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A CHAOTIC ATTRACTOR IN TIMING NOISE FROM THE VELA PULSAR?

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ABSTRACT

We have analyzed 14 years of timing residual data from the Vela pulsar in order to determine if a chaotic dynamical process is the origin of timing noise. Using the correlation sum technique, we obtain a dimension of ~ 1.5. This low dimension indicates underlying structure in the phase residuals which may be evidence for a chaotic attractor. It is therefore possible that nonlinear dynamics intrinsic to the spin-down may be the cause of the timing noise in the Vela pulsar. However, we have found that the stimulated random walks in frequency and frequency derivative often used to model pulsar timing noise also have low fractal dimension, using the same analysis technique. Recent work suggesting that random processes with steep power spectra can mimic strange attractors seems to be confirmed in the case of these random walks. It appears that the correlation sum estimator for dimension is unable to distinguish between chaotic and random processes.

Subject heading: pulsars

I. INTRODUCTION

The Vela pulsar, one of the brightest pulsating radio sources, has been closely studied for two decades and is known to exhibit a wide variety of interesting phenomena. The spin-down behavior was monitored almost continuously between 1968 November and 1983 March as part of a pulse timing program carried out at the Jet Propulsion Laboratory (Downs and Reichley 1983; Downs and Krause-Polstorff 1986). Measurement of the phase of pulse arrival times over different epochs reveals abrupt increases in spin frequency or "glitches," occurring at the rate of one every few years (Cordes, Downs, and Krause-Polstorff 1988, and references therein). The relative change in rotation frequency ν during a glitch is $\Delta\nu/\nu \sim 10^{-6}$. In addition to these large changes in spin frequency, the pulsar period also shows much smaller fluctuations, known as timing noise. Timing noise was first recognized in the Crab pulsar by Boynton *et al.* (1972), where a significant phase residual was found to remain after removal of the low-order polynomial describing the pulsar spin-down. Many other pulsars were since discovered to have excess phase residuals (Cordes and Helfand 1980), which appear to be nonstationary, random phase fluctuations on top of the systematic increase in phase of pulse arrival times from spin-down torques. Boynton *et al.* (1972) and Groth (1975) first suggested that the timing noise in the Crab pulsar could be modeled or described as a random walk in ν. Cordes and Downs (1985) analyzed timing data on 24 pulsars collected over 13 yr at JPL and concluded that in most cases idealized random walks composed of steps *only* in the phase or one of its derivatives are not consistent with the data. It appears that the observed phase fluctuations are due to a mixture of steps in phase, frequency, and/or frequency derivative.

Physical mechanisms proposed to explain pulsar timing noise include torques internal to the neutron star, magnetospheric torque variations (which produce apparent fluctuations in the rotation rate), and angular momentum changes from external sources, such as accretion from the interstellar medium (see Cordes and Greenstein 1981 for a review). Superfluid vortex pinning models (Alpar *et al.* 1984), which fall into the first category, have been successful in accounting for the glitches observed in several pulsars, but do not produce changes of both sign in the period derivative which are characteristic of timing noise. However, Cheng *et al.* (1988) have formulated a hybrid model in which the microglitches induced by vortex creep couple to the neutron star magnetic field to produce fluctuating magnetospheric torques of both sign.

In this paper, a different approach is taken in the analysis of pulsar timing data. Instead of attempting to characterize the statistical properties of timing noise as a random process, we look for evidence of structure or nonrandomness in the data which could indicate that nonlinear dynamics intrinsic to the spin-down is the cause. The study of nonlinear dynamical systems that exhibit chaotic behavior is a recently developed and currently evolving field which is finding a rapidly increasing number of applications. These systems are governed by simple equations, and thus are deterministic, but can show very complicated and seemingly random behavior. This complexity occurs when the trajectory of the dynamical system in phase space has a fractional dimension, so that simple orbits cannot close on themselves but instead become distorted into never repeating loops. One very important property of such systems, known as "strange attractors," is the extreme sensitivity of the dynamical trajectories to initial conditions. Since the initial conditions are known only to some limiting accuracy in any real situation, it is impossible to predict the state of the system beyond some future time. This has profound implications for our ability to model accurately such systems as the Earth's climate and weather patterns, turbulent fluids, and population trends. On the other hand, these ideas hold promise for uncovering simple causes behind seemingly random or very complicated phenomena.

In the last few years, a "correlation sum" technique has been

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developed to reveal the presence of a strange attractor through analysis of time-series data (Grassberger and Procaccia 1983*d*). A single variable of the system is measured at evenly spaced time intervals, and by use of time delays the trajectories of the system in phase space can be reconstructed. The dimension of the system can then be computed, and a low fractional value of the dimension is good evidence for chaos.

We present an analysis of radio pulse time-of-arrival data from the Vela pulsar using the correlation sum technique. This technique has recently been used by several other groups to analyze time-series data from astrophysical sources. Voges, Atmanspacher, and Scheingraber (1987) analyzed timing data from the pulsating X-ray binary Her X-1 and determined a dimension of 2.3 for the system. Lochner, Swank, and Szymkowiak (1988) searched for a low dimension in the highly variable light curve of the black hole candidate Cyg X-1, but found no evidence of a dimension smaller than 10. Measurement of a low dimension in a system indicates that the dynamics can be described by a smaller number of parameters, at least equal to the dimension. A system with a fractional dimension, a strange attractor, exhibits chaotic behavior in the variables describing the dynamics.

In § II, we discuss our data and error analysis techniques, and the method of applying them to the Vela timing data. We also discuss measurement of the dimension of data generated by random walk models for pulsar timing noise using the same techniques. Results of the analysis and the measurement of the dimension of the Vela data and random walks in phase, frequency, and frequency derivative are presented in § III. In § IV, we consider the interpretation of the results, the limitations of our analysis, and possible directions to explore in searching for a nonlinear dynamical model for pulsar timing noise.

II. METHOD OF ANALYSIS

a) Summary of Correlation Sum Technique

A number of methods have been developed to measure the properties of chaotic attractors (Froehling *et al.* 1981). Many of these methods require data sets much larger than those available from pulsar timing programs and are therefore not applicable to this analysis. We use the correlation sum technique to analyze the Vela pulsar timing data because it does not require large data sets for an accurate measure of the dimension of an attractor. In this technique, the phase space trajectory of the dynamical process is reconstructed from the time series of a single variable through the method of time delays (Packard *et al.* 1980; Takens 1980). Given a measured time series, $\{x(t_1), x(t_2), \ldots, x(t_N)\}$, each point on the trajectory in a phase space of n dimensions can be represented using the vectors

$$V_i \equiv \{x(t_i), x(t_i + \tau), x(t_i + 2\tau), \ldots, x[t_i + (n-1)\tau]\} , \quad (1)$$

where n is the "embedding dimension" and τ is a fixed interval between observed points in the time series. The $M = N - (n-1)$ "delay vectors" V_i are thus constructed from each point in the time series and values of the series delayed at regular intervals. To measure the dimension of the reconstructed trajectory, one calculates the correlation sum $C(r)$, defined as (Grassberger and Procaccia 1983*a*, *d*)

$$C(r) = \frac{2}{M(M-1)} \sum_{i=1;\, j>i}^{M} \Theta(r - |V_i - V_j|) , \quad (2)$$

where $\Theta(x)$ is the Heaviside function, $M = N - (n-1)$, and $|X|$ is a norm of the vector, X. We use the L_∞ norm defined as $|V_i - V_j| \equiv \max \{[x(t_i) - x(t_j)], [x(t_i + \tau) - x(t_j + \tau)], \ldots\}$.

The correlation sum is evaluated for various choices of embedding dimension and delay (expressed in sample intervals). The task is made practical by the fact that both of these parameters can be kept small, generally less than 5 or 10. The delay time is chosen empirically beforehand so that a scatter plot (a "delay plot") of $x(t_i)$ versus $x(t_i + \tau)$ shows structure. Ideally, the delay time should be comparable to the time scale of the dynamical process. If the delay chosen is too small, the phase space trajectory is collapsed onto the 45° line in the delay plot because the points and their delays are approximately equal. If the delay is too large, then successive points in the delay plot are not well correlated, and the trajectory is distorted. As long as the delay time lies within these extremes, the measured dimension should not depend on the exact value of the delay chosen (Froehling *et al.* 1981).

Given a choice of delay time, evaluation of the correlation sum for several embedding dimensions yields a family of curves of log $C(r)$ versus log r. These curves may exhibit a linear region where

$$C(r) \propto r^{\sigma} , \quad (3)$$

whose slope $\sigma \approx n$ at embedding dimensions n less than the dimension, D, of the attractor, but σ approaches an asymptote as n is increased beyond the attractor dimension. This asymptote is a good estimate of the dimension of the attractor, even for relatively small numbers of data (Abraham *et al.* 1986). A plot of log $C(r)$ versus log r for a random noise signal would be expected to have $C(r) \propto r^n$; the slope of the curves should always equal the embedding dimension (for an infinite number of data points). A finite number of data points causes the slope σ to lie below n for sufficiently large n because there will always be a phase space of dimension too large to be adequately sampled by a finite number of points (Grassberger 1986). This sets a practical limitation on the dimension which can be reliably measured using the correlation sum technique. According to various estimates (Wolf *et al.* 1985), the number of data points required to measure a dimension D is 10^D–30^D, and Abraham *et al.* (1986) suggest that the dimension can be measured using data sets with 500 points or fewer in some cases.

For this analysis, uncertainties for each correlation sum are estimated by considering the sum to be an average of contributions from all of the phase space points defined by delay vectors. To calculate the errors, the contribution

$$C_i(r) = \frac{1}{M} \sum_{j=1}^{M} \Theta(r - |V_i - V_j|) \quad (4)$$

from every fifth point is sampled, and the standard deviation of the contributions, $C_i(r)$, to $C(r)$ from each sample point i is computed and expressed as a fraction of the mean. This fraction is multiplied by the value of the total correlation sum to produce the expected uncertainty. The percent errors computed in this way increase with decreasing value of the correlation sum, becoming very large (several hundred percent) in cases where only a few points are included in $C(r)$. As discussed below, we do not include the correlation sum points having errors greater than 100% in our estimate. The error in the slope is then computed from a weighted least-squares fit.

b) Application to Vela Pulsar Timing Noise

Data from the Vela pulsar take the form of pulse arrival times obtained at roughly weekly intervals over more than 14 yr. The precision of arrival times is $\sim 25\ \mu s$ compared to the overall pulse period of 89 ms. The errors in the arrival times are white noise in character and result from additive radiometer noise and from the finite number of pulses used to estimate each arrival time. Arrival times reflect the number of rotations that the rotating neutron star makes per unit time and therefore measure both the spin of the pulsar, its deceleration (spin-down), and irregularities in the spin. In 14 yr, the pulsar would make, without spin-down, $\sim 3 \times 10^9$ rotations, while spin-down reduces the number of rotations by $\sim 2 \times 10^6$ cycles. The irregular perturbations amount to only a few cycles of phase but are several thousand times larger than measurement errors.

In order to study the irregular component, in which we are most interested, the spin-down component must be removed from the data. This is done as follows (details may be found in Cordes and Downs 1985 and Cords, Downs, and Krause-Polstorff 1988). First, an initial model for pulse phase,

$$\hat{\phi}(t) = \phi_0 + \nu_0 t + \dot{\nu}_0 t^2/2 + \ddot{\nu}_0 t^3/6 , \qquad (5)$$

is used to estimate the phase $\hat{\phi}(t_j)$ at each arrival time t_j, where ν_0, $\dot{\nu}_0$, and $\ddot{\nu}_0$ are initial guesses for the parameters. The residual phase, $\delta\phi(t_j)$, is taken as the fractional part of $\hat{\phi}$. The parameters ν, $\dot{\nu}$, and $\ddot{\nu}$ can then be reestimated by minimizing in a least-squares sense the residuals from a second polynomial fit. To obtain the phase residuals for the correlation sum analysis in this paper, we used a third-order polynomial fit to estimate phase from pulse arrival times but removed only a second-order polynomial in the second fit. For a smoothly spinning down pulsar, equation (5) should be more than adequate to produce zero phase residuals.

We begin our analysis of 564 Vela phase residuals by examining a delay plot constructed by plotting each phase residual $\delta\phi(t_i)$ versus a delayed residual, $\delta\phi(t_i + \tau)$. The method of constructing delay plots, $x(t_i)$ versus $x(t_{i+1})$, from a time series has been described by Packard *et al.* (1980) and depends for its justification on the delay between adjacent data points being fixed. We mentioned earlier, however, that the time between successive measurements is only *roughly* 7 days. In fact, it varies from 1 to more than 20 days. The variation from the mean, moreover, exhibits local trends which are clearly undesirable. To remove the influence of the uneven time intervals between measurements as much as possible, we use the following procedure. First, we choose a time delay which is large compared with the mean interval and provides clear evidence of the structure of the trajectory in phase space. We are fortunate in that structure in the Vela data is evident in the two-dimensional delay plot. A 50 day delay was chosen for our analysis. Then, rather than fixing the number of data points between delayed residuals, we allow the number of data points to vary in such a way that the time interval is as close to our chosen delay (50 days, here) as possible. Thus we opt to make the actual time delay between measurements as close to constant as possible rather than using the arbitrary time scale that a delay of a fixed number of data points would produce. We call this procedure "evening" of the data. An analysis of the quantitative effects of data evening on evaluation of the correlation dimension will appear in a separate paper. After performing this evening of the Vela data, the rms variation about the mean delay was reduced by a factor of 5 and the spurious trends were eliminated. More importantly, the delay plot for the evened data, shown in Figure 1, appears to be smoother and more regular than the unevened plot. The evened data are then used to compute the delay vectors used in the correlation sum calculation.

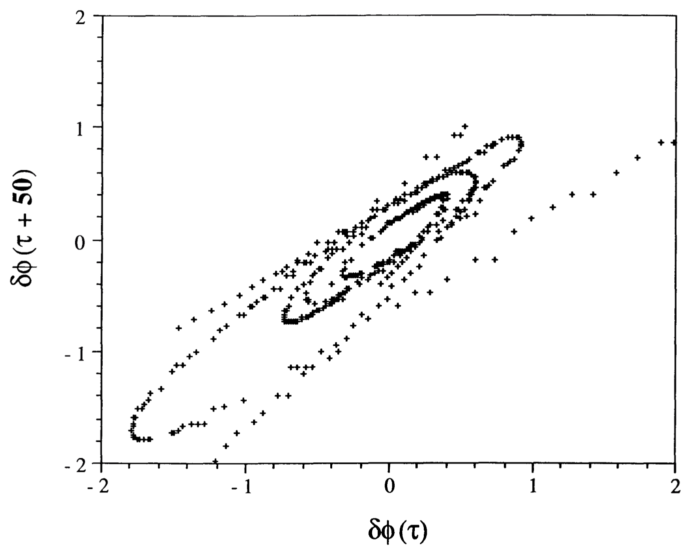

FIG. 1.—Delay plot of 564 evened Vela timing residuals, $\delta\phi(t_i)$ vs. $\delta\phi(t_i + \tau)$, with $\tau = 50$ days.

A final technical note must be included at this point. The Vela pulsar exhibits macrojumps or glitches during which the rotation period changes substantially (Cordes, Downs, and Krause-Polstorff 1988). It would not be appropriate to include timing residuals near the glitches, so such data were removed before analysis. Glitches were present in the data under consideration, and as a result we have analyzed four separate interglitch windows in the data. This does not interfere with our analysis, however, because correlation sums do not require contiguous data sets (Auvergne and Baglin 1986). Our procedure for dealing with the gaps in the data was to even the data and to compute delay vectors at successive embedding dimensions for each of the four contiguous data windows separately. The correlation sum for all of the data together is the average of the correlation sums computed for each separate window.

c) *Simulated Random Walks*

We also analyzed pseudo–timing data by constructing "random" walk processes that bear at least qualitative resemblence to actual timing data (Boynton *et al.* 1972; Groth 1975; Cordes 1980; Cordes and Downs 1985; Cordes, Downs, and Krause-Polstorff 1988). Again using the phase as the measured quantity, random walks in the kth derivative of the phase were generated according to

$$\frac{d^k\phi(t)}{dt^k} = \sum_i a_i \Theta(t - t_i) , \qquad (6)$$

where a_i is a random amplitude (with appropriate units) with zero mean. Steps in the walk occur at times t_i according to a rate R and such that the number of steps per unit time interval is a Poisson random variable. We constructed walks for $k = 0$, 1, 2 using a rate $R = 1\ \mathrm{day}^{-1}$, sampled them at the same times as the actual pulsar data, and removed a second-order polynomial to form pseudo–phase residuals analogous to those of the real data. The rate was chosen so that the mean time between random walk steps was much less than the mean sample interval. We refer to the three kinds of random walks as random walks in ϕ, ν, and $\dot{\nu}$, respectively.

Delay plots of the phase of these random processes are shown in Figure 2. The delays were evened in the same way as the delays between phase residuals in the Vela data. It is immediately obvious from these plots that the Vela phase residuals are unlike either random noise or a random walk in ϕ. The much higher correlation between successive Vela phase residuals produces a delay plot having a structure closer to that seen in the higher order random walks.

III. RESULTS

a) Vela Timing Residuals

Correlation sums, log $C(r)$ versus log r, are plotted in Figure 3 for the Vela timing residuals. The linear scaling region occurs only in the range of radius from 0.01 to 1.0. Correlation sums at radii lower than 0.01 have substantial random error, due to the limited number of points in the analysis. At radii larger than 1, all of the points are already included in the sum and the correlation sum saturates. Figure 3 reveals that the slopes of the linear regions do not get progressively larger with increasing embedding dimension, as one would expect in the case of random noise. Rather the slope saturates around embedding dimension 3 or 4, indicating that the Vela timing noise has a low dimension. Figure 4 shows the slope of the correlation sum curves from Figure 3 as a function of radius, computed from three-point unweighted least-squares fits. At each embedding dimension, the slope is roughly constant with radius in the linear scaling region ($r \approx 0.01$–1.0). Above $r \approx 1.0$ the slopes decrease to 0 as saturation occurs. The large fluctuation in the slope at $r < 0.005$ is due to stochastic noise in $C(r)$ and indicates the smallest r at which the slopes of the correlation sum can be reliably calculated.

We can quantify the results by computing the slope from a weighted least-squares fit to the points on the log $C(r)$ versus log r curves at each embedding dimension. The weights are determined from the errors in $C(r)$, as discussed above. To

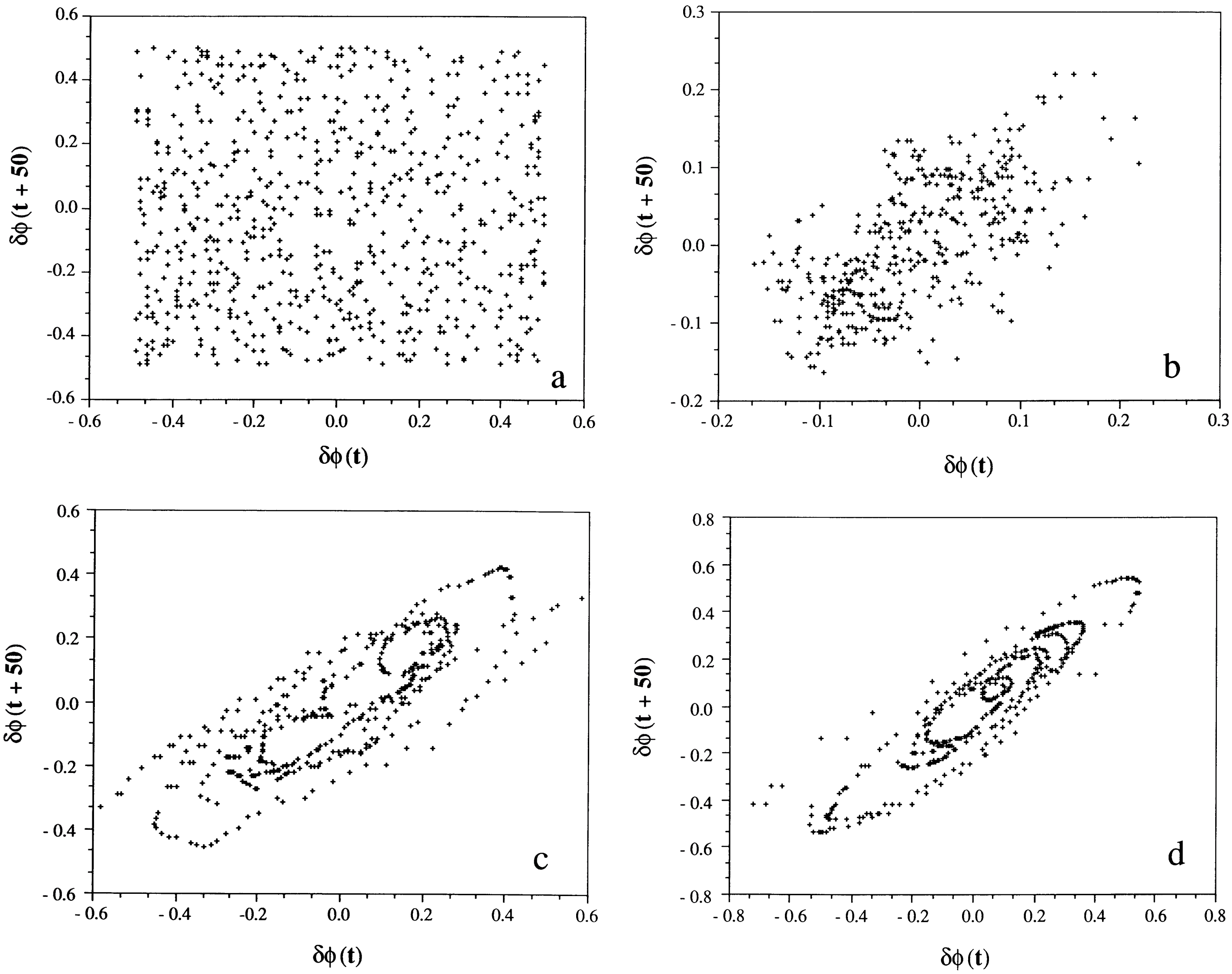

FIG. 2.—Delay plots of evened phase residuals for random processes: (*a*) random noise; (*b*) random walk in phase; (*c*) random walk in frequency; (*d*) random walk in frequency derivative.

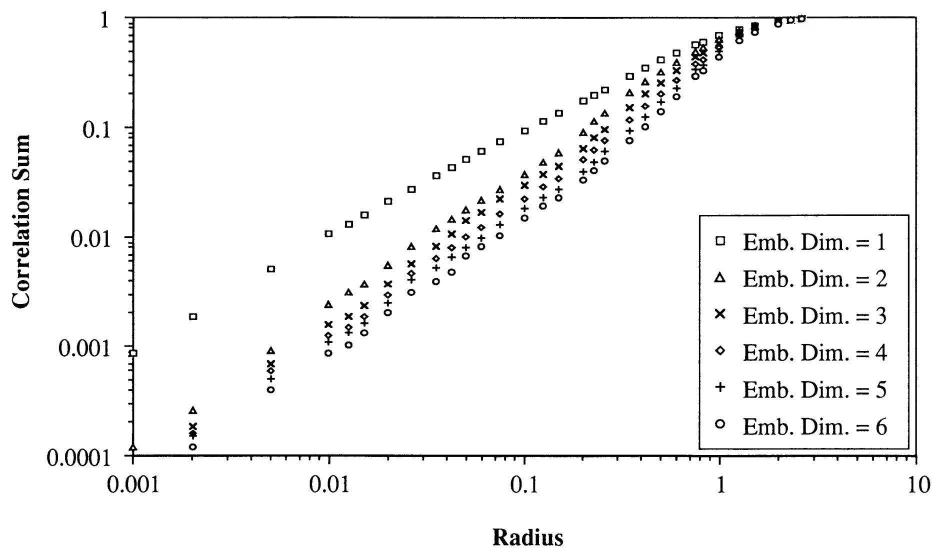

FIG. 3.—Correlation sums for 564 Vela timing residuals

minimize subjective influences, we choose the following objective standard for the points to include in each fit. The lower bound is taken to be the point at the smallest r value for which the error in $C(r)$ lies below 100%. This occurs at $C(r) \sim 0.01$, corresponding to $r \sim 0.005$ at the higher embedding dimensions. This choice seems to agree well with visual estimates of where random deviations from linear scaling begin and with the onset of random fluctuations in the slope as shown in Figure 4. The upper bound is taken to be where the rate of change of reduced χ^2 (with respect to degrees of freedom) first exceeds 0.03. This upper bound criterion, chosen to mark the point where the correlation sum saturates, again coincides with visual estimates of the turnover in the curves and is the point beyond which the reduced χ^2 values rapidly increase. The fits to correlation sums for the data sets containing random walks are done in the same way.

The computed slopes versus embedding dimension are shown in Figure 5. The error bars in the figure are the errors from the weighted least-squares fit to log $C(r)$ versus log r at each embedding dimension. Systematic and measurement errors are negligible in comparison and are not included here. It is apparent that the values of the slope are converging to a constant by embedding dimension 3. Thus we can estimate the correlation dimension of the Vela residuals to be ~ 1.5. Figure 5 also shows the slope of the correlation sum of random noise (see Fig. 6*a*) generated at the same times as the pulsar data. It does not exactly follow the 45° line expected for noise due to the limited number of data points. Nevertheless, the slope of log $C(r)$ versus log r for the Vela data clearly reaches an asymptote well below the measured noise slope.

For completeness, we have also evaluated the dimension of the data excluding macrojump ("glitch") pre- and postcursors. This estimate suggests that the pre- and postcursor data do not substantially affect the determination of dimension. Phase residual data which included removal of a third-order polynomial fit were analyzed in the same way and yielded the same dimension estimate (within the errors). However, the errors in this case were significantly larger.

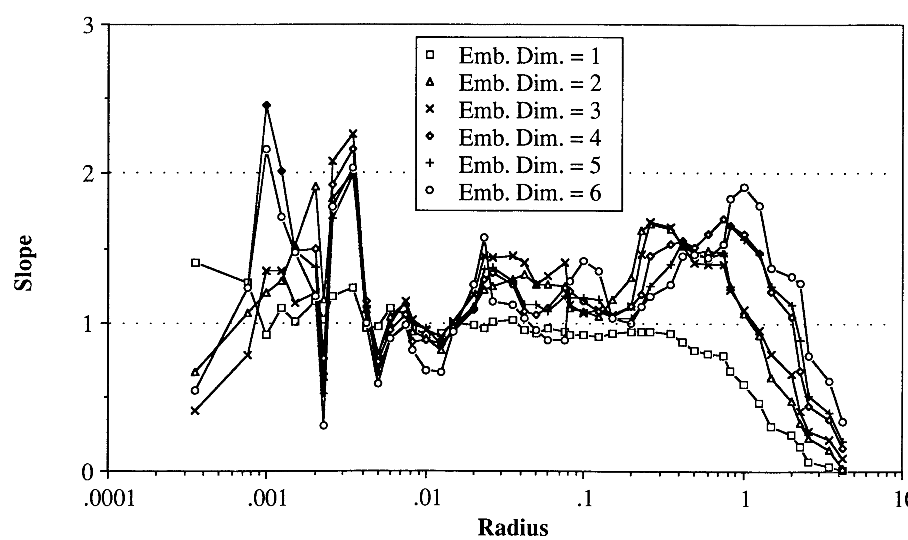

FIG. 4.—Correlation sum slopes vs. radius for Vela timing residuals

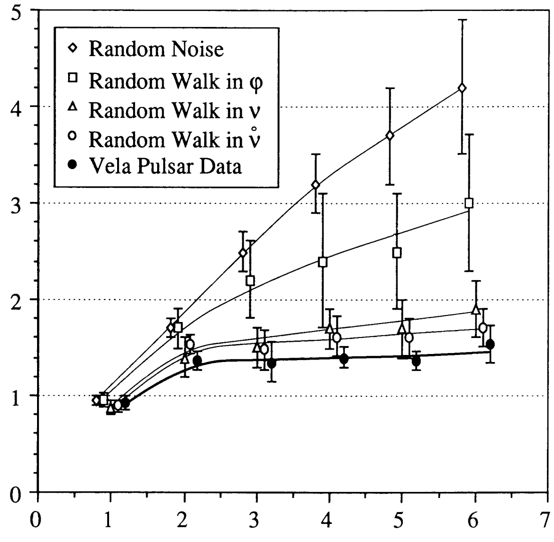

FIG. 5.—Correlation sum slopes vs. embedding dimension for Vela timing residuals and four random processes

b) Comparison with Random Walks

Although our estimation of the dimension of the Vela data stands on its own, it is desirable to follow up this estimation by an investigation of known processes which might mimic our results. In particular, recent work (Osborne *et al.* 1986) has suggested that data with steep power spectra can produce small, fractional correlation dimensions "even when no underlying attractor is present." Moreover, it has been shown (Greenstein 1979; Cordes and Downs 1985) that the Vela timing residuals share structural features compatible with the second integral of a random walk (i.e., random walk in $\dot{\nu}$)—which has a steep power spectrum. Indeed, the delay plot of a random walk in $\dot{\nu}$ (Fig. 2*d*) shows an orbital structure not unlike the Vela data.

TABLE 1

PROPERTIES OF RANDOM PROCESSES

Process	$D_\phi^{(1)}$ Slope[a]	$D_\phi^{(2)}$ Slope	$D_\phi^{(3)}$ Slope	Phase Spectral Index[b]	Dimension[c]
White noise in ϕ	0	0	0	0	4.2 ± 0.7[d]
Random walk in ϕ	1	1	1	-2	3.0 ± 0.7[d]
Random walk in ν	2	3	3	-4	1.9 ± 0.3
Random walk in $\dot{\nu}$	2	4	5	-6	1.7 ± 0.2

[a] Logarithmic slope of the structure function near the origin.

[b] Slope x of the phase power spectrum $\propto f^x$.

[c] Dimension of the process estimated from slope of the correlation sum at embedding dimension 6 (see Fig. 5).

[d] A lower limit to the true dimension, since the slope of log $C(r)$ vs. log r has not reached an asymptote.

We have evaluated correlation integrals for simulated random walks in ϕ, ν and $\dot{\nu}$, as well as for white noise. Correlation sums for these processes are plotted in Figure 6, and the resulting slopes of the linear scaling regions of log $C(r)$ versus log r are shown in Figure 5. The correlation dimensions of random walks in ν and $\dot{\nu}$ are both ~ 1.7, not very different from the Vela data. The slopes of log $C(r)$ versus log r for the random walk in ϕ are not clearly convergent (at least for an embedding dimension less than 10), but indicate a correlation dimension well above 2, significantly larger than the higher order random walks. Thus the dimension is inversely dependent on the order of the random walk. Since random walks of increasing order have increasingly steep power spectra (see Table 1), the correlation dimension seems to decrease with increasing index of the power spectrum. This finding agrees with that of Osborne and Provenzale (1989), who performed a correlation sum analysis of random noise with steep power spectra and obtained a low fractal dimension inversely dependent on the slope of the power spectrum.

It is useful to compare the properties of white noise and the random walk processes with the dimension estimates. In particular, we consider the slope of the power spectrum and the

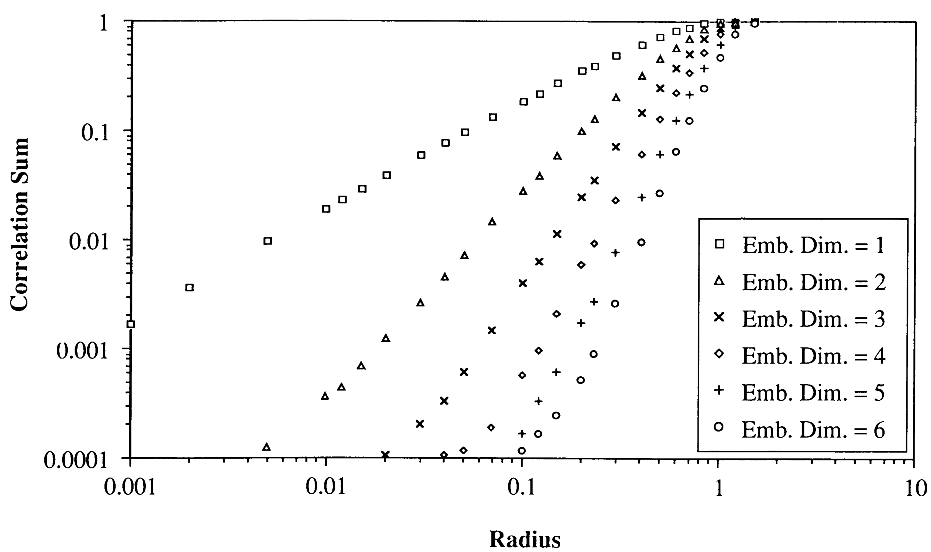

FIG. 6*a*

FIG. 6.—Correlation sums for random processes: (*a*) random noise; (*b*) random walk in phase; (*c*) random walk in frequency; (*d*) random walk in frequency derivative.

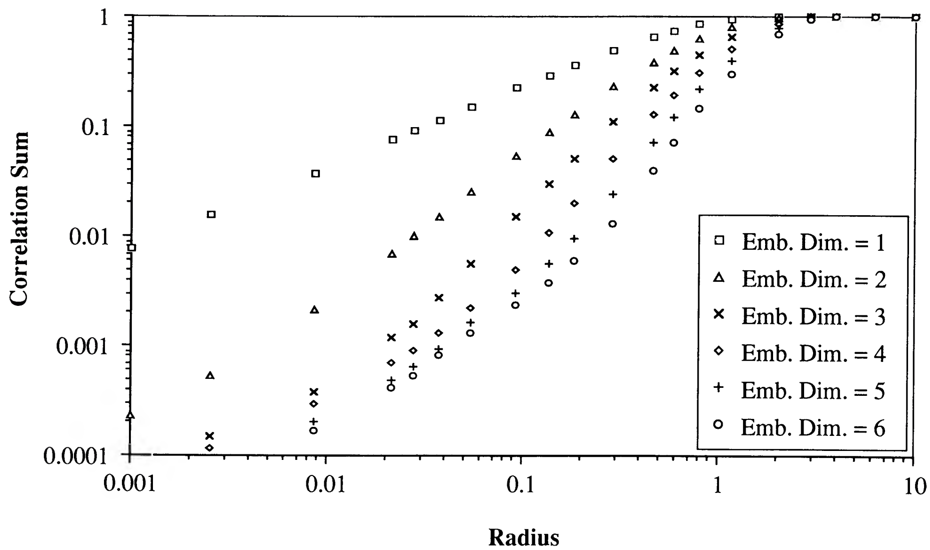

FIG. 6b

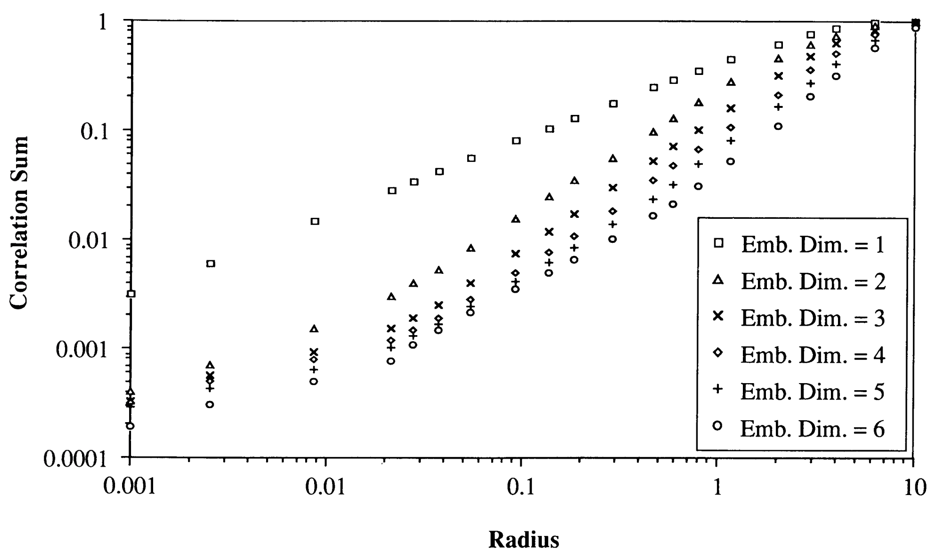

FIG. 6c

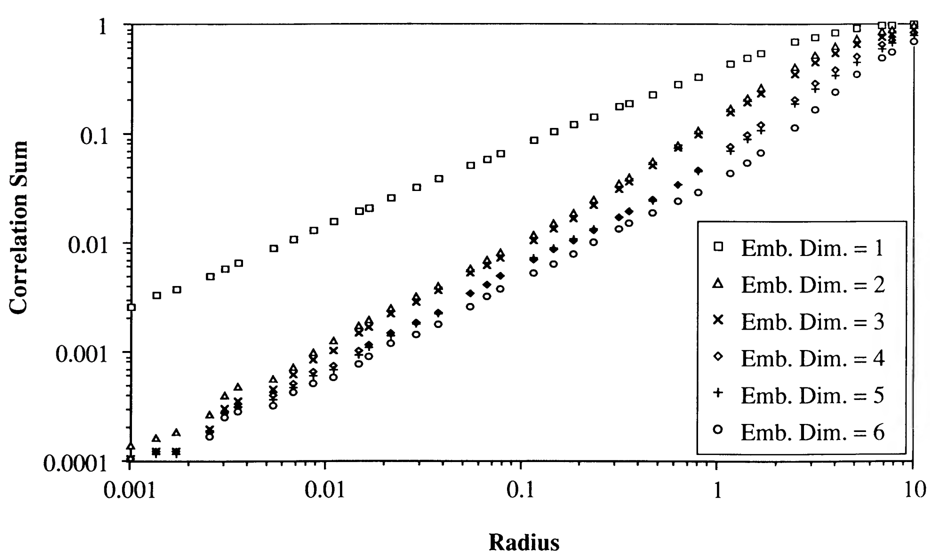

FIG. 6d

slope of structure functions of the various processes. Strictly speaking, the power spectrum for nonstationary processes is not defined (Papoulis 1965). However, the expectation of a suitable spectral *estimator* (such as the discrete Fourier transform of a finite realization) may be considered and used to describe the random walk processes. Indeed, Groth (1975) bifurcates the random walks into stationary and nonstationary parts, with the stationary parts having well-defined spectra. For the three processes we have considered, the power spectra have regions in frequency space that scale as f^{-2}, f^{-4}, and f^{-6}, respectively, compared to f^0 for white noise.

Structure functions are commonly used to describe the nonstationary fluctuations of clocks and frequency standards (e.g., Rutman 1978) and have also been used to characterize the phase fluctuations of pulsars (Cordes and Downs 1985; Cordes, Downs, and Krause-Polstorff 1988). Following Rutman (1978) and Cordes and Downs (1985) we define the mth-order structure function of the phase as the exception

$$D_\phi^{(m)}(t, \tau) \equiv \langle [\Delta_\phi^{(m)}(t, \tau)]^2 \rangle , \tag{7}$$

where the mth increment $\Delta_\phi^{(m)}(t, \tau)$ is defined as

$$\Delta_\phi^{(m)}(t, \tau) \equiv \sum_{l=0}^{m} (-1)^l \binom{m}{l} \phi[t + (m - l)\tau] . \tag{8}$$

The order of structure function (m) determines which order of polynomial ($m - 1$) is removed from the phase.

The random walks we have defined are all nonstationary. However, various *increments* of the different random walks are stationary in the sense that they depend only on the lag τ rather than the absolute time t in equation (8). For example, the random walk in phase ($k = 0$) has stationary first-order (and higher order) increments, while the other random walks we have considered have nonstationary first increments. The increment order must be at least 2 to become stationary for the random walk in frequency and must be at least 3 for the random walk in $\dot{\nu}$.

We summarize the properties of white noise and random walks in Table 1, where we give the slopes of structure functions and power spectra, along with the dimension estimates that have been determined numerically. We speculate that the apparent "saturation" of the slope of the (first-order) structure function at a value of two (the so-called "square law" structure function) may be related to the apparent convergence of the dimension toward a value ~ 1.7.

IV. DISCUSSION

The correlation dimension of 1.5 ± 0.2 determined from analysis of the Vela timing residuals indicates that at least (but possibly as few as) two variables can be expected to control the physical process responsible for the observed timing noise. The correlation results for the random walks in ν and $\dot{\nu}$ suggest that such random walks are candidates for the process. In fact, random walk models for pulsar timing noise can require only a few variables (e.g., the microglitch model of Cheng 1987). Apparently the correlation sum technique cannot be relied upon alone to confirm or refute the existence of a deterministic attractor. Since the technique merely evaluates the scaling properties of data, that should not be surprising. The correlation sum then only tells us the dimension of a deterministic attractor, given its existence. We can, nonetheless, use the correlation dimensions in conjunction with other analyses to constrain our modeling options.

Other techniques for analyzing time series data that might differentiate between random and chaotic processes require larger data sets or are not independent of the correlation sum. For example, Lyapunov exponents (Wolf *et al.* 1985) can reveal sensitivity to initial conditions by measuring how rapidly two adjacent orbits of an attractor diverge. A negative exponent indicates a chaotic rather than a random process. However, at least 10 dynamical time scales are usually required to compute a Lyaponov exponent accurately, and the Vela data seems to have at most five. The Kolmogorov entropy is another measure of chaotic processes, but since its lower bound can be calculated from the correlation sum (Grassberger and Procaccia 1983*b*), it is related to the correlation dimension and is not an independent measure. Therefore, it also would not be able to distinguish between the Vela data and the higher order random walks.

In principle, one could distinguish between a random walk and a truly chaotic process by analyzing derivatives of the phase. By taking successive derivatives, a nonstationary random walk can be reduced to stationary white noise. Thus a delay plot constructed from a time series in the third derivative of ϕ and the slopes from a correlation sum analysis should both be indistinguishable from random noise if the process is a random walk in $\dot{\nu}$. If the process is chaotic, then there should be some residual structure. In practice, however, the numerical error in taking successive derivatives is large and may mask any structure in delay plots of the third derivative.

Scargle (1989) has recently proposed an additional technique which may be able to separate the random noise component from the chaotic component of a time series. Furthermore, for simple cases it is able to separate a time series generated by a random or chaotic process from a constant filter with which it is convolved. Since a random walk is a moving average—i.e., the convolution of a fixed filter with random noise—this technique could possibly determine whether the Vela phase residuals are random or chaotic. That is, if the Vela phase residuals are a random walk, then the delay plot of the defiltered time series should look like random noise.

The possibility remains, however, that the timing noise in the Vela pulsar is due to a chaotic dynamical process. Nonlinear effects in the spin-down might be capable of producing chaotic torque variations, and there are a number of simple models which could be investigated. For example, the Euler equations for a rotating object with magnetic dipole moment not aligned with either the rotation axis or moment of inertia principal axes exhibit chaotic spin-down behavior in some parameter regimes. A periodically kicked rotator is a well-studied example of chaotic dynamics with a fractal dimension between 1 and 2 (Schuster 1984). The measured low dimension of the Vela data indicates that only a few variables would be needed to describe the system. If this is the case, then high-order random walks are just very good phenomenological models for chaotic dynamical processes. Indeed, more detailed scrutiny of the Vela data shows that a simple random walk in ν or $\dot{\nu}$ alone cannot underlie the Vela timing irregularities (Cordes *et al.* 1988).

Our analysis has also raised other questions which should be investigated in order to strengthen our result and which could be of more general application. First, since the data were not taken at evenly spaced intervals, we have introduced a technique for evening the time delays prior to performing the correlation sum. We plan to test this technique on attractors of known dimension to evaluate the limits of its applicability and to make the technique available to others. Second, since our data consists of only ~ 500 points, we intend to evaluate the errors involved in using the correlation sum analysis with small data sets, again using known attractors.

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