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RADIATIVE TRANSFER IN ASTRONOMICAL MASERS. I. THE LINEAR MASER

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ABSTRACT

This is the first in a series of three papers presenting a comprehensive, unified approach for radiative transfer in astronomical maser sources. This paper provides the general formalism and presents the detailed analytic solution of the linear maser, including the case of a source illuminated by background radiation.

Subject headings: masers — radiative transfer

I. INTRODUCTION

The structure of astronomical maser sources is being probed in greater detail all the time, thanks to continuous, ongoing progress in high-resolution interferometry (e.g., Cohen 1989; Moran 1989). Maser observations are fulfilling a long-held promise by providing detailed information about the small-scale structure of the host environments. The study of these detailed observations, which will undoubtedly continue to improve, requires a general, unified formalism for analyzing the source structure and propagation of radiation in astronomical masers. Surprisingly, a consistent overall theory has never been formulated. Some of the general properties of maser radiation were first discussed by Litvak *et al.* (1966) and Litvak (1970). These general discussions were followed by somewhat more detailed work by Litvak (1971, 1973) and culminated in the seminal study of Goldreich and Keeley (1972), which provides the only complete analytic solutions of specific geometries available to date. That paper provides much of the basic theory that has been in use; the only additional detailed solutions presented since then involve the numerical studies of linear and rectangular masers by Alcock and Ross (1985). These different studies employed a variety of approaches, suitable for the particular task at hand. Even the Goldreich and Keeley paper, which derived complete solutions for both spherical and cylindrical masers in the same work, treated each geometry as a separate case, solved anew with techniques devised specifically for it. These solutions were not obtained from general expressions where the particular properties of the desired geometry could simply be inserted; such a general solution, in fact, has never been worked out in the literature as yet. The aim of this series of papers is to address this problem and rectify the situation by formulating a unified approach to radiative transfer in astronomical masers and providing the general solution for a maser with an arbitrary geometrical shape.

Although many of the properties of maser sources are independent of the source geometry, the relation between the intensity I_ν and the angle-averaged J_ν ($\equiv \int I_\nu d\Omega/4\pi$) does depend on the geometrical distribution of the radiation. As a result, the source geometry affects both the onset of saturation and the intensity in the saturated regime. The details of these properties can therefore be studied only in the context of specific geometries. This paper, the first in a series of three, studies the simplest geometrical configuration, involving a linear setup. Although this simple geometry cannot be used to study beaming effects, it does display most of the essential features related to the structure of masers; in particular, the structure of the solution on a given ray in any geometry is always similar to that of a linear maser. And because exact solutions are available for this geometry in various limits, it provides much needed insight for the more complicated three-dimensional models studied in the following papers.

A complete solution of the linear maser has not yet been presented, the only detailed study in the literature being the numerical calculation of Alcock and Ross (1985). The aim of this paper is to provide the complete analytic solution for this model. The concepts and techniques developed for this solution are then utilized in the study of more realistic three-dimensional geometries. Paper II of the series (Elitzur 1990*a*) provides the general solution for a three-dimensional maser of any shape and applies it to the specific case of spherical geometry. The complete solution of a filamentary maser is worked out in Paper III (Elitzur, McKee, and Hollenbach 1991). Strong astronomical masers are most likely shaped like elongated tubes (e.g., Elitzur 1982; Genzel 1986; Elitzur, Hollenbach, and McKee 1989), and the filamentary model is the one most relevant for observations.

The plan of this paper is as follows: Section II presents the general background material and overall framework. Many of the expressions are familiar and are included for completeness and for clarifying the concepts and assumptions involved. Some of the results are new, in particular equation (2.16), which is a general expression (applicable also to nonmaser radiation) for the product of intensities of the two streams on any given ray. The complete analytic solution of a linear maser without background radiation is worked out in § III, with an emphasis on the development of techniques and insight that can be employed in more realistic geometries. The results of the numerical calculations of Alcock and Ross (1985) are explained. The effect of background radiation on the maser sturcture, which has never been fully studied in any geometry, is incorporated in § IV.

II. GENERALITIES AND FORMALISM

The propagation of maser radiation is controlled by the behavior of the (absolute value of the) absorption coefficient κ_ν of the inverted system, given by[1]

$$\kappa_\nu = \frac{\kappa_{0\nu}}{1 + J_\nu/J_s}, \tag{2.1}$$

[1] The notation for absolute value is omitted from the expressions for κ, τ, and T_x of the maser transition. This should not cause any confusion, as it will always be clear which are the intrinsically negative quantities.

where J_s is the saturation intensity and $\kappa_{0\nu}$ is the standard absorption coefficient in the limit $J_\nu \ll J_s$. This result follows from the level population equations of the phenomenological two-level maser model and has been discussed at length many times (e.g., Elitzur 1982). Deviations from this standard form due to various frequency redistribution mechanisms (Elitzur 1990*b*) do not affect the overall maser power and will be neglected here.

In the unsaturated regime ($J_\nu \ll J_s$), $\kappa_\nu = \kappa_{0\nu}$, and the equation of radiative transfer can be solved immediately for any geometry, yielding

$$I_\nu = (S_0 + I_e) \exp(\kappa_{0\nu} l) - S_0 , \tag{2.2}$$

where l is length along the ray path, I_e is the intensity of an external source which may illuminate the maser backside, and S_0 is the (radiation-independent) source function; it is related to the inversion efficiency η ($\equiv \Delta p/p$, where p is the sum of the pump rates into the maser levels and Δp is the difference) through

$$S_0 = \frac{2h\nu^3}{c^2} \frac{1+\eta}{2\eta} . \tag{2.3}$$

The various quantities can be expressed in terms of their equivalent temperatures. Assuming that all of these temperatures are in the Rayleigh-Jeans regime, the solution becomes

$$T_b = (T_{x0} + T_e) \exp(\kappa_{0\nu} l) - T_{x0} , \tag{2.4}$$

where T_{x0} is the (absolute value of the) excitation temperature corresponding to the unsaturated population difference. In the absence of external radiation, the intensity can be brought to the form

$$I_\nu = \left(\frac{1}{4\pi}\right) h\nu A_{21} N_{2\nu} l_0 \frac{\exp \tau_\nu - 1}{\tau_\nu} , \tag{2.5}$$

where A_{21} is the coefficient for spontaneous emission and l_0 and τ_ν are, respectively, the source dimension and gain along the ray trajectory. This result will be used to derive the escape probability for maser radiation (Paper II).

The exponential growth of the intensity slows down once J_ν becomes comparable to J_s and the maser saturates. The absorption coefficient then obeys $\kappa_\nu = \kappa_{0\nu} J_s/J_\nu$, and the standard flux divergence relation $\nabla \cdot \boldsymbol{F}_\nu = 4\pi\kappa_\nu(S_\nu - J_\nu)$ (e.g., Mihalas 1978) becomes

$$\nabla \cdot \boldsymbol{F}_\nu = 4\pi\kappa_{0\nu} J_s , \tag{2.6}$$

since $S_\nu \ll J_\nu$. Thus the maser flux increases linearly with overall dimension for sufficiently large, saturated masers if the pump rates remain constant throughout; a more exact, formal derivation of this result is provided in Paper II. The intensity scale of maser emissivity is set by J_s in the saturated domain and by the source function S_0 in the unsaturated region. The ratio of these two intensity scales,

$$\gamma = J_s/S_0 , \tag{2.7}$$

is a measure of the amount of amplification the maser has to undergo before it saturates. From previous results it follows that

$$\gamma \simeq \eta\Gamma/A_{21} , \tag{2.8}$$

where Γ is the loss rate of the maser system. It is of the same order as the collision rate, $\sim 10^{-10}$ s^{-1}. The overall density N is probably of order 10^6, 10^9, and 10^{12} cm^{-3} in the strong OH, H_2O and SiO maser regions, respectively. It then follows that $\gamma \sim 10^7\eta$–$10^8\eta$ for astronomical masers. The inversion efficiency η is probably a few percent, so $\gamma \gtrsim 10^5$. The masers are expected to saturate when their gains exceed $\sim \ln \gamma \simeq 11$.

a) Radiative Transfer

The luminosity of a saturated maser can be determined from volume integration, using the Gauss theorem, of equation (2.6), without the need to solve the equation of radiative transfer. However, this equation is still necessary for studying the intensity on individual rays. The equation for any maser can be written as

$$\frac{dI_\nu}{d\kappa_{0\nu} l} = \frac{I_\nu}{1 + J_\nu/J_s} + S_m , \tag{2.9}$$

where the maser source function is $S_m = \epsilon_\nu/\kappa_{0\nu}$ and ϵ_ν is the volume emission coefficient. This is different from the standard definition of the source function ($S = \epsilon_\nu/\kappa_\nu$), since the intensity dependence of the absorption coefficient has been explicitly removed. The maser source function is

$$S_m = S_0 \frac{1 + \eta/(1 + J_\nu/J_s)}{1+\eta} , \tag{2.10}$$

where S_0 is the source function in the unsaturated limit. Goldreich and Keeley (1972) integrate the radiative transfer equation over frequency and introduce a line width, assumed constant, utilizing frequency integrals of I_ν. This step is not necessary, and the ν-dependence is carried here explicitly.

Throughout the discussion it was assumed, and we will continue to do so, that the physical conditions in the source, e.g., pump rates, densities, etc., are uniform. Even with this assumption the maser source function S_m still varies across the source owing to the effect of saturation on the emission coefficient (i.e., the population of the maser upper level). This is a minor effect, however; S_m only varies from S_0 for an unsaturated maser to $S_0/(1+\eta)$ in the strongly saturated limit. This small variation is of little consequence and will be ignored in the subsequent discussion. Thus, equation (2.9) with S_m replaced by S_0 will be used as the equation of radiative transfer that governs the maser behavior. It is easy to show that a simple transformation of the intensity and source function leads to an exact equation of the same mathematical form with an effective source function that is intensity-independent: the functions $S' = S_0/(1+\eta)$ and $I'_\nu = I_\nu + \eta S'$ obey a radiative transfer equation identical in form to equation (2.9). The intensity variation of the source function can therefore be easily incorporated, if desired.

For rays strong enough that the source term can be neglected, the equation becomes

$$\frac{dI_\nu}{dl} = \kappa_\nu I_\nu . \tag{2.11}$$

The gain along such rays therefore obeys

$$d\tau_\nu \equiv \kappa_\nu dl = dI_\nu/I_\nu , \tag{2.12}$$

and the intensity varies according to

$$I_\nu(l) = I_\nu(l_1) \exp \tau_\nu(l_1, l) , \tag{2.13}$$

where $\tau_\nu(l_1, l)$ is the gain from the fiducial point l_1. This result shows that the saturated maser can be formally regarded as an amplifier, similar to the behavior in the unsaturated regime.

The difference is that whereas the unsaturated gain increases linearly with position along the ray trajectory, the gain growth in the saturated region is much more moderate (typically, only logarithmic) owing to saturation.

Consider now the two streams of radiation propagating in opposite directions on a given ray (Litvak 1973; Alcock and Ross 1985). Denote the appropriate intensities by $I_\nu(+)$ and $I_\nu(-)$, and the length element in the positive direction along this ray by ds. If both intensities obey equation (2.11), then

$$\frac{dI_\nu(+)}{ds} = \kappa_\nu I_\nu(+) , \quad \frac{dI_\nu(-)}{-ds} = \kappa_\nu I_\nu(-) . \tag{2.14}$$

The radiation moving outward grows with distance, while the intensity of the inward-moving stream is decreasing. In fact, from these equations it follows that in their region of applicability, the product $I_\nu(+)I_\nu(-)$ is constant along the ray path, and *the growing maser intensity increases at the expense of the oppositely moving radiation.* An exact relation for this product can be obtained by combining the full equations of radiative transfer (eq. [2.9]) for the streams $I_\nu(+)$ and $I_\nu(-)$ on any given ray. It follows immediately that

$$\frac{dI_\nu(+)I_\nu(-)}{d\kappa_{0\nu} s} = S_m[I_\nu(-) - I_\nu(+)] . \tag{2.15}$$

Thus,

$$I_\nu(+)I_\nu(-) = \int [I_\nu(-) - I_\nu(+)]S_m \kappa_{0\nu} ds . \tag{2.16}$$

This is obviously a general result, applicable also to nonmaser radiation when the product $S_m \kappa_{0\nu}$ is replaced by the emission coefficient ϵ_ν. It will be used extensively to obtain the intensity of the subordinate stream once the dominant intensity is known. As far as I can tell, this result is new.

III. THE LINEAR MASER

The linear maser is characterized by radiation flow along a single axis in the forward and backward directions. This is different from cylindrical geometry, where the rays fill a solid angle that varies with location. Positions in the linear maser are labeled by the coordinate z, which varies in the interval $[-\ell, \ell]$. The intensity can be written in the form

$$I_\nu(z, \mu) = I_{\nu+}(z)\delta(\mu - 1) + I_{\nu-}(z)\delta(\mu + 1) , \tag{3.1}$$

leading to

$$J_\nu(z) = \tfrac{1}{2}[I_{\nu+}(z) + I_{\nu-}(z)] , \quad F_\nu = 2\pi(I_{\nu+} - I_{\nu-}) . \tag{3.2}$$

The solution is thus completely characterized by the two functions $I_{\nu\pm}$. In the absence of external radiation, which is the case studied in this section, the model is symmetric under reflection about $z = 0$, so $I_{\nu+}(z) = I_{\nu-}(-z)$ and $J_\nu(z) = J_\nu(-z)$. Note that $I_{\nu\pm}$ are really components of J_ν and not the actual intensities of the linear maser. The latter, $I_\nu(z, \mu = \pm 1)$, diverge with the δ-functions. This is an anomaly of the linear geometry which arises because the radiation has been squeezed into an infinitely narrow beam. However, this is a trivial divergence, and the meaningful content of the solution *is* contained in the functions $I_{\nu\pm}$, which will therefore be referred to simply as intensities. The equations governing $I_{\nu\pm}$ are read from the coefficients of the δ-functions in the radiative transfer equation.

Consider first a maser that is unsaturated throughout, namely, $J_\nu(z) < J_s$ for all ν and z. The intensity is then given by equation (2.2), yielding

$$I_{\nu\pm}(z) = S_0\{\exp[\kappa_{0\nu}(\ell \pm z)] - 1\} \tag{3.3}$$

and

$$J_\nu(z) = \tfrac{1}{2}S_0\{\exp[\kappa_{0\nu}(\ell + z)] + \exp[\kappa_{0\nu}(\ell - z)] - 2\} . \tag{3.4}$$

These results display the required reflection symmetry. The intensity $J_\nu(z)$ has its minimum at the center and reaches maximum at the maser endpoints.

Consider next a succession of models with increasing length ℓ and with all the other parameters held fixed. When a certain length, $\ell_{s\nu}$, is reached, the end intensities $J_\nu(\pm\ell_{s\nu})$ exceed the saturation parameter J_s. The saturation length $\ell_{s\nu}$ can be found from

$$\exp(2\kappa_{0\nu}\ell_{s\nu}) = 2\gamma , \tag{3.5}$$

where we utilized the fact that $\gamma \gg 1$ (§ II). When the length is further increased, the maser develops a three-zone structure characterized by transitions at the (as yet undetermined) points $\pm z_{s\nu}$: a core ($|z| \le z_{s\nu}$), where $J_\nu < J_s$ and the maser is unsaturated, and two end regions ($|z| \ge z_{s\nu}$) with $J_\nu > J_s$ and a saturated behavior. The intensity in each zone can be obtained by solving the radiative transfer equation in the appropriate limit. The boundary conditions are $I_{\nu-}(\ell) = I_{\nu+}(-\ell) = 0$ and $J_\nu(\pm z_{s\nu}) = J_s$. The dimension of the unsaturated core can be found from the continuity of the solution across its boundaries.

In the unsaturated core the intensity is exponentially amplified, and

$$I_{\nu+}(z) = [S_0 + I_{\nu+}(-z_{s\nu})] \exp[\kappa_{0\nu}(z + z_{s\nu})] - S_0 . \tag{3.6}$$

The expression for $I_{\nu-}(z)$ can be obtained from symmetry. The core amplifies two input terms: the source function S_0 and the radiation flowing in from the $z \le -z_{s\nu}$ saturated region. At this point we cannot tell which is more important, since the intensity $I_{\nu+}(-z_{s\nu})$ has not yet been determined.

a) The Saturated Zone

The solution in the saturated zone will be obtained using the following technique: First, solve for the intensity of the dominant stream, which is equivalent to solving for the angle-averaged intensity J_ν. Next, the intensity of the subordinate stream is obtained from equation (2.16). This technique can be easily extended to other geometries as well.

Consider the saturated zone $z \ge z_{s\nu}$. The angle-averaged intensity is obviously dominated in this region by $I_{\nu+}$. To demonstrate this, consider first the boundary point $z_{s\nu}$. From equation (3.6) and the symmetry relation $I_{\nu+}(-z_{s\nu}) = I_{\nu-}(z_{s\nu})$, it follows that

$$I_{\nu+}(z_{s\nu}) > I_{\nu-}(z_{s\nu}) \exp(2\kappa_{0\nu} z_{s\nu}) \gg I_{\nu-}(z_{s\nu}) .$$

The intensity of the radiation moving outward, having undergone exponential amplification across the unsaturated core, greatly exceeds that of the opposite flow. Moving farther out, $I_{\nu+}$ keeps growing with z while $I_{\nu-}$ decreases toward $I_{\nu-}(\ell) = 0$, and the dominance of $I_{\nu+}$ is enhanced even more. Thus, the radiation propagating outward strongly dominates throughout the entire saturated region, and $J_\nu(z) = \frac{1}{2}I_{\nu+}(z)$. From equation (2.11), the radiative transfer equation for the dominant stream is then

$$\frac{dI_{\nu+}(z)}{dz} = 2\kappa_{0\nu} J_s , \tag{3.7}$$

subject to the boundary condition $I_{\nu+}(z_{s\nu}) = 2J_s$. The solution is

$$I_{\nu+}(z) = 2J_s m_\nu(z) \, , \quad J_\nu(z) = J_s m_\nu(z) \, ,$$
$$m_\nu(z) = 1 + \kappa_{0\nu}(z - z_{s\nu}) \, . \tag{3.8}$$

The intensity in the saturated zone increases linearly with length; in particular, the intensity of the emerging radiation is

$$J_\nu(\ell) = \tfrac{1}{2} I_{\nu+}(\ell) = J_s \kappa_{0\nu} \ell \, , \tag{3.9}$$

assuming $\ell \gg z_{s\nu}$. The two halves of the maser can be regarded as independent sources, each one radiating outward from its endpoint in proportion to its length. The linear increase with length of the emitted intensity is a direct consequence of equation (2.6) in the case of linear geometry. The linear increase with length of J_ν is a general result, independent of geometry (Paper II).

The intensity $I_{\nu-}$ can be obtained from the radiative transfer equation

$$\frac{dI_{\nu-}(z)}{d\kappa_{0\nu} z} = \frac{I_{\nu-}(z)}{m_\nu(z)} + S_0 \, , \tag{3.10}$$

subject to the boundary condition $I_{\nu-}(\ell) = 0$. However, it is more convenient to use the general result of equation (2.16), which becomes

$$I_{\nu-}(z) = \frac{S_0}{I_{\nu+}(z)} \int_z^\ell I_{\nu+}(z') \kappa_{0\nu} \, dz' \, . \tag{3.11}$$

The growth of $I_{\nu-}$, moving from the $z = \ell$ endpoint inward, is governed by $I_{\nu+}$ because this stream controls the inversion. The intensity scale of the subordinate stream is set by the source function S_0, while that of the dominant stream is given by the saturation intensity J_s. This demonstrates again that $I_{\nu+} \gg I_{\nu-}$ in the saturated region and the solution is self-consistent. The integration is immediate, leading to

$$I_{\nu-}(z) = S_0 \frac{m_\nu^2(\ell) - m_\nu^2(z)}{2m_\nu(z)} \, . \tag{3.12}$$

Near the maser endpoint, $I_{\nu-}(z) \simeq S_0 \kappa_{0\nu}(\ell - z)$ and the intensity of the inward-moving radiation quickly exceeds the source function. The intensity of this stream at the saturation edge is

$$I_{\nu-}(z_{s\nu}) = \tfrac{1}{2} S_0 (\kappa_{0\nu} \ell)^2 b \, , \tag{3.13}$$

where

$$b = (1 - z_{s\nu}/\ell)(1 - z_{s\nu}/\ell + 2/\kappa_{0\nu}\ell) \, ;$$

obviously, $b \simeq 1$. Thus, $I_{\nu-}(z_{s\nu}) \gg S_0$, and from equation (3.6) and the symmetry relation $I_{\nu+}(-z_{s\nu}) = I_{\nu-}(z_{s\nu})$ it follows that the angle-averaged intensity at the edge of the unsaturated core is

$$J_\nu(z_{s\nu}) = \tfrac{1}{2} I_{\nu-}(z_{s\nu}) \exp(2\kappa_{0\nu} z_{s\nu}) \, . \tag{3.14}$$

The saturation condition $J_\nu(z_{s\nu}) = J_s$ therefore becomes

$$\exp(2\kappa_{0\nu} z_{s\nu}) = 4\gamma(\kappa_{0\nu}\ell)^{-2} \, , \tag{3.15}$$

and the core location can finally be determined:

$$\kappa_{0\nu} z_{s\nu} = \kappa_{0\nu} \ell_{s\nu} - \ln(2^{-1/2} \kappa_{0\nu} \ell) \, . \tag{3.16}$$

The unsaturated core is slowly shrinking with increasing maser length. This effect is dictated by the boundary conditions. Radiation emerging from the core entered it at the opposite end with intensity that increases with source length, and subsequently underwent exponential amplification across the core. Yet its exit intensity is always $2J_s$, and thus the core gain must decrease. The core shrinkage is caused by the contribution of the inward-moving radiation to $J_\nu(z_{s\nu})$, which pushes the boundary closer to the center. The location of the boundary depends on frequency. Since $\kappa_{0\nu}$ is peaked at ν_0, the radiation produced at line center will travel the shortest distance before saturation takes effect. If the unsaturated absorption coefficient is Doppler-shaped, i.e., $\kappa_{0\nu} = \kappa_0 \exp(-x^2)$, where x is the dimensionless frequency shift from line center, then

$$\kappa_{0\nu} z_{s\nu} = \kappa_0 z_0 + x^2 \, , \tag{3.17}$$

where z_0 is the location of the saturation boundary at the line-center frequency ν_0. This result displays the explicit dependence on frequency of the boundary of the saturated region.

The integral in the expression for $I_{\nu-}$ (eq. 3.11) is dominated by the upper limit of the integration when $z < \ell$ and is thus independent of z. The product $I_{\nu+} I_{\nu-}$ is therefore constant for most of the maser, as seen earlier (§ II; this was first pointed out for the linear maser by Alcock and Ross 1985), and can be evaluated at, e.g., $z = z_{s\nu}$. Since at the center $I_{\nu+}(0) = I_{\nu-}(0) = J_\nu(0)$,

$$I_{\nu+}(z) I_{\nu-}(z) = J_\nu^2(0) = \gamma^{-1}(J_s \kappa_{0\nu} \ell)^2 \, , \tag{3.18}$$

or

$$J_\nu(0) = J_\nu(\ell)\gamma^{-1/2} \, . \tag{3.19}$$

The central and edge intensities of a linear maser are at a fixed ratio, which depends on the pumping scheme but is independent of the maser length; when the overall length increases, the central intensity grows in proportion to the intensities at the maser endpoints.

The solutions obtained here are not entirely accurate near the saturation points, since they were derived under the assumptions of either $J_\nu \ll J_s$ in the core or $J_\nu \gg J_s$ in the saturated zone. These conditions are equivalent to $z \ll z_{s\nu}$ or $z \gg z_{s\nu}$ (namely, $m_\nu \gg 1$), respectively. This is a minor point, however, which can be dealt with quite easily. The radiative transfer equation of a linear maser in the $z > 0$ half can be written as

$$\frac{dJ_\nu}{d\kappa_{0\nu} z} = \frac{J_\nu}{1 + J_\nu/J_s} \tag{3.20}$$

wherever one stream dominates and the source function is negligible. Thus,

$$J_\nu/J_s + \ln(J_\nu/J_s) = m_\nu \, . \tag{3.21}$$

This solution holds everywhere, except near the maser center. Thus, previous results for the saturated and unsaturated limits are recovered. In particular, more accurate explicit expressions for J_ν are

$$J_\nu = J_s m_\nu \left(1 - \frac{\ln m_\nu}{m_\nu + 1}\right) \tag{3.22}$$

in the saturated zone, and

$$J_\nu = J_s[1 + \tfrac{1}{2}\kappa_{0\nu}(z - z_{s\nu})] \tag{3.23}$$

near the saturation boundary.

The gain in the saturated region,

$$\tau_\nu(z) = \int_{z_{s\nu}}^{z} \kappa_\nu(z)dz \,, \tag{3.24}$$

can be obtained at once; since $\kappa_\nu(z) = \kappa_{0\nu}/m_\nu(z)$,

$$\tau_\nu(z) = \ln m_\nu(z) \,. \tag{3.25}$$

Thus, the function m_ν is simply the amplification factor exp $\tau_\nu(z_{s\nu}, z)$. The gain does not increase linearly with distance anymore because the inversion (and, thus, the absorption coefficient) is reduced by saturation. The logarithmic increase of the saturated gain ensures that the dominant intensity grows only linearly with distance. The intensity of the subordinate stream (eq. [3.12]) varies exponentially with gain when $m_\nu^2(z) \ll m_\nu^2(\ell)$, in accordance with equation (2.13).

By combining the contributions to the gain across the entire maser for frequencies obeying $z_{s\nu} \ll \ell$, it is easy to show that

$$\tau_\nu(-\ell, \ell) = \ln 4\gamma \,. \tag{3.26}$$

This result can also be derived by considering the ratio of edge to center intensities. It shows that the maser overall gain can increase by only ln 2 from the value it had when the maser had just saturated (eq. [3.5]). The total amplification grows by a factor of 2 toward the limit 4γ, obtained when $\ell \gg \ell_{s\nu}$. The fact that the overall gain is determined by the pump parameters and is independent of length is another manifestation of the self-regulating nature of the saturated maser. The shrinking of the core gets another explanation: the amplification across the core must decrease in proportion to $(\kappa_{0\nu}\ell)^2$ to offset the $\kappa_{0\nu}\ell$ growth of the amplification factor in each saturated zone.

A somewhat related result can be obtained for the absorption coefficient. In the saturated zone,

$$\kappa_\nu(z) = \kappa_{0\nu} J_s/J_\nu \simeq 1/z \,, \tag{3.27}$$

where the last equality holds for $z \gg z_{s\nu}$. Therefore, the product $\kappa_\nu z$ simply approaches unity. It is altogether independent of the source parameters, such as pumping scheme or overall length.

The solution for a saturated maser with an unsaturated core is now complete. It is worthwhile summarizing the results using dimensionless quantities. Natural scales for lengths and intensities are provided by $\kappa_{0\nu}$ and J_s, respectively, so introduce

$$\zeta_\nu \equiv \kappa_{0\nu} z \,, \quad \zeta_{s\nu} \equiv \kappa_{0\nu} z_{s\nu} \,, \quad \mathscr{L}_\nu \equiv \kappa_{0\nu}\ell \,, \quad \mathscr{I}_{\nu\pm} \equiv I_{\nu\pm}/J_s \,.$$

The solution in dimensionless form is then

$$\zeta_{s\nu} = \ln(2\gamma^{1/2}/\mathscr{L}_\nu) \,,$$

$$0 \le \zeta_\nu \le \zeta_{s\nu}: \quad \mathscr{I}_{\nu\pm} = 2 \exp(\pm\zeta_{s\nu} - \zeta_\nu) \,,$$

$$\zeta_{s\nu} \le \zeta_\nu \le \mathscr{L}_\nu: \quad \mathscr{I}_{\nu+} = 2m_\nu(\zeta_\nu) \,, \quad m_\nu(\zeta_\nu) = 1 + (\zeta_\nu - \zeta_{s\nu}) \,,$$

$$\mathscr{I}_{\nu-} = [m_\nu^2(\mathscr{L}_\nu) - m_\nu^2(\zeta_\nu)]/[2\gamma m_\nu(\zeta_\nu)] \,. \tag{3.28}$$

The solution for $\zeta_\nu < 0$ is obtained from the symmetry under reflection. The dimensionless solution depends on only two free parameters: γ $(=J_s/S_0)$, and the overall scaled length $\mathscr{L}_\nu$. The dependence on either of those is rather weak, however. These parameters affect only the profile of the inward-moving stream in the saturated zone, where it is of secondary importance, and enter logarithmically in the determination of the boundary $\zeta_{s\nu}$.

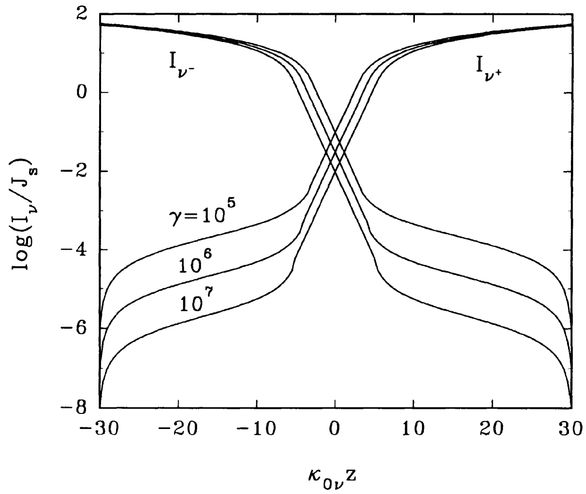

FIG. 1.—Intensities of the two rays in a linear maser as functions of position for various values of the parameter γ, as marked on the curves.

Figure 1 displays on a logarithmic scale the intensities of a linear maser whose overall length is $\mathscr{L}_\nu = 30$ for $\gamma = 10^5$, 10^6, and 10^7. These values of γ span the probable range encountered in astronomical masers, as mentioned above (§ II). The value of $\mathscr{L}_\nu$ is also representative. For example, the gain of the OH 1612 MHz maser can be obtained from

$$\kappa_{0\nu}\ell = 5.6 \times 10^{-17} N_{\rm OH}\,\eta_{-2}/\Delta v_1 \,, \tag{3.29}$$

where $N_{\rm OH}$ (cm^{-2}) is the OH column density, η_{-2} is the inversion efficiency in percent, and Δv_1 is the line width in kilometers per second. The calculations of Goldreich and Scoville (1976) show that $N_{\rm OH}$ can exceed $\sim 3 \times 10^{16}$ cm^{-2} in the 1612 MHz maser region of a late-type star. The inversion efficiency of this maser can be as high as 25%, and $\mathscr{L}_\nu$ can reach ~ 30 or so. In the case of the strong H_2O masers in star-forming regions, the recent model of Elitzur, Hollenbach, and McKee (1989), which describes the masers as filamentary structures behind shock waves, produces

$$\kappa_{0\nu}\ell \simeq 3.9a \,, \tag{3.30}$$

where a is the aspect ratio (length/width) of the maser filament. The numerical coefficient is for a prototype model with H_2 density 10^9 cm^{-3}, temperature 400 K, H_2O abundance relative to H_2 of 6×10^{-4}, and filamentary diameter 10^{13} cm (the calculated efficiency is $\eta = 8.7 \times 10^{-3}$). A filament with an aspect ratio of 25 therefore corresponds to $\mathscr{L}_\nu$ close to 100. These results for the values of $\kappa_{0\nu}\ell$ provide estimates for the degree of saturation, J_ν/J_s, in each of the relevant sources, as evident from equation (3.9).

The figure reproduces all the features of the numerical solutions of Alcock and Ross (1985). It is instructive to follow the rightward-moving stream across the maser. Entering the source from the left, its intensity is zero. Although the source function S_0 is completely negligible here in comparison with the maser intensity J_ν, some spontaneous decays do occur, and half of those produce photons that travel inward. The maser is strongly saturated by the outward-moving dominant stream $\mathscr{I}_{\nu-}$, and the $\mathscr{I}_{\nu+}$ stream builds up quickly. Initially, $\mathscr{I}_{\nu+} \simeq \gamma^{-1}(\mathscr{L} - |\zeta|)$; the subordinate stream is made up of the linearly amplified source function. A little deeper into the source

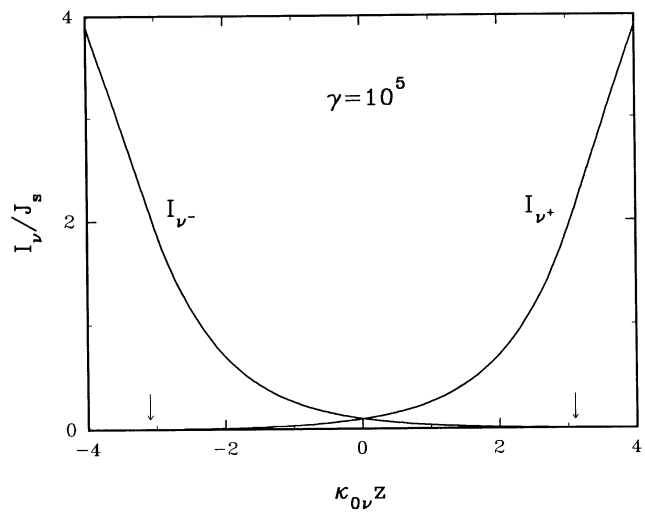

FIG. 2.—Central region of a linear maser with $\gamma = 10^5$. The arrows mark the core boundary points.

$I_{\nu+}$ exceeds S_0, becoming itself a more significant input source, and its growth with distance is quadratic. However, as the figure demonstrates vividly, $\mathscr{I}_{\nu+}$ is still completely negligible in comparison with $\mathscr{I}_{\nu-}$ in the entire left saturated zone. The $\mathscr{I}_{\nu+}$ stream becomes significant only after entering the unsaturated core (at $-\zeta \simeq 3$–5), where it undergoes exponential amplification, emerging from the core as the dominant stream. Afterward, its intensity increases linearly with distance and the dependence on the model parameters is essentially gone. The intensity of the dominant stream, which also controls the behavior of the subordinate stream, is determined by the pump. The detailed solution demonstrates why the intensity of a saturated maser does not depend on the source function, even though spontaneous decays provide the original seed photons.

The logarithmic scale, necessary for the display of the full solution, is somewhat misleading in the prominent exposure it provides to negligible intensities. Figure 2 therefore plots the central region of the $\gamma = 10^5$ maser on a linear scale. The saturation boundary points, $\zeta_s = \pm 3.05$ in this case, are marked with arrows. It is obvious that, to a good degree of approximation, each stream can be considered as if it originated from the far end of the unsaturated core. Insofar as only observed quantities are concerned, a very satisfactory solution can be obtained by neglecting altogether the inward-moving stream in each saturated zone. The core boundary is then approximated by the fixed value $z_{s\nu} = \ell_{s\nu}$. The expressions for all the quantities relevant for observations are the same as for the full solution. It is also worth emphasizing that in spite of its great significance for the buildup of maser intensity, the core occupies only a negligible fraction of the source. Neglecting the core altogether and employing the expressions $I_{\nu+} = 2J_s\kappa_{0\nu}z$, $I_{\nu-} = 0$ for $z \geq 0$ is a reasonable approximation for all the relevant properties of the saturated linear maser. Obviously, though, this approximate solution misses entirely the essence of the physics.

Some further insight into the behavior of the linear maser can be gained from an integral of motion, recently obtained by Alcock and Ross (1985). To derive it, note that for any maser, whatever the geometry, the radiative transfer equations for the two streams along any ray can always be combined to produce

$$-\frac{dI_{\nu+}}{dI_{\nu-}} = \frac{J_sI_{\nu+} + S_0(J_\nu + J_s)}{J_sI_{\nu-} + S_0(J_\nu + J_s)} . \qquad (3.31)$$

From this it follows that for the linear maser

$$[J_\nu(z) + J_s]^2 + \gamma I_{\nu+}(z)I_{\nu-}(z) = [J_\nu(\ell) + J_s]^2 , \qquad (3.32)$$

which is the Alcock and Ross result. This relation can be used to derive the intensity of the subordinate stream in the saturated zone (eq. [3.12]) and the intensity at the maser central point (eq. [3.19]). The Alcock and Ross integral of motion can also be derived from the equations for the angle-averaged intensity J_ν and the normalized flux $H_\nu \equiv F_\nu/4\pi = \frac{1}{2}(I_{\nu+} - I_{\nu-})$. The transfer equations for these two quantities are

$$\frac{dJ_\nu}{d\kappa_{0\nu}z} = \frac{H_\nu}{1 + J_\nu/J_s} , \quad \frac{dH_\nu}{d\kappa_{0\nu}z} = \frac{J_\nu}{1 + J_\nu/J_s} + S_0 , \qquad (3.33)$$

and the boundary conditions are $H_\nu(0) = 0$, $H_\nu(\ell) = J_\nu(\ell)$. These equations lead to the integral of motion

$$(J_\nu + J_s)^2 + \gamma(J_\nu^2 - H_\nu^2) = \text{const} , \qquad (3.34)$$

which is the same as equation (3.32).

b) *Core Saturation*

When the maser length is sufficiently increased, the intensity of the subordinate stream can become so high as to equal J_s at the core boundary. This happens when $\kappa_{0\nu}\ell = (2\gamma)^{1/2}$ (eq. [3.13]). Thus, for maser lengths exceeding this limit, the core saturates because of the effect of the inward-moving stream. Indeed, equation (3.18) shows that $J_\nu(0) = J_s$ when the length equals $\ell_{c\nu}$, defined by

$$\kappa_{0\nu}\ell_{c\nu} = \gamma^{1/2} . \qquad (3.35)$$

The maser is saturated throughout, and the solution must be modified. When complete saturation sets in ($\ell > \ell_{c\nu}$; J_ν exceeds J_s everywhere), the equations for J_ν and H_ν are easy to solve. The solution for all $z \geq 0$ to leading order in γ^{-1} is

$$H_\nu = J_s\kappa_{0\nu}z , \quad J_\nu = J_s\kappa_{0\nu}(z^2 + z_c^2)^{1/2} , \qquad (3.36)$$

where

$$z_c = \ell\gamma^{-1/2} .$$

The intensities can be obtained from $I_{\nu\pm} = J_\nu \pm H_\nu$. The structure of this solution undergoes a complete change across the point $z = z_c$. Inside this transition point,

$$z < z_c : \quad J_\nu = J_s\kappa_{0\nu}z_c , \quad H_\nu = 0 , \quad I_{\nu+} = I_{\nu-} = J_\nu \qquad (3.37)$$

to leading order in z/z_c. That is, the intensities of the two streams are equal, both intensities and the angle-averaged intensity are constant, and the flux vanishes. Outside the transition point,

$$z > z_c : \quad J_\nu = H_\nu = \tfrac{1}{2}I_{\nu+} = J_s\kappa_{0\nu}z \qquad (3.38)$$

to leading order in z_c/z. This solution is the same as the one obtained previously in the saturated zone of a maser with an unsaturated core. Therefore, the maser is again divided into three zones, as before: a central core ($|z| < z_c$) where the intensity J_ν is nearly constant, the flux is approximately zero, and the two streams are approximately equal, and two end regions ($|z| > z_c$) where all the expressions are essentially the same as when the core was unsaturated. The frequency-independent point z_c therefore defines the boundary of a core, which is now saturated. The solution is identical to the previous one, except inside the core. The shrinking of the core during its unsaturated phase is reversed, and its size grows with the overall length during complete saturation. However, the core still occupies only a negligible, constant fraction of the maser during this phase. Outside the core, which is most of the maser, the solution is the same as when the core was unsaturated.

The approximate properties of the core in the limit of complete saturation can also be obtained from general arguments, an approach useful for other geometries where an exact solution is not available. A core of finite size is always necessary for the structure of the maser to enable the intensities $I_{\nu+}$ and $I_{\nu-}$ to reverse their roles as dominant and subordinate streams in a continuous fashion near the center. To leading order in γ^{-1}, the only possible solutions for equations (3.33) during saturation are $H_\nu = J_\nu$, which is the solution in the saturated zones where one stream dominates, and $H_\nu \simeq 0$ and $J_\nu \simeq$ constant, which is the saturated core solution (this is also evident from the integral of motion [3.34]). The two streams are therefore equal across the saturated core, and the angle-averaged intensity can be approximated by a constant inside the core and by $J_s \kappa_{0\nu} z$ for $z \geq z_c$. The core boundary is determined by the condition that at that point both streams contribute equally to J_ν. For the linear maser this implies $I_{\nu+}(z_c) = I_{\nu-}(z_c) = J_\nu(z_c)$. The core equation is thus

$$I_{\nu+}(z_c) I_{\nu-}(z_c) = J_\nu^2(z_c) . \tag{3.39}$$

But this product must also obey equation (3.18) because that relation follows from the properties of the saturated region and is independent of the behavior at the core. Therefore, $z_c = \ell\gamma^{-1/2}$, as before. The equality of the two streams across the core follows also from the fact that the relation $I_{\nu+} = I_{\nu-}$ holds both at the core's endpoints and at the center $z = 0$ (from symmetry). Thus, $H_\nu \simeq 0$ across the core and J_ν is constant.

It is unlikely that core saturation ever occurs in astronomical masers because $(2\gamma)^{1/2}$ is at least ~ 500. In contrast, $\kappa_{0\nu}\ell$ probably never exceeds ~ 50 in most sources other than the strong H_2O masers in star-forming regions. For those, the estimate of equation (3.30) shows that $\kappa_{0\nu}\ell$ can reach ~ 500 only for aspect ratio a in excess of ~ 130. Such high aspects are unlikely, though not impossible (Elitzur, Hollenbach, and McKee 1989). However, it must be noted that this is a hybrid estimate, since the results of a filamentary model calculation are inserted in a linear maser model. When the proper result for a filamentary maser is used, the relevant gain is across rather than along the filament and it is evident that core saturation is essentially impossible in H_2O masers (Paper III). In addition, core saturation has no effect at all on the maser's observed properties, since the solution in the end regions is independent of the core properties. This limit is therefore mostly only of theoretical interest.

IV. BACKGROUND RADIATION

The only source of input radiation considered in the discussion so far was spontaneous decays inside the maser itself. An additional source of seed photons is sometimes provided by external radiation entering the maser. The effects of such input radiation are discussed now.

Consider first an unsaturated maser illuminated by external radiation I_e. For any geometry, the intensity along rays that intersect the external source is then given by equation (2.2) (or, equivalently, eq. [2.4]). The intensity modification is comprised of $S_0 \rightarrow S_0 + I_e$, and the effect of the background source on the solution is thus measured by the parameter

$$\gamma_e = I_e/S_0 , \tag{4.1}$$

equivalent to T_e/T_{x0}. An extended, unsaturated maser cloud in front of a point source will therefore display enhanced emission in the directions that intersect the background source, resulting in the appearance of a bright spot. The contrast χ at the bright spot is defined as

$$\chi \equiv \frac{I_\nu}{I_{\nu nb}} \tag{4.2}$$

where the subscript nb denotes the intensity in the absence of external radiation. For an unsaturated maser,

$$\chi = 1 + \gamma_e . \tag{4.3}$$

The bright spot contrast is thus determined exclusively by the parameter γ_e.

When the maser is saturated, the effects of the background source depend on the model geometry, and the discussion is confined from now on to the linear maser, which can be solved exactly. The behavior of the solution can actually be derived from some simple arguments, without even solving the equation of radiative transfer. To do that, consider first doubling the maser length to 4ℓ while keeping all the other parameters constant. The core is then located at the center of the new structure, its size shrunk by $2 \ln 2$ from the value given in equation (3.16). From equation (3.13), the intensity of the internally generated rightward-moving stream at the core entrance is

$$I_{\rm in} = 2S_0(\kappa_{0\nu}\ell)^2 . \tag{4.4}$$

Now remove the left half of this maser and replace it with external radiation whose intensity is equal to $I_{\rm in}$, entering from the left. All the properties of the $I_{\nu+}$ intensity remain exactly the same, and since this stream controls the maser structure, this too remains unchanged. Therefore, external radiation whose intensity is as high as $I_{\rm in}$ will cause the core to move all the way to the left edge and shrink by $\ln 4$. The maser emergent intensity $I_{\nu+}(\ell)$ would obviously double. When the external intensity is slowly turned off, the solution with $I_e = 0$ must be recovered with the core back at the center and in its original size. Therefore, the reverse process, wherein the external radiation is slowly turned on, obviously causes a leftward motion of the core, coupled with its slow shrinkage. The external radiation begins to affect the maser structure once its intensity at core entrance,

$$I_{\rm ex} = I_e \kappa_{0\nu}\ell \tag{4.5}$$

(as can be easily seen from eq. [2.16]), becomes comparable to $I_{\rm in}$. The impact of external radiation on the solution is therefore expected to be measured by the parameter

$$\mathscr{R}_e \equiv \frac{J_{\rm ex}}{J_{\rm in}} = \frac{\gamma_e}{2\kappa_{0\nu}\ell} . \tag{4.6}$$

In this definition, the ratio of intensities I_ν was replaced by that of angle-averaged J_ν, which is the same for a linear maser. This substitution ensures the general nature of the definition, making it suitable also to three-dimensional geometries.

These properties are displayed by the detailed solution of the equation of radiative transfer. This equation contains no reference to the external radiation, so the structural form of its solution must remain the same. The information about the external radiation is contained only in the boundary condition $I_{\nu+}(-\ell) = I_e$, for a maser illuminated from the left, whose effects propagate through the various maser zones. The starting point of the discussion is therefore a linear maser strongly saturated by its own radiation. This implies that $\ell \gg \ell_{s\nu}$, which also ensures that $\ell \gg z_{s\nu}$. Consider now a series of models with an increasing external intensity I_e (i.e., γ_e) and with all the other parameters held fixed. Since the model is not symmetric upon reflection anymore, the core need not be centered on the origin. The left and right core boundary points will be denoted by $z_{s\nu-}$ and $z_{s\nu+}$, respectively; in the absence of external radiation, $z_{s\nu\pm} = \pm z_{s\nu}$ obviously. The radiation emerging from the core in either direction corresponds to saturated intensity, so

$$I_{\nu+}(z_{s\nu+}) = I_{\nu-}(z_{s\nu-}) = 2J_s . \qquad (4.7)$$

Both streams are amplified across the core by the same amount, so their intensities must also be equal upon entry to the core on either end:

$$I_{\nu+}(z_{s\nu-}) = I_{\nu-}(z_{s\nu+}) = 2J_s \exp[-\kappa_{0\nu}(z_{s\nu+} - z_{s\nu-})] . \qquad (4.8)$$

But the intensity $I_{\nu+}$ in the $z \leq z_{s\nu-}$ saturated zone contains also the external radiation and is therefore larger than $I_{\nu-}$ at the reflected point in the $z \geq z_{s\nu+}$ region. The only way to reconcile this asymmetry with equation (4.8) is for the core to shift to the left. The subordinate $I_{\nu+}$ stream requires less amplification to meet the core entry condition, and the core is situated closer to the left edge of the maser. The core boundaries thus correspond to leftward shifts by the amounts $s_{\nu\pm}$ from their locations in the absence of external radiation, namely,

$$z_{s\nu\pm} = \pm z_{s\nu} - s_{\nu\pm} . \qquad (4.9)$$

The intensities at the core entry points can be determined from the equation for the subordinate stream (eq. [2.16]), which becomes

$$I_{\nu+}(z)I_{\nu-}(z) = I_{\nu+}(-\ell)I_{\nu-}(-\ell) + S_0 \int_{-l}^{z} [I_{\nu-}(z') - I_{\nu+}(z')]\kappa_{0\nu} dz' \qquad (4.10)$$

in the left saturated zone ($z \leq -z_{s\nu-}$); note again that $I_{\nu+}(-\ell) = I_e$. Thus,

$$\begin{aligned} I_{\nu+}(z_{s\nu-}) &= I_e m_{\nu-} + \tfrac{1}{2}S_0(m_{\nu-}^2 - 1) , \\ I_{\nu-}(z_{s\nu+}) &= \tfrac{1}{2}S_0(m_{\nu+}^2 - 1) , \\ m_{\nu\pm} &= 1 + \kappa_{0\nu}(\ell - z_{s\nu} \pm s_{\nu\pm}) . \end{aligned} \qquad (4.11)$$

At the core entry, the amplification of the external radiation is linear, while that of the source term is quadratic. The reason is that the source term generates photons throughout the entire maser. The intensity emerging from the right end of the maser is

$$I_{\nu+}(\ell) = 2J_s m_{\nu+} . \qquad (4.12)$$

Because $z_{s\nu} \ll \ell$, it can be neglected in the expression for $m_{\nu+}$ always, and also in $m_{\nu-}$ so long as $\ell \gg s_{\nu-}$. The maser contrast factor χ, the intensity enhancement due to the external radiation (eq. [4.2]), is thus

$$\chi = 1 + s_{\nu+}/\ell . \qquad (4.13)$$

This result simply reflects the fact that the saturated maser intensity is proportional to the length of the saturated region. The maser contrast depends only indirectly on I_e through its effect on the location of the core boundary. Obviously, $\chi \leq 2$; the contrast of a saturated maser is limited because the length of the saturated region can only increase by a factor of 2 when the core shifts all the way to the left.

The core-amplification relation for $I_{\nu-}$ (eq. [4.8]) becomes

$$\exp[\kappa_{0\nu}(2z_{s\nu} + s_{\nu-} - s_{\nu+})] = \frac{4\gamma}{(\kappa_{0\nu}\ell)^2}\left(1 + \frac{s_{\nu+}}{\ell}\right)^{-2} , \qquad (4.14)$$

where we utilized the fact that $\kappa_{0\nu}\ell \gg 2$. Inserting the expression for $z_{s\nu}$ (eq. [3.15]) yields

$$\kappa_{0\nu}(s_{\nu+} - s_{\nu-}) = 2 \ln \chi . \qquad (4.15)$$

From equations (4.13) and (4.15) it follows that $(s_{\nu+} - s_{\nu-})/s_{\nu+} \lesssim 2/\kappa_{0\nu}\ell \ll 1$, so

$$s_{\nu+} \simeq s_{\nu-} \equiv s_\nu . \qquad (4.16)$$

To lowest order, the core moves to the left as a whole unit. Combining equations (4.8), (4.11), and (4.16) produces

$$s_\nu = \ell \frac{\mathscr{R}_e}{1 + \mathscr{R}_e} \qquad (4.17)$$

as expected (note the discussion leading to eq. [4.6]). The emergent intensity is enhanced over its value in the absence of external radiation by

$$\chi = 1 + \frac{\mathscr{R}_e}{1 + \mathscr{R}_e} . \qquad (4.18)$$

The contrast factors for saturated and unsaturated behavior can be written in a combined form:

$$\chi = 1 + \gamma_e \begin{cases} 1 , & \text{unsaturated} , \\ 1/(\gamma_e + 2\kappa_{0\nu}\ell) , & \text{saturated} . \end{cases} \qquad (4.19)$$

This displays the unlimited contrast that can be obtained for an unsaturated maser (as long as the external radiation is not so strong as to cause saturation itself) and the bound provided by the saturation process.

The behavior of the contrast factor χ and the maser structure with increasing I_e are best analyzed in different intensity regimes:

1. $\gamma_e < 2\kappa_{0\nu}\ell$: In this regime, $I_e < 2S_0\kappa_{0\nu}\ell$, that is, $I_{ex} < I_{in}$ or $\mathscr{R}_e < 1$. The external intensity at core entrance is a perturbation on the internally generated radiation. The core shifts steadily to the left in proportion to γ_e. This shift is accompanied by a slow (logarithmic) shrinkage of the core and a steady increase in the contrast factor χ.

2. $2\kappa_{0\nu}\ell < \gamma_e < 2(\kappa_{0\nu}\ell)^2$: Because γ_e exceeds $2\kappa_{0\nu}\ell$, $\mathscr{R}_e > 1$ and $s_\nu \simeq \ell$. The intensity enhancement factor is $\chi = 2$, or

$$I_{\nu+}(\ell) = 4J_s\kappa_{0\nu}\ell . \qquad (4.20)$$

The saturation is dominated by the external radiation, which needs to be amplified in the left saturated zone before it can induce saturation. The core has shifted all the way to the left edge of the source and shrunk by the maximum amount of ln 4 (eq. [4.15]), as expected. Various terms have to be modified now because $z_{s\nu}$ cannot be neglected in the expression for $m_{\nu-}$ anymore. The intensities entering the core are

$$I_{\nu+}(z_{s\nu-}) = I_e(1 + \kappa_{0\nu}d_\nu) , \quad I_{\nu-}(z_{s\nu+}) = 2S_0(\kappa_{0\nu}\ell)^2 , \qquad (4.21)$$

where

$$d_\nu = \ell + z_{s\nu-}$$

is the core distance from the maser left end. The value of d_ν can be obtained by equating the two core entry intensities. As long

TABLE 1

VARIATION OF CORE POSITION AND SIZE WITH I_e

I_e (1)	$1 + z_{sv-}/l$ (2)	$(z_{sv+} - z_{sv-})/(2z_{sv})$ (3)
0	1	1
$[0, 2S_0 \kappa_{0\nu} \ell]$	$1/(1 + \mathscr{R}_e)$	$1 - (1/\kappa_{0\nu} z_{sv}) \ln [1 + \mathscr{R}_e/(1 + \mathscr{R}_e)]$
$[2S_0 \kappa_{0\nu} \ell, 2S_0(\kappa_{0\nu}\ell)^2]$	$2\kappa_{0\nu}\ell/\gamma_e - 1/\kappa_{0\nu}\ell$	$1 - (1/\kappa_{0\nu} z_{sv}) \ln 2$
$[2S_0(\kappa_{0\nu}\ell)^2, 2J_s]$	0	$(1/2\kappa_{0\nu} z_{sv}) \ln (2J_s/I_e)$
$> 2J_s$	0	0

Note.—Col. (1) lists the appropriate intensity range. Cols. (2) and (3) list the distance of the left core boundary from the edge and the core size, respectively, both normalized to their values in the absence of external radiation and assuming $z_{sv} \ll l$. The dimensionless parameters γ_e and $\mathscr{R}_e$ are defined in eqs. (4.1) and (4.6), respectively.

as $\gamma_e < 2(\kappa_{0\nu}\ell)^2$, the resulting equation can be solved and the core distance from the maser left end obeys

$$\kappa_{0\nu} d_\nu = \frac{2(\kappa_{0\nu}\ell)^2}{\gamma_e} - 1 . \qquad (4.22)$$

This result is self-consistent because $d_\nu/\ell \simeq 2\kappa_{0\nu}\ell/\gamma_e < 1$, as it should. The core-amplification condition, obtained by inserting the intensities from equation (4.21) in equation (4.8), becomes

$$\kappa_{0\nu}(z_{sv+} - z_{sv-}) = 2\kappa_{0\nu} z_{sv} - \ln 4 . \qquad (4.23)$$

This demonstrates once again that the core has shrunk by ln 4, verifying the consistency of the solution.

3. $2(\kappa_{0\nu}\ell)^2 < \gamma_e < 2J_s$: Since $\gamma_e > 2(\kappa_{0\nu}\ell)^2$, the external radiation can saturate the maser without any amplification in the left saturated zone. Equivalently, $I_{\nu-}(z_{sv+}) < I_e$: the intensity of the leftward-moving stream at its core entry point is smaller than I_e; thus $I_{\nu-}$ is too weak to saturate. The left saturated zone disappears, and the unsaturated region shifts all the way to the left edge.

4. $\gamma_e > 2\gamma$: The unsaturated region disappears altogether, since $I_e > 2J_s$. The external radiation does not require even core amplification to saturate the entire maser. Since the maser is fully saturated by I_e, its structure can best be obtained from equation (3.33). The angle-averaged intensity J_ν and the normalized flux H_ν obey

$$J_\nu(z) = H_\nu(z) = \tfrac{1}{2} I_e + J_s \kappa_{0\nu}(\ell + z) . \qquad (4.24)$$

Note that $\ell + z$ is simply the distance into the maser from its left end. The maser intensity $I_{\nu+} = 2J_\nu$ reflects the combined contributions of background radiation and saturated maser emission generated over the entire length $[-\ell, z]$. The observed intensity is

$$I_{\nu+}(\ell) = I_e + 4J_s \kappa_{0\nu} \ell , \qquad (4.25)$$

which finally displays again an explicit dependence on the external radiation. The intensity of the opposite stream can be obtained from equation (3.32); the intensity emerging from the $z = -\ell$ end is

$$I_{\nu-}(-\ell) = 2S_0 \kappa_{0\nu} \ell \left(1 + \frac{2J_s}{I_e} \kappa_{0\nu} \ell\right) . \qquad (4.26)$$

Note that $I_{\nu-}(-\ell)$ decreases with I_e, demonstrating explicitly that the $I_{\nu+}$ stream grows at the expense of the opposite stream. When the external radiation is so intense that $I_e > 2J_s \kappa_{0\nu} \ell$, $I_{\nu-}(-\ell)$ reaches a lower limit of $2S_0 \kappa_{0\nu} \ell$. This is the intensity of a saturated maser whose emissivity is characterized by $\tfrac{1}{2}S_0$ instead of J_s.

Table 1 summarizes the variation of d_ν/ℓ, the relative distance of the core from the edge of the maser, and of $(z_{sv+} - z_{sv-})/(2z_{sv})$, the relative core size, with increasing external intensity. The results derived here can be analyzed in light of the general properties of the maser, derived in § II. The luminosity of a saturated maser depends only on its volume and the pumping scheme. The saturated region does not amplify input radiation but rather converts pumping events to maser photons. The dependence of luminosity on input radiation should thus disappear during saturation. Indeed, the emergent intensity of the linear maser shows no explicit dependence on the external radiation (eq. [4.12]); a weak dependence on I_e is only caused by the length increase of the right saturated region from ℓ to 2ℓ. The emergent intensity remains unchanged, at $\sim 4J_s \kappa_{0\nu} \ell$, while I_e varies over a rather large range from $2S_0 \kappa_{0\nu} \ell$ all the way to $2J_s$. The reason for this insensitivity to I_e is that the solution carries no reference to it in the $z \geq z_{sv+}$ zone; neither the radiative transfer equation nor the boundary condition $J_\nu(z_{sv+}) = J_s$ depends on it. The only dependence on I_e is caused by its effect on z_{sv+}, the location of the saturation edge. But $\ell - z_{sv+} \simeq 2\ell$ over this entire range of I_e, since the right saturated zone essentially encompasses the whole maser. The I_e-dependence finally displayed by the fully saturated maser (eq. [4.25]) does not reflect any additional photon generation. Rather, this parameter region corresponds to $I_e > 2J_s$, so the input radiation itself must be the result of maser emission. This is therefore the case of a maser amplifying input maser radiation, corresponding to an effective increase of the maser volume.

The fact that the intensity amplification is limited to a factor of 2 is a peculiarity of the linear geometry which arises because it does not have a true intensity; the functions $I_{\nu\pm}$ are actually angle-integrated intensities. The linear maser solution provides a correct indication of the behavior of J_ν, but not of I_ν. Indeed, external radiation can only increase the maser J_ν by about a factor of 2 in any geometry, larger values being precluded by fundamental maser properties, but the intensity I_ν can be enhanced indefinitely along selected rays as long as their overall contribution to J_ν is negligible. The contrast at the bright spot can be arbitrarily high, provided that the spot is sufficiently small. The explicit expressions appropriate for cylindrical geometry illustrate this point. They are presented in Paper III.

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Note added in proof.—Equation (2.15) has been previously derived by G. Rybicki (*Ap. J.*, **213**, 165 [1977], eq. [16]). In addition, equation (3.34) appears to be an example of what has been called a quadratic integral by Rybicki (in *Methods of Radiative Transfer*, ed. W. Kalkofen [New York: Cambridge University Press, 1984], eqs. [6.9] and [6.10]). I thank Dr. Rybicki for bringing these points to my attention.

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