDM is not accurate. In particular, as shown by numerical simulations, the more noisy are the data the less reliable the estimates become.

At the moment the only way to partially overcome the first two problems is to try low-order SDM and to see if the corresponding parameters "surfaces" give indications of a determi-

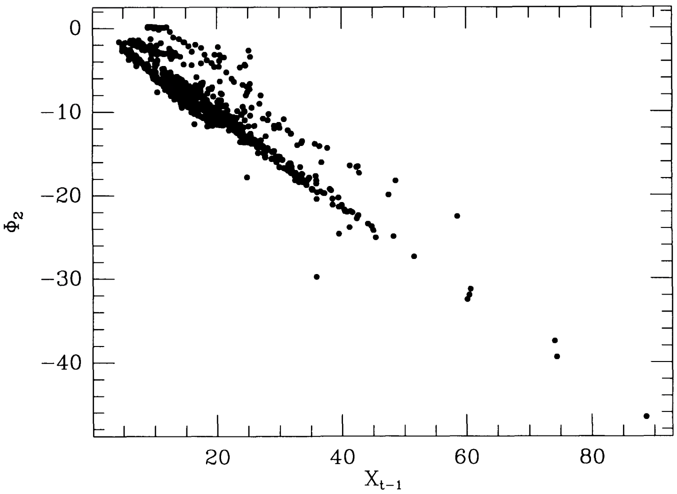

FIG. 2.—Diagram of ϕ_2 vs. X_{t-1} produced by the SDM

nate type of nonlinearity. On the contrary the third difficulty is not so serious, since the principal aim of SDM is to identify the type of nonlinearity present in the data. However in presence of " noisy " data it is safer to use an SDM of type AR:

$$X_t + \sum_{u=1}^{p} \phi_u(x_{t-1})X_{t-u} = \mu(x_{t-1}) + e_t \;, \qquad (4)$$

since, as numerical simulations have shown, it is more robust than the ARMA and MA ones. The reason is that in the case of an " ARMA " SDM, contrary to an " AR " one, the state vector x_{t-1} depends on the e_t quantities that, being unknown, can be only estimated.

3.3. *Results*

We have fitted a SDM of type AR(2) to the regularized 3C 345 time series. AR models have been chosen because, as stated above, they make possible a more robust representation of a stochastic process with respect to the ARMA and MA ones. At the same time, lacking at the moment a general criterion, the choice of the order is more subjective and dictated by the necessity of simplicity in the interpretation of results. Figure 2 shows the plot of the ϕ_2 parameter versus the X_{t-1} values. A linear structure is visible in the plot. Such a linear structure is typical of the so-called bilinear models of order (1, 0, 1, 1) (Priestley 1988, p. 130):

$$X_t = a^* X_{t-1} + bX_{t-1}e_{t-1} + e_t \;,$$

where a^* and b are constant coefficients and $\{X_t\}$ and $\{e_t\}$ are the observed time series and a strict white-noise process (innovation), respectively. In this model the a^* coefficient represents the linear component of the process, whereas the b represents the nonlinear one. If the innovation has mean μ_e and the process has a "pedestal" μ_0, the model given above can be written as

$$X_t = X_t^* - \mu_0$$

$$X_t = \mu_e + aX_{t-1} + bX_{t-1}e_{t-1} + e_t \;,$$

where $a = a^* + b\mu_e$ and $\{X_t^*\}$ is the observed time series.

Bilinear models have been introduced into the statistical literature by Granger & Andersen (1978). A more general and systematic study is given in the monograph of Subba Rao & Gabr (1984). Bilinear models are often used in the field of system control since they arise in a natural manner for many physical processes (Mohler 1973). An interesting characteristic of the bilinear models is their capability of producing strong bursts (Tong 1983).

In order to determine the parameter values of the bilinear representation of the 3C 345 light curve, we have tried to apply the least-squares algorithm developed by Subba Rao & Gabr (1984). Our attempt, however, has not been successful because the algorithm does not converge. To avoid this problem we have developed a method based on the singular value decomposition (SVD) technique (see Appendix). An interesting characteristic of this method is that, although equally spaced data are required, the filling of the occurring gaps is not necessary. Using continuous sequences of data with minimum length of five points (see Appendix), the parameter values found are as follows:

$$\hat{a}^* \simeq 0.90 \;, \quad \hat{b} \simeq 0.79 \;, \quad \hat{\sigma}_e^2 \simeq 0.15 \;, \quad \hat{\mu}_e \simeq 0.07 \;, \quad \hat{\mu}_0 \simeq 5.5 \;,$$

where the caret denotes estimate.

Figure 1*c* shows a typical simulation that is obtained with these parameter values. It has to be noticed that Figure 1*c* is *not* a fit to the 3C 345 data but simply a realization of a bilinear process.

When comparing Figure 1*c* with Figure 1*b*, one must consider that the light curve of 3C 345 is irregularly sampled. For example, the trend visible in Figure 1*b* between 600 and 800 time units is probably an appearance due to the scarcity of data in that interval. With this in mind, the similarity between the two figures is striking; the model has been able to get the " explosivity " of the 3C 345 light curve.

The values determined for the parameters provide a model both nonstationary and noninvertible (Granger & Andersen 1978). This means that the time series of 3C 345 has statistical properties (essentially mean and variance) which vary with time and that it is not possible to estimate the $\{e_t\}$ sequence from the recurrent formula:

$$\hat{e}_t = X_t - (\mu_e + aX_{t-1} + bX_{t-1}\hat{e}_{t-1}) \,.$$

Since this formula is fundamental for the least-squares methods, it can be easily understood why the algorithm of Subba Rao & Gabr has failed to fit the 3C 345 data.

The bilinear models obtained from the light curves of 3C 345 regularized with different bin sizes (from 5 to 20 days), are still able to reproduce the observed bursts, although with some variation of the value parameters, as could be expected (see Pandit & Wu 1975). In particular the parameters $\hat{a}^*$ and $\hat{\mu}_e$ vary in a narrow range (respectively, 0.9–1.0 and 0.05–0.2), while the parameter $\hat{b}$ is subject to larger variations (0.6–1.2) anticorrelated with the variance $\hat{\sigma}_e$ (0.7–0.1) and $\hat{\mu}_0$ is less constrained (1–6). In any case the basic characteristics of the time series (noninvertibility and nonstationarity) are preserved. We stress that this characterization of the process under study is the relevant point in our statistical analysis rather than the particular values obtained for the parameters of the discrete model. A more detailed study of the identifiability of continuous dynamical system from discrete time series is under development (Vio et al. 1991).

4. DISCUSSION AND CONCLUSIONS

As emphasized above, the importance of the result obtained in the previous paragraph resides in the fact that a simple nonlinear stochastic model is able to reproduce the main characteristics of the light curve of 3C 345.

The physical implication is that the luminosity of 3C 345 is regulated by a quantity subject to random variations. No periodicities, or "transient periodicities" as suggested by some authors (e.g., Webb et al. 1988), are necessary. The model, being nonstationary, generates time series whose power spectra show strong peaks at low frequency changing both their position and amplitude with time, as observed in 3C 345. In this framework, the strong luminosity bursts are due to small perturbations of a nonlinear system close to an unstable state: quasars could manifest the tendency for large-scale systems to become organized into a state in which they are just on the edge of the instability (cf. Bak, Tang, & Wiesenfeld 1987; Maddox 1990). These conclusions are not necessarily in contrast with the results obtained by Barbieri et al. (1990), who interpreted the light curve of the QSO 3C 446 in terms of a periodic nonstationary process with a period of about 1540 days. In fact, many nonlinear processes can generate periodic or semiperiodic time series.

In the last years it has become evident that also some simple nonlinear systems, the so-called chaotic systems, can show "noisy" time series, even if their dynamics is governed by deterministic equations. However the chaotic processes are stationary ones (Scargle 1990) and therefore they cannot explain time series characterized by sudden bursts of large amplitude.

Many of the usual statistical methods, suitable for linear cases, have to be dropped in the nonlinear context. For example, the power spectra of nonlinear processes generally show peaks at certain frequencies and at their sum and/or difference. This phenomenon, called *frequency multiplication* (Priestley 1988), is due to the interaction of the harmonics characteristic of the process. While the power spectrum is not able to find out which peaks are harmonically related, the bispectrum, and in general all the polyspectra, can do it and therefore help to understand the statistical characteristics of the process (Priestley 1988; Nikias & Raghuveer 1987). However, without "a priori" information on the process, these methods provide results which are of difficult interpretation. Further work on the statistical algorithms is therefore necessary, but at the same time it would be of extreme interest to translate the models for compact accreting objects (see, for example, Abramowicz & Szuszkiewcz 1989) in terms of light curves to be analyzed with the present algorithms.

It is a pleasure to thank C. Barbieri for helpful discussions and suggestions and P. Andreani and F. La Franca for carefully reading the manuscript.

APPENDIX

THE SVD ALGORITHM

For a bilinear model, once the possible pedestal μ_0 has been removed from the observed data (i.e., $X_t = X_t^* - \mu_0$ where X_t^* is the observed time series), we can write a set of equations:

$$X_t - aX_{t-1} - \mu_e = bX_{t-1}e_{t-1} + e_t \tag{A1}$$

forming, for $t = 2, \ldots, N$ (the first equation is lost since e_0 is not known), a system of $N - 1$ nonlinear equations in $N + 3$ unknowns. If for a moment we suppose to know the a, b, and μ_e parameters, the system (A1) becomes linear in the unknowns e_t, and with the additional condition:

$$\sum_{t=1}^{N} e_t = 0$$

it can be written in matricial form:

$$\Gamma\epsilon = \zeta \, , \tag{A2}$$

where

$$\epsilon^T = \{e_1, e_2, \ldots, e_N\}$$

$$\zeta^T = \{X_2 - \mu_e - aX_1, X_3 - \mu_e - aX_2, \ldots, X_N - \mu_e - X_{N-1}, 0\}$$

$$\Gamma = \begin{pmatrix} bX_1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & bX_2 & 1 & \ldots & 0 & 0 \\ 0 & 0 & bX_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & bX_{N-1} & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

(the symbol ^{T} denotes transposed).

In principle by solving the system (A2) it is possible to obtain the $\{e_t\}$ sequence; however, due to round-off errors, often the matrix Γ can be singular. In this case the solution is not unique. Usually there will be an $N - M$ dimensional family of solutions, where M is the rank of the matrix Γ. The singular value decomposition (SVD) is a particular technique that is suited for solving this kind of problem (see Press et al. 1988). More precisely, the SVD picks, among all possible solutions, the one with the smallest length $|\epsilon|^2$. Moreover if the elements of the vector ζ are affected by measurement errors, the system (A2) could not have solution. In this second case the SVD is able to construct a "solution" vector ϵ that will not exactly solve $\Gamma\epsilon = \zeta$, but among all possible vectors ϵ, it will do the closest possible job in the least-squares sense. In other words the SVD finds the solution that minimizes the quantity:

$$S \equiv |\Gamma\epsilon - \zeta| .$$

In reality the a, b, μ_e and μ_0 coefficients must be determined. To do so the following procedure has been adopted:

1. Starting "guesses" for a, b, μ_e, and μ_0 are chosen.
2. The corresponding $\{e_t\}$ are calculated with the system (A2) solved through the SVD.
3. The "badness" of the solution is estimated by computing a "penalty" described below.
4. The "penalty" is minimized by using a nonlinear fitting algorithm (cf. the routine GRIDLS of Bevington 1969), which varies a, b, μ_e, and μ_0 and for each combination of these parameters compute, as in step (2), the corresponding $\{e_t\}$.

The central point of this procedure is the choice of the "penalty". Since in the model (A1) the e_t are independent, we have searched for the solution which fulfills this condition by adopting as "penalty" the product of two factors CORR and GAUSS, that give the degree of correlation and of "non-gaussianity" for the innovations, respectively. The quantity CORR is defined as

$$\text{CORR} = \left(\prod_{i=1}^{n} \hat{e}_i\right)^{1/n} - \hat{\mu}_e ,$$

and GAUSS is the statistics of D'Agostino (D'Agostino 1986).

An interesting property of this approach is the possibility of applying it to data with gaps. In principle, in fact, it is sufficient to eliminate from the system (A2) the equations corresponding to the missing data, and to solve it as usual. In practice, especially with "noisy" data, it is more convenient to maintain in system (A2) only the equations relative to continuous sequences of points with a minimum length of at least four points.

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