THE ASTROPHYSICAL JOURNAL, 367:601–611, 1991 February 1

ADIABATIC PROPERTIES OF PULSATING DA WHITE DWARFS. I. THE TREATMENT OF THE BRUNT-VÄISÄLÄ FREQUENCY AND THE REGION OF PERIOD FORMATION

P. BRASSARD, G. FONTAINE, AND F. WESEMAEL
Département de Physique, Université de Montréal
S. D. KAWALER
Center for Solar and Space Research, Yale University; and Department of Physics, Iowa State University
AND
M. TASSOUL
6000 Chemin Deacon, Montréal, Québec
Received 1989 October 3; accepted 1990 July 25

ABSTRACT

We present the first results of a series of investigations aimed at exploring the systematics of the gravity-mode period structure of models of pulsating DA white dwarfs in the adiabatic approximation. In this first paper, we consider the basic issue of the region of period formation in a degenerate star. This is motivated by a paper by Pesnell which challenges the usually accepted concept of envelope modes for white dwarfs. It is essential here to treat carefully the Brunt-Väisälä frequency, which is the oscillation frequency of fluid elements when buoyancy is the restoring force. We show that it *must* be appropriately transformed in order to obtain reliable numerical results in degenerate stellar models. This is because the Brunt-Väisälä frequency is defined in terms of a *difference* between two numbers which become nearly equal in highly degenerate matter, thus causing serious numerical problems and systematic errors. We derive an alternative expression which is generally valid for multicomponent, nonideal, partially degenerate, and partially ionized plasmas such as those encountered in white dwarf envelopes. We use this expression to compute the period structure of the same white dwarf considered by Pesnell. For one particular model he investigated we find a period and a weight function distribution that are quite different from his results. However, in a numerical experiment based on the usual relation for the Brunt-Väisälä frequency (which leads to unreliable values in degenerate matter), we recover *exactly* the results of Pesnell. We conclude that the implicit numerical differencing used in the Lagrangian pulsation code of Pesnell leads to very serious difficulties when used with models of degenerate stars, and reaffirm the correctness of the fundamental result of other investigations: gravity-mode pulsations in white dwarfs are truly envelope modes. The implications of our findings on the work of Cox *et al.*, based on the same Lagrangian approach, are also discussed. We find that the basic period structure of *all* their models suffers from obvious symptoms of theory breakdown; this implies that their nonadiabatic results must be regarded as questionable.

Subject headings: stars: interiors — stars: pulsation — stars: white dwarfs

I. INTRODUCTION

It is now well established that ordinary DA white dwarfs (i.e., stars with an almost pure hydrogen surface composition) go through an instability strip during their cooling history. The instabilities manifest themselves in terms of multiperiodic luminosity variations which result from the superposition of excited nonradial pulsation modes of the g (gravity) type. Typical light curves of pulsating DA white dwarfs (also known as ZZ Ceti stars) show peak-to-peak variations ranging from $\sim$0.005 mag to upward of 0.30 mag. The periods of the observed pulsation modes range between 100 and 1200 s. Although more than a dozen different pulsation modes are simultaneously observed in the more complex ZZ Ceti pulsators, it is clear that these modes are selectively chosen by a filtering mechanism because the nonradial g-mode spectra of DA white dwarfs are very rich and *many* more modes are available. Recent analyzes locate the empirical ZZ Ceti instability strip in the range of effective temperature 13,000 K $\gtrsim T_e \gtrsim$ 11,400 K (Wesemael, Lamontagne, and Fontaine 1986; Lamontagne, Wesemael, and Fontaine 1987; Daou *et al.* 1990). By the time a cooling DA white dwarf first enters this strip, its effective temperature has dropped sufficiently that hydrogen in the superficial layers is recombining. It is this partial ionization phenomenon that triggers instabilities against nonradial pulsation modes in this type of star. Indeed, it has been shown independently by several groups that the pulsation modes are driven near the base of the thin surface convection zone which forms as a consequence of the partial ionization of hydrogen and the concomitant increase of the opacity in the outer layers (Winget 1981; Dziembowski and Koester 1981; Dolez and Vauclair 1981; Winget *et al.* 1982; Winget and Fontaine 1982; Starrfield *et al.* 1982; Cox *et al.* 1987). Hence, the blue edge of the ZZ Ceti instability strip can be seen as a natural consequence of the recombination of hydrogen as cooling proceeds.

The ZZ Ceti instability strip provides us with a "window" through which we can seismologically probe the interiors of white dwarfs. As was well put by Winget (1986), such an opportunity is of intense interest because "the white dwarfs contain an archeological record of the history of star formation in our Galaxy." For instance, the period structure of a ZZ Ceti star depends on its total mass, its effective temperature, and the masses of its outer helium and hydrogen layers which surround the core in a typical DA white dwarf. While the effective temperature as well as the gravity (and, thus, the mass through a

suitable mass-radius relationship) of a pulsating white dwarf can be obtained through classical model atmosphere techniques in conjunction with properly time-averaged observations of its spectrum, inference about its chemically stratified structure can be made by comparing its observed periods with those of theoretical models. The question of how much hydrogen and helium are left over in evolutionary phases prior to the white dwarf stage is an important one which is currently the subject of a lively debate (see, e.g., Fontaine and Wesemael 1987; Shipman 1989). Likewise, it is possible to infer the core composition of a pulsating white dwarf through a comparison of measurements of rates of period change with rates predicted by evolution theory as a result of structural changes due to cooling. Other benefits of white dwarf seismology include the determination of rotation periods and magnetic field strengths through observations of frequency splittings in the Fourier spectrum of a pulsator.

While our understanding of ZZ Ceti stars has improved tremendously in the last decade (see, for example, the reviews of Winget 1986, 1988; Kawaler and Hansen 1989 for recent developments in this field), we have not yet reached the point where the full potential of astroseismology can be exploited and the internal constitution of white dwarfs can be inferred. This is because the g-mode spectrum of a white dwarf is so dense that mode identifications in actual pulsators have remained ambiguous. More progress needs to be made on both observational and theoretical fronts. On the one hand, the advent of the Whole Earth Telescope (Nather 1989) promises a revolution of its own, delivering data of unprecedented quality and with a density some 10 times larger than before. On the other hand, progress in theory has more or less stalled since the successful efforts by many groups to identify the driving mechanism. As a matter of fact, the most detailed investigation remains that of Winget (1981; see also Winget *et al.* 1982 and Winget and Fontaine 1982), a nonadiabatic survey mostly geared toward the identification of the driving mechanism. If seismological tools are to be developed at their fullest for ZZ Ceti stars (i.e., if mode identification is to be achieved), a very *complete* and *thorough* investigation of the pulsation properties of appropriate models must be carried out.

In this context, we have recently embarked in a series of investigations aimed at exploring the systematics of the g-mode period structures of models of ZZ Ceti stars. We thus focus, in this series, on the most fundamental aspect of white dwarf seismology. To carry out our program, we develop, as needed, new analytic and numerical tools. We also take advantage of the availability of the large grid of evolutionary equilibrium models computed by Tassoul, Fontaine, and Winget (1990) in order to explore parameter space thoroughly. We restrict our analysis to the *adiabatic approximation* because the period structures of the models can be most efficiently and quite accurately studied in that approximation. Obviously, we cannot say anything about the stability of a given mode in a particular model but, in any case, it appears that the stability analyses currently available are sometimes conflicting and inconclusive (see, e.g., Cox *et al.* 1987; Bradley, Winget, and Wood 1989; Bradley 1989; but see also below). Our strategy is similar to the approach successfully followed by Kawaler (1986) in his investigations of the properties of the pulsating PG 1159 stars: a thorough adiabatic survey was completed before investigating the problem of self-excitation of the various modes. We feel that, in the past, the power of the adiabatic approach has been somewhat overlooked for the ZZ Ceti stars.

In the present paper, the first in the series, we first explore the fundamental issue of the region of period formation in a degenerate star (§ II). This is motivated by a paper by Pesnell (1987) which challenges the usually accepted concept of envelope modes for white dwarfs. To examine quantitatively this surprising suggestion and, more generally for our survey, to compute the period structure of a white dwarf model, we derive a numerically satisfactory expression for the Brunt-Väisälä frequency, a crucial property that determines the period spectrum for g-modes. The expression derived in § III is valid for an arbitrary equation of state, and is thus applicable to the multicomponent, nonideal, partially degenerate, and partially ionized plasmas encountered in white dwarf envelopes. We next use (§ IV) this general expression to compute the period structure of the *same* ZZ Ceti star model which was used by Pesnell (1987). We then compare the results with those obtained in a numerical experiment which mimics the approach followed by that author. This comparison demonstrates the importance of accurate evaluation of the Brunt-Väisälä frequency in seismological studies. The implications of our findings on the work of Pesnell and his collaborators, in particular the paper by Cox *et al.* (1987), are examined in the last section. We thus confirm that white dwarf pulsation modes are concentrated in the surface layers. This is the crucial factor which enables us to use the pulsations to probe the compositional stratification of white dwarfs (Brassard *et al.* 1989).

II. WHAT REGIONS OF A WHITE DWARF ARE PROBED BY g-MODE PULSATIONS?

The g-mode period structure of a stellar model is largely specified by the spatial distribution of the square of the Brunt-Väisälä frequency which is defined by

$$N^2 \equiv -g\left[\frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{\Gamma_1 P}\frac{dP}{dr}\right], \tag{1}$$

where g is the local gravity, r is the radial coordinate, ρ is the density, P is the pressure, and Γ_1 is the first adiabatic exponent. The most basic structural feature of a white dwarf is its highly degenerate interior. As is well-known, this leads to a decoupling of the mechanical properties of a star (largely determined by the degenerate electron gas pressure) from its thermal properties (mostly governed by the thermodynamic behavior of the ions). In the narrow ZZ Ceti instability strip, one can thus expect that the period structure of a DA white dwarf—a mechanical property—does not change very much. There *are* small structural changes, however, caused by the cooling of the ionic plasma as the star evolves across the strip. These changes are extremely important since they cause a slow period evolution and provide us with a powerful tool—through measurements of rates of period change—for inferring the core composition.

The high degeneracy which characterizes the interior of a white dwarf also has far-reaching consequences for mode propagation. Indeed, in a typical ZZ Ceti star, electron degeneracy leads to nearly isothermal and nearly isentropic stratifications in the core region containing more than 99% of its mass. Thus, because the density gradient is almost adiabatic throughout the interior of a white dwarf, the Brunt-Väisälä frequency (cf. eq. [1]) is very small there and low-order g-modes cannot propagate at large amplitudes. As a result, g-modes are essentially envelope modes in white dwarfs, with large amplitudes occurring only in the nondegenerate outer layers (see Winget

and Fontaine 1982). One can thus expect that g-modes in white dwarfs are extremely sensitive to envelope properties such as compositional stratification and partial ionization phenomena.

Compositional stratification is, in fact, the second structural feature of a white dwarf model which has strong effects on the period structure. Composition changes in transition zones produce sharp features in the profile of the Brunt-Väisälä frequency, which are responsible for the main filtering and mode selection capabilities of a ZZ Ceti star model. Indeed, as first investigated by Winget, Van Horn, and Hansen (1981), trapped modes result when a resonance or near-resonance occurs between the local g-mode radial wavelength and the thickness of one of the composition layers. This results in a period structure which strongly bears the signature of compositional stratification in the outer layers. Hence, it has been widely accepted that white dwarf pulsations probe primarily the outer layers of these stars. This point of view has been borne out by detailed calculations carried out by several independent groups (see, e.g., Winget 1988 and references therein).

A surprisingly different stance has been taken recently by Pesnell (1987; see also Cox *et al.* 1987) who claims that g-modes in white dwarfs probe much deeper than previously thought. Although such modes have generally very small core amplitudes, the point is made that the only appropriate method to describe where the period of a mode is determined is to consult the weight function of the pulsation mode (this has been done, for example, by Kawaler, Hansen, and Winget 1985 in their study of the pulsating PG 1159 stars). As an illustrative example, Pesnell (1987) shows the weight function of a particular g-mode (noted g_1^2 here, $l = 2$, $k = 1$) for an evolutionary DA white dwarf model computed by Tassoul, Fontaine, and Winget (1990). The model has a total mass of 0.6 $M_\odot$. It has a C-rich core surrounded by a He-rich layer itself surrounded by a H-rich layer. The composition transition layers are treated under the assumption of diffusive equilibrium, which, for the particular model of interest, implies that there are some small traces of helium extending down to the center of the star. The effective temperature of the model is $T_e = 13{,}969$ K, the hydrogen (helium) layer has a mass equal to 10^{-10} (10^{-2}) times the total mass of the star, and convection is treated with the standard version of the mixing-length theory due to Böhm-Vitense (1958). The results of Pesnell (1987) indicate, surprisingly, that the period of the g_1^2 mode (which is found to be 59.0 s) is determined *in the central regions* of the model; indeed the weight function shows a maximum at about the half-way point in radius (approximately corresponding to the half-way point in mass also). The results are the same for both the Lagrangian pulsation code developed by Pesnell and for an Eulerian version which he used. In the latter case, the square of the Brunt-Väisälä frequency has been evaluated from equation (1), which implies the numerical evaluation of the derivative $d \ln \rho / dr$. In principle, this procedure takes into account the varying chemical composition in transition zones, including the deep core which contains traces of helium. Alarmingly, the period of the g_1^2 mode and, more generally, the complete period spectrum of the model are found by Pesnell (1987) to be *quite different* from that of earlier calculations carried out by Winget (1981) using the same model. Pesnell suggests that his taking into consideration the changing chemical composition in the deep core (ignored in Winget's calculations) accounts for this difference. If correct, this would mean that the basic period structure cannot be computed with any amount of confidence because it seems so sensitive to the presence of small traces of helium in the core. For simplicity, the unperturbed model had been computed under the assumption of diffusive equilibrium, but actual time-dependent diffusion calculations carried out by Pelletier (1988) show that this assumption breaks down in the very deep core of a white dwarf with an age characteristic of pulsating DA stars. Thus the helium distribution in the core of a DA model can only be specified by time-dependent calculations and, until detailed results of such calculations exploring a large volume of parameter space become available, white dwarf seismology would remain next to useless within the interpretation given by Pesnell (1987).

Other implications of the suggestion that white dwarf periods are formed deep in the core are given by Pesnell (1987). Among others, mode trapping caused by resonance effects at the composition interfaces (located in the outermost layers) would have greatly reduced efficiency. As compared with previous Eulerian calculations of several groups, the white dwarf computations based on the Lagrangian code of Pesnell have generally their region of period formation shifted toward the deep interior. For example, Cox *et al.* (1987) show the weight function of a particular mode in their Figure 11, and mention that the period of this mode is largely determined in the very deep layers, whereas driving is found, as usual, very near the surface.

Motivated by the past successes of white dwarf seismology based on Eulerian calculations (in particular the prediction and subsequent discovery of DB variable stars; see Winget and Fontaine 1982), we have felt that this new concept of period formation in the core of a white dwarf should be carefully examined. We have therefore embarked, with a fresh look, on a detailed investigation of the period structure of the model analyzed by Pesnell. In the process, we have discovered a basic shortcoming of the methods used by that author which (1) explains the discrepant results, and (2) has far-reaching implications for the results of his Lagrangian calculations in degenerate stars in general. We present our results in detail below (§ IV) but first discuss the treatment of the Brunt-Väisälä frequency which must be used in white dwarfs.

III. THE BRUNT-VÄISÄLÄ FREQUENCY IN PRESENCE OF VARYING COMPOSITION

In ordinary stars, the run of the Brunt-Väisälä frequency can be directly computed from equation (1), which necessarily involves the numerical evaluation of the spatial derivative $d \ln \rho / dr$ at each shell of the model. In degenerate stars, it is useful (indeed *essential* in the deep core; see below) to transform equation (1). We thus consider a plasma containing N different atomic species in *thermodynamic equilibrium*. The pressure of such a plasma may generally be written in the form

$$P = P(\rho, T, \{X_i\}), \qquad i = 1, \ldots, N - 1, \tag{2}$$

where ρ is the density, T is the temperature, and X_i represents the mass fraction of atoms of species i. Note that the index i runs only from 1 to $N - 1$ because of the constraint

$$\sum_{i=1}^{N-1} X_i + X_N = 1 . \tag{3}$$

Of course, other choices of independent variables are possible, but the set $(\rho, T, \{X_i\})$ is particularly convenient for our purposes.

Written as in equation (2), the pressure has a quite general form and may include nonideal effects, degeneracy effects, as

well as partial ionization effects. In the latter case, note that it is the *condition of thermodynamic equilibrium* that ensures that the pressure depends explicitly *only* on $(\rho, T, \{X_i\})$ and not on the populations of the various ionized states. In terms of our independent variables, the first adiabatic exponent may be written as

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{\mathrm{ad},\{X_i\}}, \tag{4}$$

where (against the common practice; e.g., Cox and Giuli 1968), we have explicitly indicated that this derivative is evaluated at *constant chemical composition.* This constaint is implicit in most astrophysical textbooks. With this in mind, we also remark that the usual relationships between the adiabatic exponents and other thermodynamic derivatives are generally valid, even for plasmas in which the chemical composition may change. In particular, a useful general relation in the present context is that provided by equation (34d) of Fontaine, Graboske, and Van Horn (1977),

$$\Gamma_1 = \frac{\chi_\rho}{1 - \nabla_{\mathrm{ad}} \chi_T}, \tag{5}$$

where

$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{T,\{X_i\}}, \tag{6}$$

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rho,\{X_i\}}, \tag{7}$$

and

$$\nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\mathrm{ad},\{X_i\}}. \tag{8}$$

From equation (2), we can immediately write

$$d \ln P = \chi_\rho d \ln \rho + \chi_T d \ln T + \sum_{i=1}^{N-1} \chi_{X_i} d \ln X_i, \tag{9}$$

where we have defined generalized compressibilities

$$\chi_{X_i} = \left(\frac{\partial \ln P}{\partial \ln X_i}\right)_{\rho,T,\{X_{j\neq i}\}}. \tag{10}$$

From equation (9), we obtain

$$\frac{d \ln \rho}{d \ln P} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho} \nabla - \frac{1}{\chi_\rho} \sum_{i=1}^{N-1} \chi_{X_i} \frac{d \ln X_i}{d \ln P}, \tag{11}$$

where the short-hand notation $\nabla = d \ln T / d \ln P$ has been used for the actual temperature gradient.

Next, we transform equation (1) slightly with the help of the equation of hydrostatic equlibrium ($dP/dr = -\rho g$),

$$N^2 = -g \frac{d \ln P}{dr} \left[\frac{d \ln \rho}{d \ln P} - \frac{1}{\Gamma_1}\right] = \frac{g^2 \rho}{P} \left[\frac{d \ln \rho}{d \ln P} - \frac{1}{\Gamma_1}\right]. \tag{12}$$

Substituting (5) and (11) into (12), we finally obtain

$$N^2 = \frac{g^2 \rho}{P} \frac{\chi_T}{\chi_\rho} \left[\nabla_{\mathrm{ad}} - \nabla - \frac{1}{\chi_T} \sum_{i=1}^{N-1} \chi_{X_i} \frac{d \ln X_i}{d \ln P}\right]. \tag{13}$$

This is the desired general expression for the Brunt-Väisälä frequency. Ignoring for the moment the effects of varying the chemical composition, we immediately see why equation (13) is to be preferred to equation (12) from a numerical point of view. The latter equation shows that N^2 is small in degenerate matter because $d \ln \rho / d \ln P \approx 1/\Gamma_1$. Differencing these two nearly equal terms poses numerical difficulties in degenerate matter. By contrast, equation (13) indicates that N^2 is small in degenerate matter primarily because χ_T (a multiplying factor in front of the brackets) becomes small. This last form is far more numerically satisfactory and reliable than equation (12).

In the *particular* case of compositionally stratified DA white dwarf models, there are two separate and distinct composition transition zones: the H/He and He/C buffer regions. In both cases, we are dealing with the simple situation of a *two-ion buffer zone* ($N = 2$). We thus may choose Y, the mass fraction of helium, as a convenient and *unique* indicator of composition change in both zones. Equation (13) then reduces to

$$N^2 = \frac{g^2 \rho}{P} \frac{\chi_T}{\chi_\rho} (\nabla_{\mathrm{ad}} - \nabla + B), \tag{14}$$

with

$$B \equiv -\frac{\chi_Y}{\chi_T} \frac{d \ln Y}{d \ln P}, \tag{15}$$

where

$$\chi_Y = \left(\frac{\partial \ln P}{\partial \ln Y}\right)_{\rho,T}. \tag{16}$$

The Brunt-Väisälä frequency in the white dwarf models reported by Tassoul, Fontaine, and Winget (1990) has been computed with the help of these last equations. Written in this form, the explicit contribution of a change of chemical composition to the Brunt-Väisälä frequency is contained in the term B. In white dwarf models, this quantity is always positive and assumes (in presence of diffusive equilibrium) non-negligible values *only* in regions where the abundances of the two atomic species are comparable, i.e., in the composition transition zones themselves.

Figures 1*a* and 1*b* (the latter is a blowup of the outermost layers) illustrate this point for the particular DA white dwarf model referred to above. They show the run of the quantity B as a function of radius. This quantity features two relatively narrow spikes, one centered on the He/C transition zone ($r/R \approx 0.92$, Fig. 1*a*), and a larger one centered on the H/He transition zone ($r/R \approx 0.9968$, Fig. 1*b*). The H/He spike is larger because the H/He transition zone is narrower than the He/C transition zone (see Tassoul, Fontaine, and Winget 1990 for a discussion of this).

Away from the composition transition zones, B assumes negligible values. In particular, in the deep isothermal core where $\nabla \approx 0$ and $\nabla_{\mathrm{ad}} \approx 0.4$, B is totally negligible, which implies that the presence of small traces of helium in the core of this model *cannot* affect the value of N^2 there. This is in contrast to the statement of Pesnell (1987), who has proposed that these small traces were responsible for the large differences found between his pulsation results and those of Winget (1981) for the very same model. For highly degenerate matter of material with $\mu_e = 2$, the pressure is the *same* for a given set (ρ, T) whatever (in the present case) the proportions of the He/C mixture, so $\chi_Y = (d \ln P / d \ln Y)_{\rho,T} \to 0$. This phenomenon

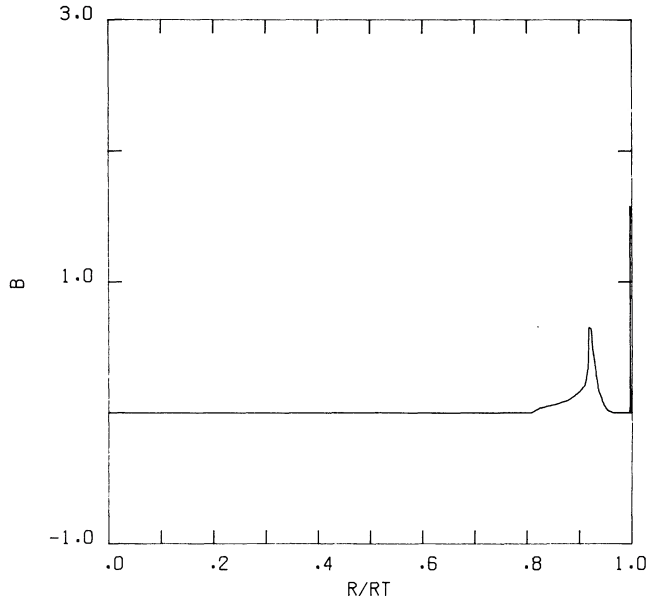

FIG. 1a

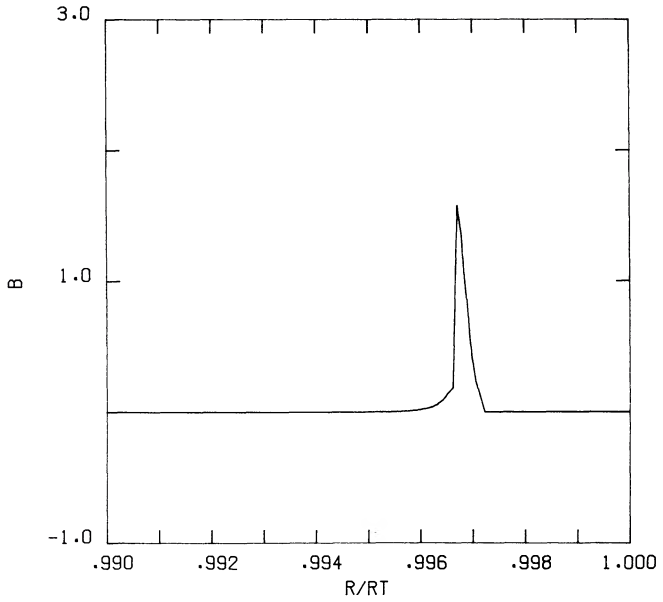

FIG. 1b

FIG. 1.—(*a*) Run of the quantity B as a function of radius for our DA white dwarf model. B contains the explicit contribution of a change of chemical composition to the Brunt-Väisälä frequency. It has nonnegligible values only in the composition transition zones. The structure centered on $r/R \approx 0.92$ corresponds to the He/C transition zone. A very sharp peak corresponding to the H/He transition zone is barely visible very near the surface. (*b*) Same as Fig. 1*a*, but blowup of the outermost layers of the model. The feature is due to the H/He transition zone.

completely overwhelms the small values of χ_T which appears in the denominator of B (see eq. [15]), hence B vanishes in the deep interior.

The form of equation (14) is also reminiscent of the well-known relationship between the Brunt-Väisälä frequency and the Ledoux criterion for convective stability in the case where the plasma forms an ideal and nondegenerate gas. If we were to specialize to this particular case, the mean molecular weight μ (which is the usual variable appearing in the Ledoux criterion) would still not be an appropriate variable in the present context. This is because partial ionization regions can sometimes overlap with composition transition zones in white dwarf models, which causes μ to vary not only with composition there but also with varying ionization as function of depth. The difficulty is easily circumvented by replacing μ by *any* variable which measures *only* a change in chemical composition; the functional form of the term B in equation (14) is the same as that of the term involving μ in the Ledoux criterion. In the present case of a two-ion transition zone, the helium mass fraction Y specifies uniquely the run of the chemical composition, is independent of ionization effects, and is, thus, an acceptable variable. The *ionic* mean molecular weight would also be appropriate in equations (15) and (16) because it is a unique function of Y; other choices are also possible. In practice, the computation of B in white dwarf models is straightforward: χ_Y is obtained by differentiating in composition in the equation-of-state tables, χ_T is another equation-of-state variable, and $d \ln Y/d \ln P$ is numerically evaluated from the actual composition profile, which is usually well-resolved in stellar models.

It is of interest to discuss briefly the more usual expression of the Brunt-Väisälä frequency in presence of varying composition which is found in textbooks (e.g., Cox 1980). To recover that expression, it is necessary to specialize to the case of a *completely ionized, nondegenerate, and ideal gas* (an approximation valid in the deep interiors of main sequence stars for example). Here, however, the plasma may comprise more than two atomic species, and we must return to the more general expression given by equation (13). In addition, the pressure is now given by

$$P = \frac{N_0 k \rho T}{\mu} + \frac{4\sigma}{3c} T^4 , \tag{17}$$

where the symbols have their standard meanings. The mean molecular weight is related to the mass fractions of the different elements through the relation

$$\frac{1}{\mu} = \sum_{i=1}^{N-1} \frac{(Z_i + 1)X_i}{A_i} + \left(1 - \sum_{i=1}^{N-1} X_i\right) \frac{Z_N + 1}{A_N} . \tag{18}$$

where Z_i and A_i are, respectively, the atomic number and the atomic weight of element of species i. Note that equation (18) is only valid if the ionization of all elements is complete. By combining (17) and (18), one can compute the quantities

$$\chi_{X_i} = \left(\frac{\partial \ln P}{\partial \ln X_i}\right)_{\rho, T, \{X_{j \neq i}\}} = X_i \beta \mu \left(\frac{Z_i + 1}{A_i} - \frac{Z_N + 1}{A_N}\right) , \tag{19}$$

where $\beta \equiv P_{\rm gas}/P$ and $P_{\rm gas} = N_0 k \rho T/\mu$. This last relation is a direct consequence of the *particular* form for the pressure (eq. [17]). From equation (18), one can also write generally

$$\begin{aligned} -d \ln \mu &= \sum_{i=1}^{N-1} X_i \mu \left(\frac{Z_i + 1}{A_i} - \frac{Z_N + 1}{A_N}\right) d \ln X_i \\ &= \frac{1}{\beta} \sum_{i=1}^{N-1} \chi_{X_i} d \ln X_i , \end{aligned} \tag{20}$$

where use has been made of equation (19). We thus obtain the

result,

$$-\beta \frac{d\ln\mu}{d\ln P} = \sum_{i=1}^{N-1} \chi_{X_i} \frac{d\ln X_i}{d\ln P} . \quad (21)$$

By noting further that $(\partial \ln P/\partial \ln \mu)_{\rho,T} = -\beta$, we finally get

$$\left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T} \frac{d\ln\mu}{d\ln P} = \sum_{i=1}^{N-1} \chi_{X_i} \frac{d\ln X_i}{d\ln P} . \quad (22)$$

Substituting (22) into (13), we obtain the usual expression of the Brunt-Väisälä frequency which is related to the Ledoux criterion for convection,

$$N^2 = \frac{g^2\rho}{P}\frac{\chi_T}{\chi_\rho}\left[\nabla_{ad} - \nabla - \frac{1}{\chi_T}\left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T} \frac{d\ln\mu}{d\ln P}\right]. \quad (23)$$

Since μ is uniquely related to the $\{X_i\}$ (via eq. [18]), we may write for the logarithmic pressure derivatives

$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{T,\{X_i\}} = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{T,\mu} , \quad (24)$$

and

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rho,\{X_i\}} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rho,\mu} . \quad (25)$$

Equations (23), (24), and (25) give collectively the usual form of the Brunt-Väisälä frequency. We may note that $N^2 > 0$ in radiative regions, which corresponds to the usual Ledoux condition for convective stability

$$\nabla_{ad} - \nabla - \frac{1}{\chi_T}\left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T} \frac{d\ln\mu}{d\ln P} > 0 . \quad (26)$$

Once again, however, this particular form of the Brunt-Väisälä frequency (and of the Ledoux criterion) has been obtained for the special case of a completely ionized, nondegenerate, and ideal gas. For more general cases, equation (13) (or an equivalent one since the set $[\rho, T, \{X_i\}]$ is not unique) must be used.

We may also remark that our equation (14) is consistent with equation (8) of another paper by Pesnell (1986) dedicated precisely to a discussion of the appropriate form of the Brunt-Väisälä frequency in presence of partial ionization and varying composition. It should be noted, however, that the equation obtained by Pesnell (1986) has been derived under the unnecessary restriction of a nondegenerate, ideal gas. By contrast, our equation (14) is generally valid for a two-ion plasma. The variable that measures a change in chemical composition in Pesnell's paper is the ionic mean molecular weight (denoted μ_0 in his paper). His equation (8) gives

$$N^2 = -gA = \frac{g^2\rho}{P}\frac{\chi_T}{\chi_\rho}\left[\nabla_{ad} - \nabla - \frac{\chi_{\mu_0}}{\chi_T}\frac{\mu}{\mu_0}\frac{d\ln\mu_0}{d\ln P}\right], \quad (27)$$

where (in Pesnell's notation),

$$\chi_{\mu_0} = -\left[1 + \left(\frac{\partial n_e}{\partial n_0}\right)_{\rho,T}\right]. \quad (28)$$

We can slightly rearrange equation (27) to write

$$N^2 = \frac{g^2\rho}{P}\frac{\chi_T}{\chi_\rho}\left[\nabla_{ad} - \nabla - \frac{1}{\chi_T}\left(\frac{\partial \ln P}{\partial \ln \mu_0}\right)_{\rho,T} \frac{d\ln\mu_0}{d\ln P}\right] \quad (29)$$

after noting that

$$\left(\frac{\partial \ln P}{\partial \ln \mu_0}\right)_{\rho,T} = -\frac{\mu}{\mu_0}\left[1 + \left(\frac{\partial n_e}{\partial n_0}\right)_{\rho,T}\right] = \frac{\mu}{\mu_0}\chi_{\mu_0} . \quad (30)$$

Equation (29) has exactly the same functional form as our equation (14), as it should. We have already noted that the exact choice of the variable measuring a change in chemical composition is immaterial as long, of course, as this variable depends *only* on the composition.

Even though the equation obtained by Pesnell (1986) is consistent with our results, we cannot help but notice that a serious error of interpretation has, unfortunately, been put forward in that paper. Indeed, *in the limit of complete ionization*, the claim is that equation (29) (or the equivalent expression in Pesnell 1986) does not lead to the usual expression for the Brunt-Väisälä frequency in the case of a nondegenerate, ideal gas with varying composition. This is surprising because, quite clearly from equation (17), one must have

$$\left(\frac{\partial \ln P}{\partial \ln \mu_0}\right)_{\rho,T} = \frac{\mu_0}{P}\left(\frac{\partial P}{\partial \mu_0}\right)_{\rho,T} = \frac{\mu_0}{P}\left(\frac{\partial P}{\partial \mu}\right)_{\rho,T}\frac{d\mu}{d\mu_0}$$
$$= \left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T}\frac{d\ln\mu}{d\ln\mu_0} , \quad (31)$$

where advantage has been taken of the fact that μ and μ_0 are uniquely related and are independent of ρ and T in the limit of complete ionization. From equation (31), we may write

$$\left(\frac{\partial \ln P}{\partial \ln \mu_0}\right)_{\rho,T}\frac{d\ln\mu_0}{d\ln P} = \left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T}\frac{d\ln\mu}{d\ln P} . \quad (32)$$

Substituting (32) into (29) gives back, contrary to Pesnell's claim, the standard expression (eq. [23]) for the Brunt-Väisälä frequency. Hence, the suggested reduction of the Brunt-Väisälä frequency for a completely ionized, ideal, and nondegenerate multicomponent plasma is clearly incorrect.

We strongly suspect that the confusion arises from the fact that the quantity χ_{μ_0} (see eq. [28]), defined by Pesnell (1986) is not a true compressibility (i.e., a logarithmic derivative such as our quantities χ_{X_i} for example). In fact, from equation (30), one sees how the standard compressibility $(\partial \ln P/\partial \ln \mu_0)_{\rho,T}$ is related to the quantity χ_{μ_0}. It is the term μ/μ_0 (< 1 for complete ionization) which makes the functional dependence *appear* different from that of the usual expression in equation (27) and which suggests a *reduction* of the Brunt-Väisälä frequency. As we have seen, however, μ/μ_0 and χ_{μ_0} should be lumped together as in equation (30).

In a specific example related to the region around the hydrogen exhausted core of a nondegenerate star, Pesnell (1986) suggests that the Brunt-Väisälä frequency should be reduced by some 17% as compared to the value computed by the standard expression related to the Ledoux criterion. For reasons just given, this suggestion is incorrect. In fact, *for that particular example*, we can rigorously write

$$\frac{5}{6}\frac{d\ln\mu_0}{d\ln P} = \frac{d\ln\mu}{d\ln P} , \quad (33)$$

which, substituted in Pesnell's equation (21), gives back the usual expression of the Brunt-Väisälä frequency contrary to the claim and major conclusion of that paper.

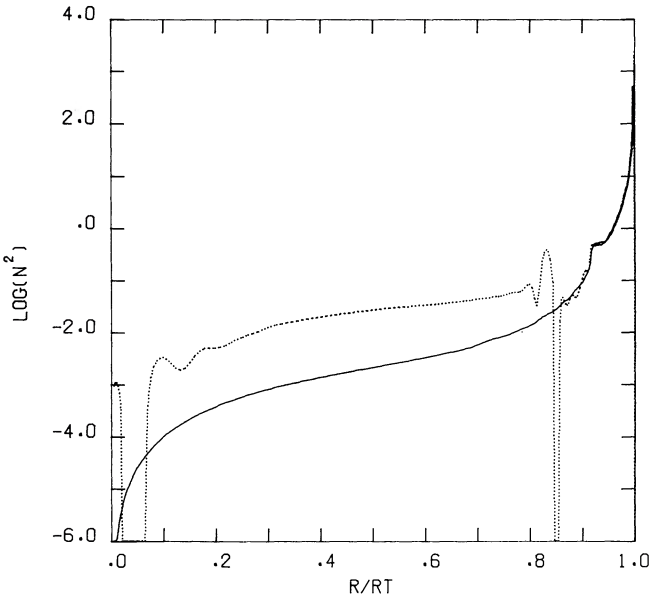

FIG. 2a

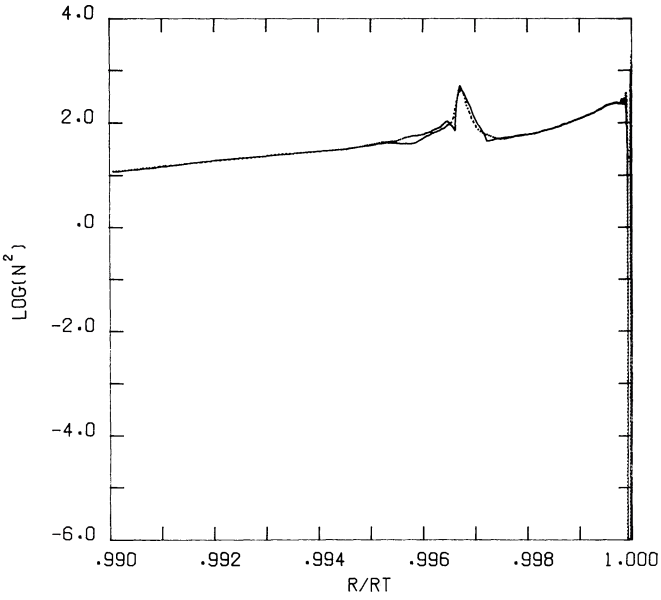

FIG. 2b

FIG. 2.—(*a*) The square of the Brunt-Väisälä frequency as a function of the radius for our reference model. The continuous curve corresponds to the computations based on equation (14), the dotted curve to those based on eq. (1) with numerical derivatives. (*b*) Same as Fig. 2*a*, but blowup of the outermost layers to emphasize the H/He transition region.

IV. A COMPARISON OF TWO DIFFERENT APPROACHES TO THE PERIOD STRUCTURE OF ZZ CETI STAR MODELS

We have used the Eulerian code developed by Hansen (see Kawaler, Hansen, and Winget 1985) to analyze the adiabatic period structure of the DA model discussed by Pesnell (1987). The continuous curve in Figures 2*a* and 2*b* (the latter is again a blowup of the outermost layers) shows the distribution of N^2 in terms of the radius. The small localized structure around $r/R \approx 0.92$ (Fig. 2*a*) is associated with the He/C transition zone (log $\Delta M/M \approx -2$). A sharp feature, visible around $r/R \approx 0.9968$ (Fig. 2*b*), is associated with the H/He transition zone located in the outermost layers (log $\Delta M/M \approx -10$). The N^2 profile is generally quite smooth and any discontinuities or quasidiscontinuities (such as the structures associated with the composition transition zones and contained in the term B) are potential sources of resonance effects. Our calculations indicate that the period of the g_1^2 mode is 122.8 s in reasonable agreement with the older calculations of Winget (1981) which give 142.3 s, but a far cry from the 59.0 s obtained by Pesnell (1987). Note that we recover *exactly* the results of Winget (1981) by ignoring the effects of the composition transition layers on the Brunt-Väisälä frequency as was done in that study. In practice, this is accomplished by ignoring the contribution of B to N^2 (see eq. [14]).

Interestingly, however, we can reproduce the results of Pesnell (1987) by performing the following experiment: we now calculate N^2, not with equation (14), but rather directly with equation (1) which involves the numerical evaluation of $d\ln\rho/dr$. This procedure is implicit in the Lagrangian formulation of Pesnell and was *required* by him in his Eulerian code in order to recover the results of the Lagrangian calculations. The dotted curve in Figures 2*a* and 2*b* shows the resulting N^2 profile for our reference model. As compared to the continuous curve, additional structure and two spurious "convection" zones (negative values of N^2) have appeared. It is obvious that the N^2 profile presented by Pesnell (1987) in his Figure 1 is quite similar to our dotted curve. Apparently, the spurious "convection" zones were suppressed by Pesnell but important structure was left after this operation as can clearly be seen in his figure. All other features of Pesnell's figure are accurately reproduced, however. As noted before, the quasidiscontinuities that are present in our dotted curve and in the N^2 profile shown by Pesnell (1987) can isolate and trap certain modes if resonance conditions are met. These modes are trapped in the *deep interior*, however, implying very large kinetic energies and very low growth rates. More importantly, it is the large systematic differences observed between the dotted and continuous curves in our Figures 2*a* and 2*b* that are directly responsible for the large differences found in the period of the same mode. For instance, in the range $0.1 \lesssim r/R \lesssim 0.8$, the dotted curve suggests systematically larger values of N^2 in that particular model. Hence, the model appears "less degenerate" than before. From asymptotic theory (cf., Tassoul 1980), the period of a g-mode is inversely proportional to $\int |N|/r\,dr$, so, if $|N|$ is larger, we can expect a *smaller* value of the period. We find, in fact, in our altered calculation, a period of 58.7 s for the g_1^2 mode, very close to the value of Pesnell (1987) but quite different from our original estimate.

As hinted previously, the fundamental reason for this discrepancy is that a numerical evaluation of the difference appearing in equation (1) encounters serious difficulties in degenerate matter. Indeed, with a slight rearrangement of terms, we can write

$$N^2 = -\frac{g}{P}\frac{dP}{dr}\left[\frac{d\ln\rho}{d\ln P} - \left(\frac{d\ln\rho}{d\ln P}\right)_s\right], \tag{34}$$

where s is the specific entropy per gram. In the nearly adiabatic interior of a degenerate star, the numerical evaluation of derivatives boils down to computing N^2 by *taking the difference* of two nearly equal quantities. By contrast, our formulation of N^2

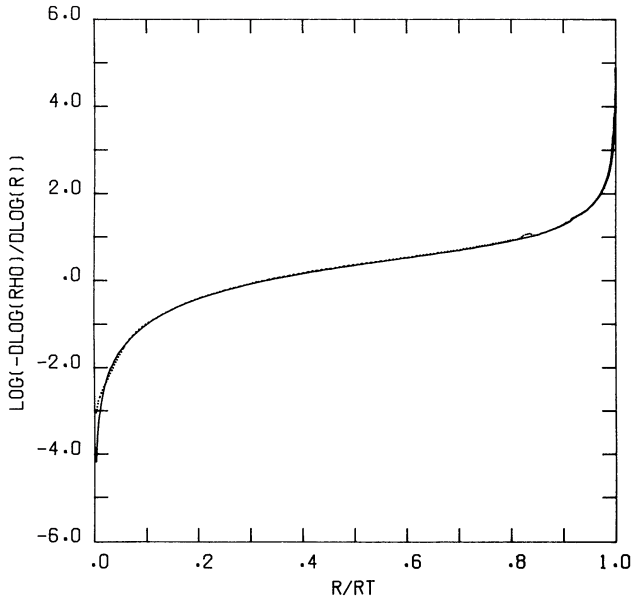

FIG. 3*a*

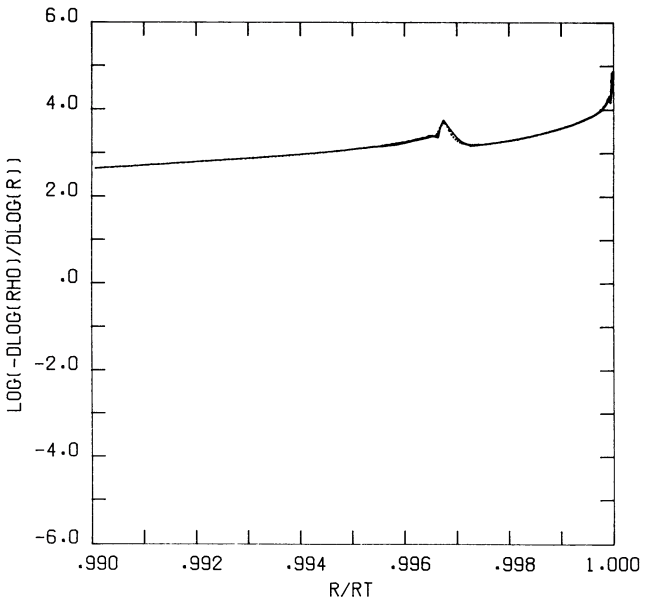

FIG. 3*b*

FIG. 3.—(*a*) The logarithmic derivative of the density ρ as a function of radius for our reference model. The continuous curve corresponds to the values inferred from combining eqs. (1) and (14), while the dotted curve corresponds to the values obtained numerically using a 3 point Lagrange differentiation scheme. (*b*) Same as Fig. 3*a*, but blowup of the outermost layers to emphasize the H/He transition region.

given by equation (14) is reliable everywhere. In the deep core, as mentioned previously, the term B does not contribute, ∇ is also negligibly small because of the nearly isothermal conditions, ∇_{ad} remains a number with typical values slightly under 0.4, and N^2 is then evaluated by *multiplying* various quantities. Of course, the two formulations are equivalent in principle, and we indeed observe that the two curves shown in Figure 2*a* and 2*b* merge together for large values of r/R, i.e., in the outer layers where the degree of degeneracy decreases substantially. The small differences that are observed in the H/He transition zone (Fig. 2*b*) are not significant and are attributed to a different accuracy of the numerical scheme in a region where the variables change rapidly.

Figure 3*a* and 3*b* emphasize the fact that the limitation of equation (1) is truly due to the use of a *difference* between two numbers which become nearly equal in degenerate matter. What is plotted is the logarithm of the quantity $d\ln\rho/d\ln r$ as a function of radius. The dotted curve corresponds to our numerically evaluated values, while the continuous curve corresponds to the values of $d\ln\rho/d\ln r$ *inferred* from equation (1) with N^2 evaluated from equation (14), i.e.,

$$\frac{d\ln\rho}{d\ln r} = -\frac{N^2 r}{g} - \frac{\rho g r}{\Gamma_1 P}. \tag{35}$$

The figures show that the agreement is very good; only small differences are observed in certain particular regions. This means that the absolute value of $d\ln\rho/d\ln r$ evaluated numerically is not bad, but is still not good enough to give reliable results for N^2 because it is *subtracted* from another number ($\rho g r/\Gamma_1 P$) which is nearly equal. It may be of interest to point out that very similar results are obtained whether or not dP/dr is itself evaluated numerically in equation (1) or replaced by $-\rho g$. The fundamental limitation of equation (1) in degenerate matter rests indeed with its particular form involving a difference of two nearly equal numbers.

Thus, numerical differencing implicit in the Lagrangian formalism of Pesnell leads to unreliably noisy N^2 profiles in white dwarf models with concomitant dramatic consequences on the period structure. Not surprisingly, the region of period formation is also affected by these problems. For example, Figure 4 contrasts the two weight functions which we have computed for the g_1^2 mode of interest. The continuous curve refers to the weight function for the eigenmode computed with our equation (14) for N^2 and leading to a period of 122.8 s. Quite clearly, the mode is an envelope mode as can be expected for a degenerate star. By contrast, the dotted curve, based on the use

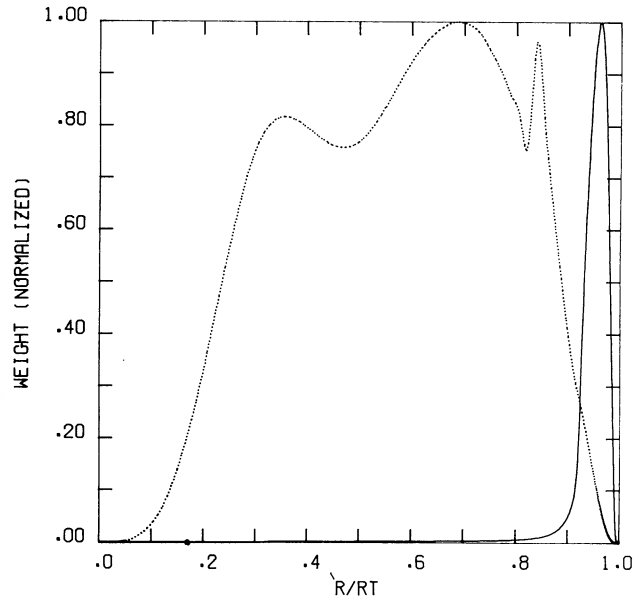

FIG. 4.—The weight function of the g_1^2 mode in terms of the radius. The continuous and dotted curves correspond to the two sets of calculations referred to in the caption of Fig. 2*a*.

of equation (1), leads to the conclusion that the period is formed in the deep interior as in Pesnell (1987). We have thus clearly identified the origin of the discrepant period and weight function of the eigenmode discussed by Pesnell (1987). We feel that this numerical experiment should put to rest the idea that white dwarf pulsations probe the deep core.

V. DISCUSSION AND CONCLUSION

We have found it both interesting and instructive to investigate the complete adiabatic period structure of our white dwarf model for all modes with $l = 1$, 2, and 3 and with periods in the range 100 s $\leq \Pi \leq$ 1000 s. Figure 5 summarizes our results which are based on the computation of the Brunt-Väisälä frequency as given by equation (14). The figure shows the normalized kinetic energy of a mode as a function of the period. Each plotted point corresponds to a radial overtone starting to the left with $k = 1$, except for the $l = 3$ modes which start with $k = 2$ because the g_1^3 mode has a period smaller than 100 s. The positions of the $l = 1$, 2, and 3 modes are joined together by a continuous, dashed, and dotted line, respectively. As discussed by Winget, Van Horn, and Hansen (1981) as well as Winget and Fontaine (1982), the primary minima in these curves correspond to modes with a node at the H/He interface. Such modes show very small amplitudes below the H/He buffer zone (in essence they resonate with the H layer thickness, or, more accurately, they are efficiently reflected at the composition interface), so we can think of these modes as effectively *trapped* in the outer H layer. Because the kinetic energy of a mode is proportional to the displacement vector integrated over the whole star, modes trapped in the outer H envelope show a characteristic signature corresponding to minima in kinetic energy. Thus, the g_8^1 mode with a period $\Pi = 542.5$ s and (normalized) kinetic energy log $E = 42.17$ is the first trapped mode of the $l = 1$ sequence. Note that the H/He transition zone has a finite width imposed by diffusion consideration, and is therefore not a true discontinuity. This explains why the kinetic energy minima have finite widths; modes can be *partially trapped* in ZZ Ceti star models.

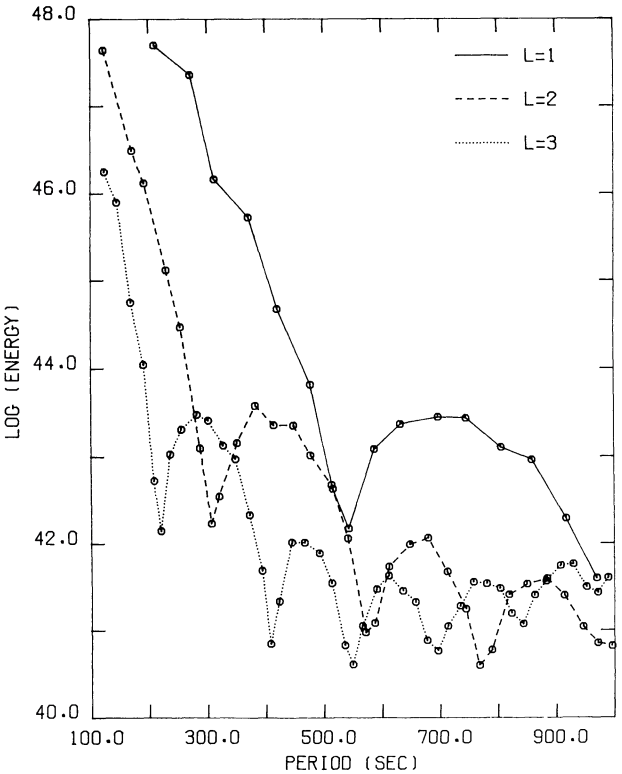

FIG. 5.—Kinetic energy vs. period for all g-modes of our reference model in the period range 100 s $\leq \Pi \leq$ 1000 s and with $l = 1$ (*continuous curve*), and $l = 2$ (*dashed curve*), and $l = 3$ (*dotted curve*). Each plotted point corresponds to a radial overtone starting to the left with $k = 1$ ($l = 1$ and 2) and $k = 2$ ($l = 3$). This period structure is obtained with the Brunt-Väisälä frequency computed with the help of equation (14).

As can be further observed in the figure, the number of trapped modes in the period window 100–1000 s increases with increasing l. This is because, for a given period, the radial overtone (the number of radial nodes) increases with l. Hence, more modes have nodes falling at the H/He interface for larger values of l. At the same time, the contrast in kinetic energy between the trapped modes and the other modes tend to decrease with increasing order k (as can be expected). Note, in addition, that beyond the primary minima shown in Figure 5, there are also secondary features *which are real*, and which correspond to modes with nodes at the He/C interface. Because the He/C transition zone is located much deeper (log $\Delta M/M \approx -2$, $r/R \approx 0.92$) than the H/He transition zone (log $\Delta M/M \approx -10$, $r/R \approx 0.9968$) in our particular model, the effect on the kinetic energy is much reduced.

It is also interesting to reconsider briefly the g_1^2 mode discussed in the previous section. Figure 5 indicates that this mode ($\Pi = 122.8$ s, log $E = 47.64$) does not resonate with the H outer layer. Nevertheless, it has been established that it is a true envelope mode, as its weight function is confined into the outer 10% of the radius (see Fig. 4). The difference with a trapped mode such as g_8^1 is marked, however. Indeed, if we consult the weight function of this latter mode, we find that it is confined in the outer 0.3% of the radius, i.e., in the H outer layer (as expected from the previous arguments).

In sharp contrast to the results we just discussed, Figure 6 shows a clearly unphysical period structure. Here, we have considered the same reference model, but the Brunt-Väisälä frequency has been computed directly from equation (1) which involves the numerical evaluation of $d \ln \rho/dr$. As determined in the previous section, this treatment of N^2 makes the model appear less degenerate, so the lowest order modes have periods less than 100 s. In Figure 6, we have therefore plotted the results in the period range 0–1000 s so as to include the lowest order modes ($k = 1$) for $l = 1$, 2, and 3. In particular, note that the g_1^2 mode ($\Pi = 58.7$ s log $E = 50.37$) discussed previously is included. Note also the much wider range of kinetic energy caused by the presence of sharp peaks.

The period structure illustrated in Figure 6 is plagued by several defects. We can isolate at least four of them: (1) The distribution of kinetic energy with respect to period for a given value of l shows a jagged appearance, with no obvious effects caused by transition zone mode trapping which, as we have seen, should manifest themselves in the form of well-defined minima in the distribution. (2) Convergence difficulties in finding eigensolutions were frequently experienced; many modes are missed (but we have nevertheless joined the points together in the figure). (3) Certain isolated modes show very large values of the kinetic energy while modes with adjacent values of k show the "normal" behavior; these sharp peaks are associated with modes that are trapped in the core. This phenomenon is possible because the Brunt-Väisälä frequency suffers from *unphysical structure* in the deep core which can

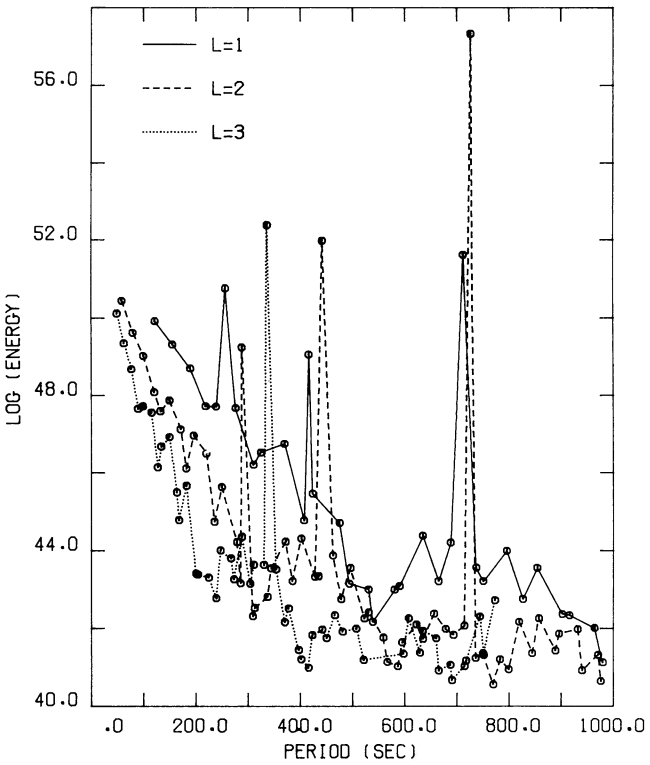

FIG. 6.—Same as Fig. 5, but, this time, the Brunt-Väisälä frequency is computed directly from equation (1) with numerical derivatives.

cause potential resonance conditions there. We have explicitly verified this assertion by consulting the weight functions of the mode with the largest kinetic energy in Figure 6 (a $l = 2$ mode). We find, as expected, that this particular mode is confined to the *inner* 5% of the radius. (4) Comparison with models with different effective temperatures shows that the period-luminosity relationship for a given mode is muddled, in contrast to the expected behavior of a *slow* period evolution (see above).

In comparing our results with the calculations of Cox *et al.* (1987) based on Pesnell's Lagrangian formalism, *we cannot escape the important conclusion that the period structures of their models suffer from all of the above symptoms.* This is particularly evident when looking at their growth rate versus period diagrams (specifically their Figs. 2*a*, 2*b*, 3*a*, 3*b*, 4, 5*a*, 5*b*, 5*c*; see also their Table 2). The sharp *dips* that are obtained in these diagrams (for given values of l) correspond exactly to the sharp *peaks* which are found in our Figure 6 as the growth rate (a nonadiabatic property) is inversely proportional to the kinetic energy. These dips are unfailing signs of unphysical structure of the Brunt-Väisälä frequency in the deep degenerate cores of the models computed by these authors. In addition, as already pointed out previously, they also find that g-modes are formed deep in the core of a white dwarf (see their Fig. 11), another signature of numerical problems in the evaluation of the Brunt-Väisälä frequency in degenerate matter. Furthermore, Cox *et al.* (1987) report missing many modes because of convergence problems (another symptom of an excessively noisy N^2). Hence we must conclude that the period structures of the models discussed by Cox *et al.* (1987) are largely incorrect.

To summarize, the implicit numerical differencing used in the Lagrangian pulsation code of Pesnell leads to very serious difficulties when used with models of degenerate stars. These difficulties are at the origin of his suggestion (Pesnell 1987) that white dwarf periods are formed in the deep interior. We reaffirm the prior results of all other investigations; g-mode pulsations in white dwarfs are truly envelope modes. The implications of our findings on the work of Cox *et al.* (1987) are important and far-reaching. We find, in particular, that the *basic period structure* of their models (i.e., the most fundamental aspect of astroseismology) is unreliable and, in large part, unphysical. Because of this, their *nonadiabatic* results concerning primarily driving and damping must be considered premature; thus, their controversial conclusions about the complete insensitivity of the ZZ Ceti theoretical blue edge to the hydrogen layer mass remain clearly questionable.

We are grateful to A. Talon for useful discussions. This work was supported in part by the NSERC Canada, by the fund FCAR (Québec), and by the NASA grant NAGW-778 at Yale University.

REFERENCES

Böhm-Vitense, E. 1958, *Zs. Ap.*, **46**, 108.
Bradley, P. A. 1989, M.Sc. thesis, University of Texas at Austin.
Bradley, P. A., Winget, D. E., and Wood, M. A. 1989, in *IAU Colloquium 114, White Dwarfs*, ed. G. Wegner (New York: Springer), p. 286.
Brassard, P., Fontaine, G., Wesemael, F., and Kawaler, S. D. 1989, in *IAU Colloquium 114, White Dwarfs*, ed. G. Wegner (New York: Springer), p. 263.
Cox, A. N., Starrfield, S. G., Kidman, R. B., and Pesnell, W. D. 1987, *Ap. J.*, **317**, 303.
Cox, J. P., 1980, *Theory of Stellar Pulsations* (Princeton: Princeton University Press).
Cox, J. P., and Giuli, R. T. 1968, *Principles of Stellar Structure* (New York: Gordon and Breach).
Daou, D., Wesemael, F., Bergeron, P., Fontaine, G., and Holberg, J. B. 1990, in press.
Dolez, N., and Vauclair, G. 1981, *Astr. Ap.*, **102**, 375.
Dziembowski, W., and Koester, D. 1981, *Astr. Ap.*, **97**, 16.
Fontaine, G., Graboske, H. C., Jr., and Van Horn, H. M. 1977, *Ap. J. Suppl.*, **35**, 293.
Fontaine, G., and Wesemael, F. 1987, in *IAU Colloquium 95, The Second Conf. on Faint Blue Stars*, ed. A. G. D. Philip, D. S. Hayes, and J. Liebert (Schenectady: Davis), p. 319.
Kawaler, S. D. 1986, Ph.D. thesis, University of Texas at Austin.
Kawaler, S. D., and Hansen, C. J. 1989, in *IAU Colloquium 115, White Dwarfs*, ed. G. Wegner (New York: Springer), p. 97.
Kawaler, S. D., Hansen, C. J., and Winget, D. E. 1985, *Ap. J.*, **295**, 547.
Lamontagne, R., Wesemael, F., and Fontaine, G. 1987, in *IAU Colloquium 95, The Second Conference on Faint Blue Stars*, ed. A. G. D. Philip, D. S. Hayes, and J. Liebert (Schenectady: Davis), p. 677.
Nather, R. E. 1989, in *IAU Colloquium 115, White Dwarfs*, ed. G. Wegner (New York: Springer), p. 109.
Pelletier, C. 1988, private communication.
Pesnell, W. D. 1986, *Ap. J.*, **301**, 204.
———. 1987, in *Stellar Pulsations: A Memorial to John P. Cox*, ed. A. N. Cox, W. M. Sparks, and S. G. Starrfield (New York: Springer), p. 363.
Shipman, H. L. 1989, in *IAU Colloquium 115, White Dwarfs*, ed. G. Wegner (New York: Springer), p. 220.
Starrfield, S. G., Cox, A. N., Hodson, S. W., and Pesnell, W. D. 1982, in *Pulsations in Classical and Cataclysmic Variable Stars*, ed. J. P. Cox and C. J. Hansen (Boulder: University of Colorado Press), p. 78.
Tassoul, M. 1980, *Ap. J. Suppl.*, **43**, 469.
Tassoul, M., Fontaine, G., and Winget, D. E. 1990, *Ap. J. Suppl.*, **72**, 335.
Wesemael, F., Lamontagne, R., and Fontaine, G. 1986, *A.J.*, **91**, 1376.

Winget, D. E. 1981, Ph.D. thesis, University of Rochester.
———. 1986, in *Highlights of Astronomy*, ed. J. P Swings (Dordrecht: Reidel), p. 221.
———. 1988, in *IAU Symposium 123, Advances in Helio- and Asteroseismology*, ed. J. Christensen-Dalsgaard and S. Frandsen (Dordrecht: Reidel), p. 305.
Winget, D. E., and Fontaine, G. 1982, in *Pulsations in Classical and Cataclysmic Variable Stars*, ed. J. P. Cox and C. J. Hansen (Boulder: University of Colorado Press), p. 46.
Winget, D. E., Van Horn, H. M., and Hansen, C. J. 1981, *Ap. J. (Letters)*, **245**, L33.
Winget, D. E., Van Horn, H. M., Tassoul, M., Hansen, C. J., Fontaine, G., and Carroll, B. W. 1982, *Ap. J. (Letters)*, **252**, L65.

P. Brassard, G. Fontaine, and F. Wesemael: Département de Physique, Université de Montréal, C.P. 6128, Succ. A, Montréal, Québec, Canada, H3C 3J7

S. D. Kawaler: Astronomy Program, Department of Physics, Iowa State University, Ames, IA 50011

M. Tassoul: 6000 Chemin Deacon, Apt. J2, Montréal, Québec, Canada, H3S 2T9