

MODELING SOLAR FORCE-FREE MAGNETIC FIELDS

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ABSTRACT

A class of nonlinear force-free magnetic fields is presented, described in terms of the solutions to a second-order, nonlinear ordinary differential equation. These magnetic fields are three-dimensional, filling the infinite half-space above a plane where the lines of force are anchored. They model the magnetic fields of the Sun over active regions with a striking geometric realism. The total energy and the free energy associated with the electric current are finite and can be calculated directly from the magnetic field at the plane boundary using the virial theorem. In the study of solar magnetic fields with data from vector magnetographs, there is a long-standing interest in devising algorithms to extrapolate for the force-free magnetic field in a given domain from prescribed field values at the boundary. The closed-form magnetic fields of this paper open up an opportunity for testing the reliability and accuracy of algorithms that claim the capability of performing this extrapolation. The extrapolation procedure as an ill-posed mathematical problem is discussed.

Subject headings: hydromagnetics — Sun: flares — Sun: magnetic fields

I. INTRODUCTION

Several solar vector magnetographs are being planned or built for the decade ahead (e.g., Hagyard 1985; Canfield and Dennis 1988; Canfield and Mickey 1989; Lites and Skumanich 1989; Sakurai 1989). Magnetic fields on the Sun are measured by remote sensing, based on the observation and interpretation of the influence of the *in situ* magnetic field on spectral line formation in the solar atmosphere. The magnetic field in the photosphere and, possibly, the lower chromosphere can be measured with useful results. The magnetic field in the upper chromosphere and corona cannot at present be measured with any useful spatial resolution for reasons of low light intensity and limited availability of suitable magnetically sensitive line radiation in the upper solar atmosphere. There is thus considerable interest in inferring the magnetic field in the upper atmosphere by the use of theoretical models subject to the constraint posed by field measurements made at the lower levels. Interest in this kind of modeling has been renewed recently by the new generation of vector magnetographs that will soon be available (e.g., Sakurai 1981; Cuperman, Ofman, and Semel 1989; Wu *et al.* 1990; Aly 1989). A popular model assumes that the magnetic field is force-free, with the field everywhere aligned parallel to the electric current density. This model is a reasonable approximation for the quasi-static, intense (low- β) magnetic field over a photospheric active region. The extrapolation of the force-free field from prescribed or measured boundary values is not a well-posed mathematical problem, and most if not all procedures of extrapolation are intrinsically empirical. The problem is nonlinear, in general, and at the present stage of development it is important to test all proposed schemes against known closed-form, nonlinear, force-free field solutions. This is seldom done because few closed-form solutions with the desired physical features are known. In this paper we offer a whole class of such solutions to challenge and test the reliability and accuracy of existing

numerical extrapolation schemes as well as new ones that may be proposed in the future.

The force-free magnetic field is described by the equations

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (1)$$

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (2)$$

where α is the proportionality between the magnetic field and the electric current density. Equation (2) follows from $\nabla \cdot \mathbf{B} = 0$ and requires α to be constant along the magnetic lines of force. We are interested in generating closed-form solutions of equations (1) and (2) in the infinite half-space $z > 0$ taken to be the solar atmosphere. To simulate realistic solar magnetic fields, we require the solutions to be varying in all three spatial dimensions, to describe magnetic fields that are spatially localized in $z > 0$ with finite total energies, and to be mathematically simple for convenient use in the testing of numerical codes. We are interested in situations where electric currents flow in the atmosphere, so that potential fields are of limited interest in our consideration. Equations (1) and (2) reduce to the linear Helmholtz equation if we take the scalar α to be uniform in space. This class of force-free field generally does not have bounded total energy in $z > 0$ (Chiu and Hilton 1977; Seehafer 1978). We are therefore driven to seek three-dimensional, nonlinear solutions which are in general not tractable without using numerical methods. A special class of this type of solution is presented in §§ II and III. The results are discussed in § IV.

II. MULTIPOLAR FORCE-FREE MAGNETIC FIELDS

We are interested in constructing simple theoretical examples of multipolar magnetic fields in $z > 0$ due to force-free electric currents in $z > 0$, with lines of force anchored at both ends of the plane boundary $z = 0$, such as in a solar active region. Equations (1) and (2) are formidable to solve in three dimensions, and we resort to constructing a set of axisymmetric global force-free fields to be placed in the physical domain $z > 0$ at various orientations of the axis of symmetry. To illustrate the procedure, the potential field of a point dipole

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is simple and symmetric about the dipole moment. On the other hand, the field takes on a three-dimensional form when viewed in the space $z > 0$, with a field geometry varying according to the location and orientation of the dipole moment of the point source in $z < 0$. There are few known solutions describing nonlinear force-free fields, and the first task of our construction is to generate a class of such solutions describing axisymmetric fields in the infinite space associated with point sources located at the origin. In the physical sense, the magnetic field is due to both the volume force-free electric current and the discrete current at the point source according to Ampere's law. At the point source, the force-free condition breaks down and the magnetic field is singular. The presence of such a singularity is unavoidable, since force-free fields in the infinite space are forbidden to be everywhere regular by the virial theorem (Chandrasekhar 1961)—a point to which we will return in § IV. The solutions to be constructed are analogous to the axisymmetric multipolar harmonic functions of potential theory. When the solutions are obtained, we shall show how they may be used to simulate active region magnetic fields in $z > 0$. Explicit examples then follow in the next section.

The magnetic field in axisymmetry can be expressed, using spherical coordinates, in a manifestly divergence-free form in terms of two scalar functions A and Q :

$$\mathbf{B} = \frac{1}{r \sin \theta} \left(\frac{1}{r} \frac{\partial A}{\partial \theta} \hat{r} - \frac{\partial A}{\partial r} \hat{\theta} + Q \hat{\phi} \right). \quad (3)$$

It should be emphasized that the spherical coordinates we use are for the purpose of generating nonlinear force-free fields of the form given by equation (3), and, at this point of our construction, these coordinates are yet to be related to the Cartesian coordinates in terms of which the domain of physical interest $z > 0$ is defined. The force-free equations require Q to be a strict function of A , with

$$\alpha = \frac{dQ}{dA} \quad (4)$$

and

$$\frac{\partial^2 A}{\partial r^2} + \frac{1 - \mu^2}{r^2} \frac{\partial^2 A}{\partial \mu^2} + Q \frac{dQ}{dA} = 0, \quad (5)$$

where we have $\mu = \cos \theta$. Inspection of equation (5) and a little algebra show that separable solutions of the following form exist:

$$A = \frac{P(\mu)}{r^n}, \quad (6)$$

$$Q(A) = aA^{1+1/n}, \quad (7)$$

where a and n are constants and the scalar function P satisfies the nonlinear differential equation

$$(1 - \mu^2) \frac{d^2 P}{d\mu^2} + n(n+1)P + a^2 \frac{1+n}{n} P^{1+2/n} = 0. \quad (8)$$

Let us demand that the magnetic field vanish at $r \rightarrow \infty$, which is ensured by the forms of A and Q given by equations (6) and (7) with n taken as positive, and is well behaved along the axis of symmetry except at the origin. The latter condition requires that B_θ and B_ϕ vanish along the axis $\mu = 1, -1$, which by equations (3), (6) and (7) implies that

$$P = 0 \quad \text{at} \quad \mu = -1, 1. \quad (9)$$

The solutions to the boundary-value problem posed by equations (8) and (9) generate the force-free fields we seek.

Not all prescriptions of the free constants a and n lead to a solution of equation (8) satisfying the boundary condition (9), and we have an eigenvalue problem to which we now turn our attention. The first thing to note is that the case $a = 0$ corresponds to a potential field with $\alpha = 0$. Equation (8) simplifies to

$$(1 - \mu^2) \frac{d^2 P}{d\mu^2} + n(n+1)P = 0. \quad (10)$$

Subject to boundary condition (9), the potential field solutions are generated by

$$P = (1 - \mu^2)^{1/2} P_n^1(\mu), \quad (11)$$

where P_n^1 is an associated Legendre function of the first kind for each integer n . The multipolar potential fields generated by equations (6) and (11) are well known (Batchelor 1970). Moreover, equation (10) does not have a regular solution that satisfies boundary condition (9) for noninteger n .

For the nonpotential case $a \neq 0$, we need to treat the nonlinear equation (8). To satisfy boundary condition (9), n is no longer required to be an integer. To generate solutions, an approach would be to fix the value of n and seek solutions to equations (8) and (9) as an eigenvalue problem, treating a^2 as a parameter. We adopt such an approach, but limit our attention to the cases where n is an integer. We omit an exhaustive study of equations (8) in order not to detract from the purpose of this paper, which is to generate some explicit simple solutions for force-free magnetic fields. Since the function P can change sign, the terms containing $A^{1/n}$ in equation (7) and $P^{2/n}$ in equation (8) becomes imaginary and physically not meaningful for an even n when A and P are negative. This can be remedied by replacing equation (7) by

$$Q(A) = aA|A|^{1/n}, \quad (12)$$

for even n , with a corresponding modification of equation (8). For our purpose we shall consider only the case of odd n , unless otherwise specified. Our treatment of the case of odd n can easily be extended to the case of even n given by equation (12), as will be clear as we proceed. There is no general theory of the nonlinear equation (8). We have performed a numerical study of this equation as an eigenvalue problem posed by boundary condition (9). We find that for each given integer n , boundary condition (9) admits a discrete, possibly infinite, spectrum of a^2 as an eigenvalue. The spectrum of course includes the eigenvalue $a^2 = 0$. Let us concentrate on the positive values of a^2 , since the negative values of this parameter have no physical significance. Boundary condition (9) is homogeneous, so that the eigenfunction P admits a free normalization. The eigenvalue a^2 is related to this normalization by the simple scaling law $P \rightarrow P_0 P$, $a^2 \rightarrow a^2 P_0^{2/n}$ for any constant P_0 , which is evident from equation (8). For convenience, the dependence of a^2 as an eigenvalue on the normalization P_0 can be removed by fixing the normalization in some suitable manner. For our numerical computations, we fix the free normalization to be such that $dP/d\mu = 10$ at $\mu = -1$. Denote the eigenvalues by $a_{n,m}^2$ and the respective eigenfunctions by $P_{n,m}(\mu)$ in ascending order of the magnitudes of the eigenvalues, with $m = 0, 1, 2, 3, \dots$ and $a_{n,0} = 0$. Figure 1 shows some examples of the eigenfunctions for the set $n = 1$ with $a_{1,0}^2 = 0$, $a_{1,1}^2 = 0.425$, $a_{1,2}^2 = 2.55$. To have an overview of the forms of these eigenfunctions, consider first the properties of the

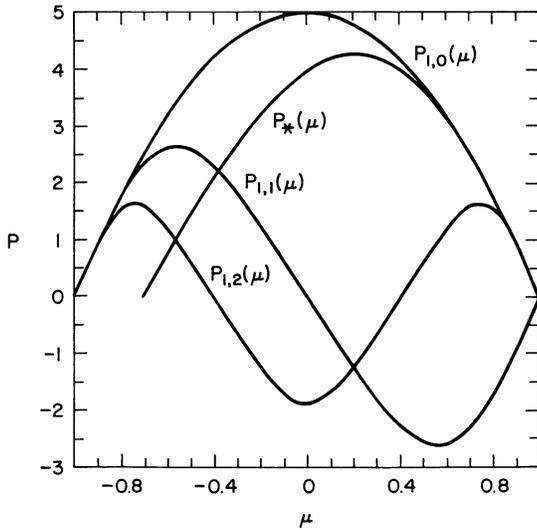


FIG. 1.—Angular functions $P_{n,m}$ for $n = 1, m = 0, 1, 2$ with $a_{1,0}^2 = 0, a_{1,1}^2 = 0.425, a_{1,2}^2 = 2.55$. The function P_* is not an eigenfunction, being a solution generated with $a^2 = 0.032$ as described in § III. P_* diverges to infinity at $\mu = -1$.

potential case with a general odd n and $m = 0$. The eigenfunction $P_{n,0}$ given by equation (11) has $n - 1$ zeros in the interval $-1 < \mu < 1$. Going back to equation (3) to note that magnetic lines of force are curves of constant A , it follows from equation (6) that a zero of the function P in that interval represents a purely radial line going out to infinity, not coincident with the two polar axes. The case of $n = 1$ gives the dipole potential field with no field lines going to infinity other than the lines on the polar axes. Hence, $P_{1,0}$ has no zero in $-1 < \mu < 1$. For $n > 1$, radial field lines not coincident with the axes appear as the result of the multipolar nature of the potential field. Our numerical study of the nonlinear eigenvalue problem shows that for a fixed odd integer n , as m increases from 0 through the integers, the number of zeros of the corresponding eigenfunctions increases one at a time from the $n - 1$ zeros obtained when $m = 0$; see Figure 1. From this result it follows that the potential dipole field, $n = 1, m = 0$, is the only magnetic field in the set of separable solutions generated above that has the dipolar field topology. All the other fields of the set are of higher multipolar orders.

Let us digress to explain physically why no members of the set of separable solutions with integer $n \geq 1$ other than the dipole potential field can have a dipolar field topology. In assuming the separable form given by equations (6) and (7), the magnetic field intensity is fixed to decline with radial distance as $r^{-(n+2)}$. This implies that the electric current density along the force-free magnetic field declines with radial distance as $r^{-(n+3)}$. If the magnetic field is dipolar in form, its geometry and the rapid decline of the electric current density would imply that the global current system has a nonzero, finite total dipolar moment. Quite irrespective of the force-free nature of the magnetic field, Ampere's law alone would then imply that the asymptotic field at infinity is a pure dipole potential field. Such a field has a radial fall-off that agrees with none of the separable solutions except when $n = 1$, and in that case only if $m = 0$, so that the angular dependence of the far field would agree. However, solutions with noninteger $n < 1$ may admit dipolar magnetic topologies. We have examined the case of

$n = \frac{1}{2}$ in detail and obtained a series of alternately even and odd eigenfunctions satisfying equations (8) and (9), corresponding to a set of increasing eigenvalues a^2 . The eigenfunction with the smallest eigenvalue a^2 is an even function of μ with no zero, or node, in $-1 < \mu < 1$. This solution generates a dipolar magnetic field. The magnetic field, with $n = \frac{1}{2}$, declines radially as $r^{-5/2}$ and has bounded total energy in the infinite space outside any unit sphere. On the other hand, the electric current density declines radially as $r^{-7/2}$, giving an infinite magnetic dipole moment. The electric current density is not localized in this sense. The usual multipole expansion for the far field does not apply in this case, and no contradiction with the above far-field consideration arises.

We have thus a set of separable solutions describing nonlinear force-free fields in axisymmetric geometry associated with point sources placed at the origin. It is easy to solve equation (8) by standard numerical methods, and we may take these fields to be available to any degree of accuracy as desired. Let us take one of these solutions and place its point source somewhere in $z < 0$ in our model of the solar corona taken to be the half-space $z > 0$. The magnetic field is of course axisymmetric relative to its point source but is no longer symmetric in an obvious manner in the physical domain $z > 0$. For our purpose, we consider the point source to be placed at the Cartesian coordinates $(0, 0, -l)$ and take the axis of symmetry of the magnetic field to lie in the z - x plane, making an angle of $0 < \Phi < \pi/2$ with the z -axis as shown in Figure 2. The relation between the spherical coordinates centered on the point source and the physical Cartesian coordinates is given by

$$\begin{aligned} X &= x \cos \Phi - (z + l) \sin \Phi, \\ Y &= y, \\ Z &= x \sin \Phi + (z + l) \cos \Phi, \end{aligned} \quad (13)$$

where $X = r \sin \theta \cos \phi$, $Y = r \sin \theta \sin \phi$, $Z = r \cos \theta$ are the Cartesian coordinates associated with the spherical coordinate system centered at the point source. The magnetic field in the domain $z > 0$ can be obtained in two steps by first writing the magnetic field in Cartesian components parallel to the system (X, Y, Z) and then transforming the latter into Cartesian components in the (x, y, z) system, as described by

$$\begin{aligned} B_x &= B_r \sin \theta \cos \phi + B_\theta \cos \theta \cos \phi - B_\phi \sin \phi, \\ B_y &= B_r \sin \theta \sin \phi + B_\theta \cos \theta \sin \phi + B_\phi \cos \phi, \\ B_z &= B_r \cos \theta - B_\theta \sin \theta, \end{aligned} \quad (14)$$

$$\begin{aligned} B_x &= B_x \cos \Phi + B_z \sin \Phi, \\ B_y &= B_y, \\ B_z &= -B_x \sin \Phi + B_z \cos \Phi. \end{aligned} \quad (15)$$

Taking any of the separable solutions and prescribing the location and orientation of its point source through l and Φ , equations (13)–(15) generate a three-dimensional model of a solar force-free field in the region $z > 0$.

III. EXAMPLES

Let us consider the $n = 1, m = 1$ solution. In this and other examples, except in the consideration of energy, we shall not specify the physical units, since these units are not important for our morphological study. In particular, we take all lengths to be measured in a yet to be specified unit r_0 . Locating the

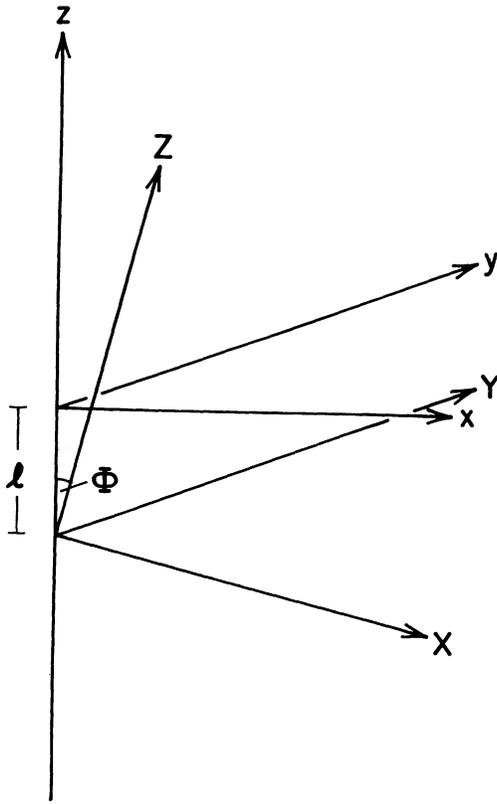


FIG. 2.—Local coordinate system (X, Y, Z) relative to physical coordinate system (x, y, z) .

point source at $l = 0.3$ and orientating the axis of symmetry parallel to the x -axis, i.e., $\Phi = \pi/2$, we obtain the three-dimensional magnetic field in $z > 0$ displayed in Figure 3. Figure 3a shows the distribution of the normal field component B_z at the plane boundary $z = 0$, given as contours at some unspecified constant interval. In Figure 3a the solid and broken contour lines denote positive and negative B_z , respectively. There are three regions of alternating magnetic polarities, separated by a pair of inversion or “neutral” lines where $B_z = 0$. This polarity structure arises from the quadrupolar nature of the $n = 1, m = 1$ force-free solution in axisymmetric geometry. Note that although this solution is symmetric about its local axis that runs parallel to the x -axis, the distribution of B_z at $z = 0$ shown in Figure 3a is not symmetric about the x -axis. A simple consideration of the magnetic field line topology reveals that the lack of symmetry about the x -axis is the result of the contribution to B_z from the toroidal component B_ϕ that circulates in a fixed direction about the symmetry axis. The symmetry of the $n = 1, m = 1$ solution about its equator translates into a symmetry of the field in Figure 3 about the y -axis, as is apparent in Figure 3b showing the magnetic lines of force seen projected on the plane $z = 0$. The normal field distribution is reproduced lightly in the background in Figure 3b as a visual aid. A total of 49 lines of force are shown originating from an evenly spaced 7×7 grid points covering a square centered at the origin. Each line of force is integrated from a footpoint, taken at one of the grid points, until it returns to the plane $z = 0$ or leaves a cubical volume centered at the origin. Note the shear of the magnetic field as indicated by the rotation of the field vector as we go from the

longer lines of force to the shorter underlying lines of force, and by the angle made at the inversion line (where $B_z = 0$) by the low-lying short lines of force. To give a feeling for the three-dimensionality of these lines of force, Figure 3c displays the same set of lines of force seen projected on the vertical plane $x = 0$. A subset of these lines of force is labeled with numerals to aid the identification of the individual lines of force in Figures 3b and 3c. In the identification of the individual lines of force in these two panels, particular attention must be given to lines that overlie one another in a given projection. For example, the line labeled 1 in Figure 3b is actually a set of three lines in a vertical plane, namely, the three labeled 1 in Figure 3c. The coincidence of projected lines of force arising from the particular symmetries of the magnetic field results in less than seven lines of force originating from each of the seven projected footpoints in Figure 3c. Some lines of force leave the cubical domain of interest at locations marked by dark dots in Figure 3c. Finally, in Figure 3d we display the distribution of the scalar α on the boundary $z = 0$ given as contours separated at some fixed interval. Since α is constant along a line of force, the field is organized in such a manner that each line of force originating from a point at $z = 0$ on a contour of constant α in Figure 3d must return to another point on the same contour.

The sheared force-free magnetic field in Figure 3 has a total energy which is finite. The total energy E can be obtained in terms of the field value at $z = 0$ by the use of the virial theorem (Chandrasekhar 1961; Molodensky 1974; Low 1982a) to give

$$\begin{aligned} E &= \frac{1}{8\pi} \int_{z>0} dV B^2 \\ &= \frac{1}{4\pi} \int_{z=0} dx dy (xB_x + yB_y) B_z, \end{aligned} \quad (16)$$

allowing for the rapid decline of B at infinity in $z > 0$. For the magnetic field in Figure 3 this surface integral reduces to an integral in μ :

$$E = \frac{B_0^2 r_0^3}{15\pi L^3} \int_{-1}^1 d\mu (W_1^2 - W_2^2 - 5\mu W_1 W_3), \quad (17)$$

where we write $l = Lr_0$ explicitly in terms of the unit r_0 and measure magnetic intensity in units of B_0 . W_1 , W_2 , and W_3 are functions of μ defined in terms of the eigenfunction $P_{1,1}$ as given in the Appendix. Direct numerical integration gives approximately

$$E = 32B_0^2 r_0^3, \quad (18)$$

where we have set $L = 0.3$ in application to the field in Figure 3. If we take r_0 to correspond to 10^5 km, characteristic of an active region, and choose B_0 so that the peak normal field component in Figure 3a is about 1000 G, equation (18) gives an energy of 6×10^{32} ergs. Not all of this energy can be liberated by the dissipation of electric currents flowing in the atmosphere $z > 0$. If all the electric currents in $z > 0$ are dissipated away without any displacement of the magnetic footpoints at the boundary $z = 0$, the relaxed state is a potential field in $z > 0$ having the same distribution of B_z at $z = 0$ as given in Figure 3a. The difference in energy between E and the energy E_0 of this potential field is the free energy ΔE associated with the electric currents. It is easy to use the classical Green's theorem to show that E_0 can be expressed in terms of the

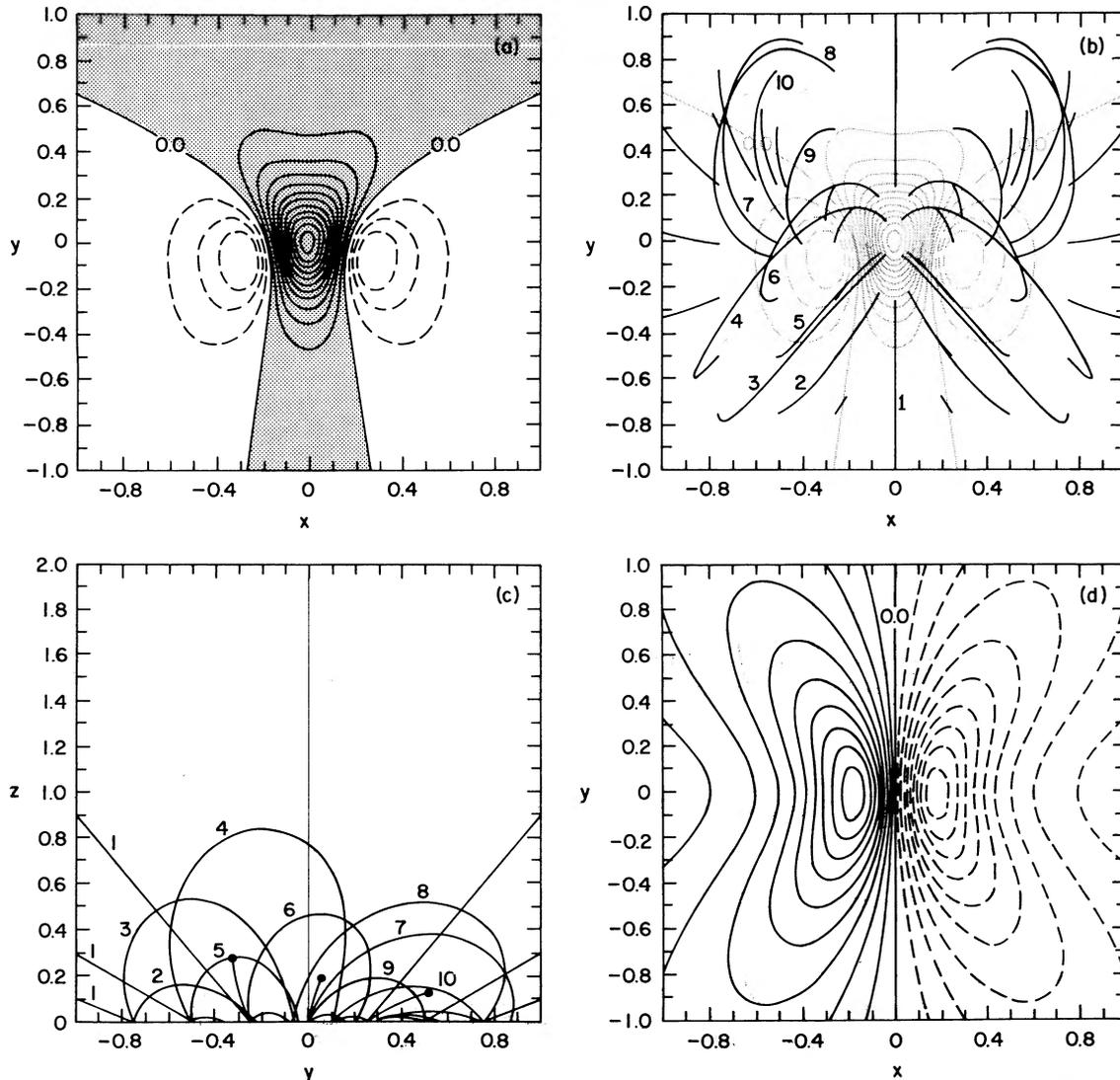


FIG. 3.—Magnetic field generated by $P_{1,1}$ with $l = 0.3$, $\Phi = \pi/2$. (a) The distribution of the normal field component on $z = 0$. (b) Magnetic lines of force projected on $z = 0$ shown against a light background of the normal flux distribution. (c) The same set of lines of force viewed in projection on the plane $x = 0$. (d) The distribution of the scalar α on $z = 0$.

boundary values of B_z at $z = 0$ as follows (Jackson 1962):

$$E_0 = \frac{1}{(4\pi)^2} \int_{z=0} dx dy \times \int_{z'=0} dx' dy' \frac{B_z(x, y, z) B_z(x', y', z')}{[(x-x')^2 + (y-y')^2]^{1/2}}. \quad (19)$$

Direct numerical evaluation of this integral over a square on $z = 0$ large enough to cover the main part of the magnetic field yields the approximate energy

$$E_0 = 24B_0^2 r_0^3 \quad (20)$$

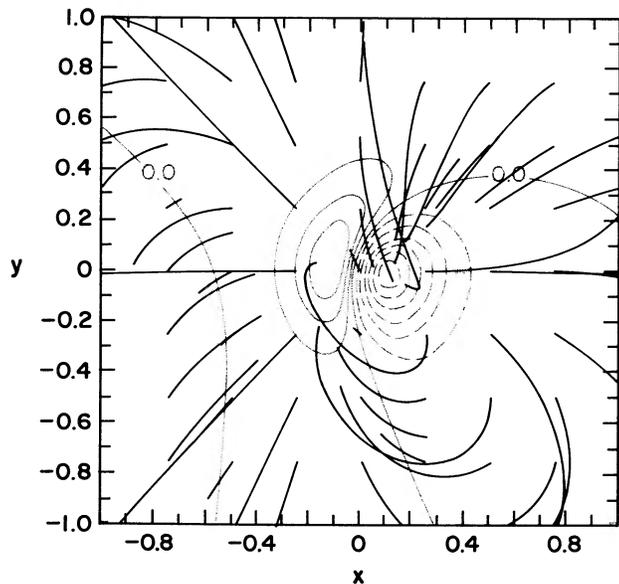
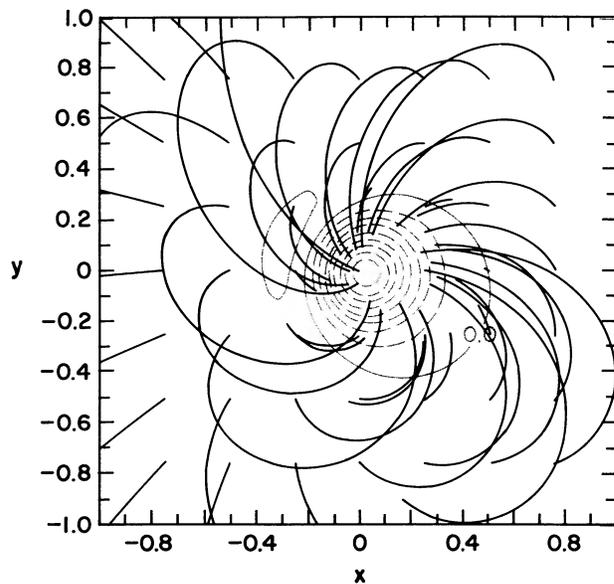
for the field in Figure 3. This shows that the free energy ΔE is about one-fourth of the total energy of the force-free field in Figure 3, of the order of 1.5×10^{32} ergs using the above set of values for B_0 and r_0 .

For the other examples we present below, we shall not study them in the same detail. We shall be interested primarily in the

magnetic field morphology. To that end, we shall present only a single figure for each example, showing the magnetic lines of force seen projected on the plane $z = 0$ with the normal flux distribution on that plane lightly contoured in the background.

Figure 4 shows the field in $z > 0$ obtained by changing the orientation of the symmetry axis of the $n = 1$, $m = 1$ force-free field such that $l = 0.3$ and $\Phi = \pi/4$. The three regions of alternating magnetic polarities are now rather distorted, and associated with this distortion are some highly twisted magnetic lines of force. As we decrease the angle of inclination Φ (see Fig. 2), we approach the configuration of a cylindrically symmetric twisted field. Figures 5 and 6 show two examples with conspicuous twists in their lines of force.

Similar constructions can be made with the other force-free fields having $n > 1$ or $m > 1$. The additional zeros of the angular eigenfunctions $P_{n,m}$ introduce additional regions of magnetic polarities, and we leave these more complex cases for the interested reader to explore. Unfortunately, linear superposition does not apply, and we do not have the facility of

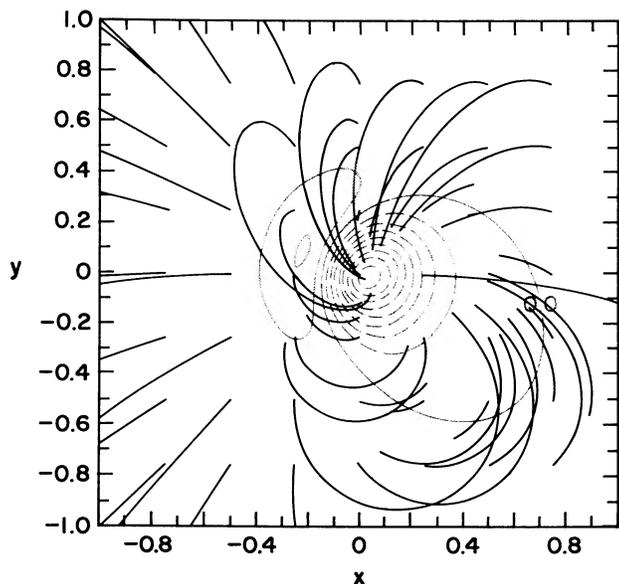
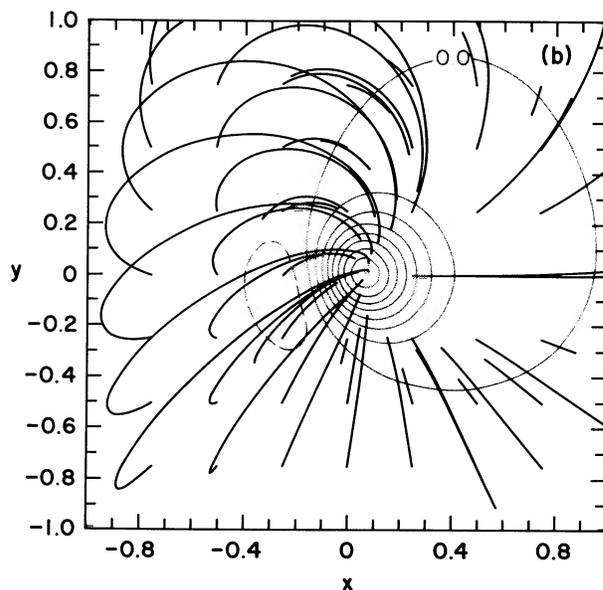
FIG. 4.—Magnetic field generated by $P_{1,1}$ with $l = 0.3$, $\Phi = \pi/4$ FIG. 6.—Magnetic field generated by $P_{1,1}$ with $l = 0.3$, $\Phi = 0.27$ (15:5)

generating complex magnetic fields out of simple ones, a procedure commonly employed with potential or constant- α force-free fields. With nonlinear fields, more complex fields than those constructed in § II require independent construction of associated discrete sources that are more complex than mere point sources.

In situations where the symmetry axis of the force-free field is oriented such that $\Phi < \pi/2$, the part of the symmetry axis where $\theta > \pi/2$ points into the region $z < 0$ away from the boundary plane $z = 0$. Given that we are interested in regular fields only in the physical domain $z > 0$, we can relax boundary condition (9) at $\theta = \pi$ ($\mu = -1$). The singularities in \mathbf{B} on the part of the axis where $\theta > \pi/2$ constitute additional discrete magnetic sources of the global field. Figure 7 is a weakly

sheared force-free field in $z > 0$ constructed with a solution P_* of equation (8) setting $a^2 = 0.032$, not one of the admissible eigenvalues. The function P_* is set to vanish at $\mu = 1$, so that the magnetic field is well behaved on the axis $\theta = 0$ but reverses sign at $\theta = 3\pi/4$ to diverge to infinity on the axis $\theta = \pi$. Figure 1 contains a graph of the angular function P_* . Orienting the axis of symmetry such that $\Phi < \pi/2$, the singularities of P_* are confined within $z < 0$ and the force-free field in $z > 0$ is everywhere well behaved.

As a final extension of the model, we note that composite magnetic fields can be constructed, patching different force-free field solutions having the same index n but different m in different sectors of an r - θ plane separated by purely radial lines of force along which P vanishes. Since the solutions on either side

FIG. 5.—Magnetic field generated by $P_{1,1}$ with $l = 0.3$, $\Phi = 0.47$ (27°)FIG. 7.—Magnetic field generated by P_* with $a^2 = 0.032$, $l = 0.3$, $\Phi = 0.61$ (35°).

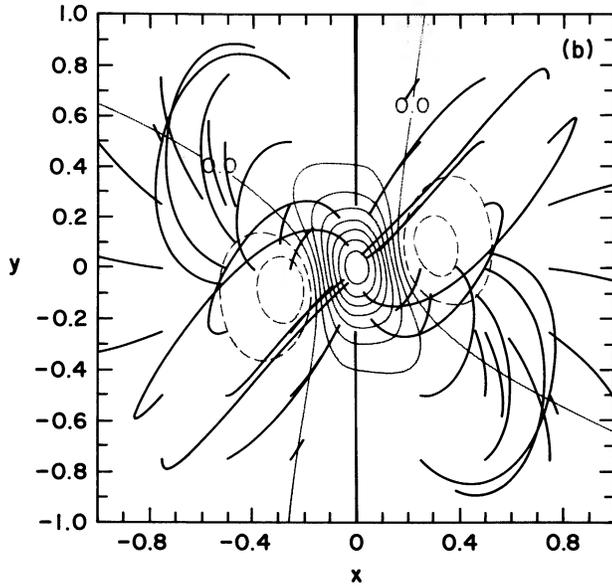


FIG. 8.—Composite magnetic field generated by $P_{1,1}$ with $l = 0.3$, $\Phi = \pi/2$, taking Q to be equal and of opposite sign in the two hemispheres $\theta < \pi/2$ and $\theta > \pi/2$.

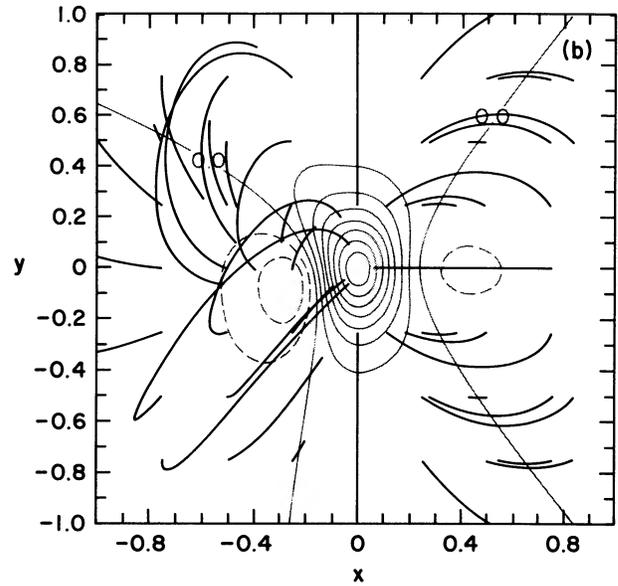


FIG. 9.—Magnetic field with $l = 0.3$, $\Phi = \pi/2$, composed of the field generated by $P_{1,1}$ in $\theta > \pi/2$ matched at $\theta = \pi/2$ to a potential field in $\theta < \pi/2$.

of the radial boundary have the same index n , the fields on either side fall with radial distance with the same power of r . The force equilibrium of the boundary is easily ensured by normalizing the magnetic fields on either side so that the total magnetic pressure is continuous across the boundary. For the $n = 1$, $m = 1$ solution, $Q = aA^2$ is everywhere positive if a is positive. The $n = 1$, $m = 1$ solution can also be associated with $Q = -|a|A^2$, since Q appears only as a quadratic term in the force-free equation (5). The simplest composite solution is that generated from the $n = 1$, $m = 1$ solution by taking Q to be positive in $\theta > \pi/2$ and to be equal and of the opposite sign in $\theta < \pi/2$. This field is shown in Figure 8. This construction is equivalent to taking the field in Figure 3b and reflecting the field in $x > 0$ about the vertical plane $y = 0$. A more interesting example is shown in Figure 9, where we match the $n = 1$, $m = 1$ solution taken in the interval $\pi/2 < \theta < \pi$ ($-1 < \mu < 0$) to a potential field in the complementary interval $0 < \theta < \pi/2$ ($0 < \mu < 1$). The regular dipole potential field has no purely radial line of force off-axis. However, with the boundary condition for regularity at $\mu = 1$ relaxed, a potential field with $n = 1$ having a purely radial line of force along the equator $\mu = 0$ can be constructed. The desired field is generated by

$$P = P_0 \left[(1 - \mu^2) \log \left(\frac{1 + \mu}{1 - \mu} \right) + 2\mu \right], \quad (21)$$

where we choose an amplitude P_0 so that the total magnetic field is continuous at $\theta = \pi/2$. This composite field is not well behaved along the axis $\theta = 0$, and it is necessary to keep that part of the axis excluded from the physical domain $z > 0$ when using this magnetic field as a model, namely, by restricting interest to $\Phi > \pi/2$. Careful study shows that the singularity of the magnetic field on the axis $\theta = 0$ is so weak that as long as this axis lies below $z = 0$, no singularity in the magnetic field in $z > 0$ is introduced even if $\Phi = \pi/2$ with the symmetry axis parallel to the boundary $z = 0$. In the limit of $\theta \rightarrow 0$ in $z > 0$, the weak singularity of P in that limit does not appear in $z > 0$

because of the rapid decay of the magnetic field with radial distance. Figure 9 displays the force-free field in $z > 0$ generated by this composite field with $\Phi = \pi/2$. Note the absence of shear in the field lines in the potential part of the global magnetic field in $x > 0$.

IV. DISCUSSION

The twist in solar magnetic fields over an active region results largely from the photospheric transport of the magnetic footpoints. At the photosphere, the plasma is so dense as to completely dominate over the Lorentz force that the magnetic field, to a first approximation, may be regarded to be transported passively. Almost any kind of photospheric motion would take the magnetic field in the corona into a force-free state of the nonlinear type with a spatially varying α (Sakurai 1979; Low 1982b). There are few known force-free fields of the nonlinear type that can model the three-dimensional active region magnetic fields. The solutions presented in the two preceding sections are very specialized, being based on the mathematical assumption of separability. On the other hand, these solutions are simple, and they are well suited for simulating the realistic geometry of solar magnetic fields when treated as fields in Cartesian geometry associated with discrete sources. These solutions merit further investigation. In particular, they present an opportunity to study hydromagnetic stability of nonlinear force-free fields by linear perturbations. The analytic form of these solutions should be exploited by using the usual perturbation expansion, although the stability problems posed by these solutions are not expected to be simple. The force-free field taken to fill the space $z > 0$ and to be rigidly anchored on the boundary $z = 0$ presents an equilibrium state that is truly three-dimensional. The stability of this kind of equilibrium is a formidable mathematical problem. The force-free field taken to fill the infinite space outside a unit sphere centered at the point source, modeling some other astrophysical situation than the one treated in this paper, presents a more tractable stability problem. In the treatment of this problem, the axisymmetry of

the equilibrium state can be exploited to simplify the perturbation equations.

The force-free fields presented in § III open up an opportunity to test extrapolation algorithms devised to generate the force-free fields in a given domain from field values prescribed at the boundary. Such algorithms are essential if we are to make optimal use of data from solar vector magnetographs. The reason is that vector field measurement by the use of magnetographs is feasible only for the fields in the photosphere and chromosphere. Unfortunately, the mathematical problem posed by the desired extrapolation is ill posed. In attempting to devise methods of extrapolation, it is important to understand the inherent obstacles that arise from an ill-posed problem. In fact, very many problems of interest to physics are ill posed, requiring special methods or treatment in order to render them meaningful (Tikhonov and Arsenin 1977).

Let us conclude with a brief discussion of ill-posed problems in the case of extrapolating for solar magnetic fields. Our interest in the extrapolation problem comes from the peculiar situation that the magnetograph gives all three components at the boundary, say, the plane $z = 0$, and we like to determine the field in $z > 0$ from these data. Take the special case, for the purpose of illustration, where the field is assumed to be potential. If we know \mathbf{B} everywhere on $z = 0$, a sensible way to solve for the field in $z > 0$ is to recognize that the given information is redundant and take only the normal field component of the boundary field to solve the potential equation as a classical Neumann boundary-value problem. For this problem, we close the boundary "at infinity" by demanding that the field vanish at infinity. This boundary-value problem is well posed in the sense of Hadamard (1932) in that the solution has a continuous dependence on the boundary value. A small change or inaccuracy in the prescription of the boundary value gives a correspondingly small change in the interior solution. This is an important consideration, for, in practice, the magnetic field is not measured exactly, nor is it completely force-free, because of the presence of weak pressure and other dynamical forces. In practice, it is also never possible to measure the field everywhere on the plane $z = 0$, or, in more realistic terms, only a finite area on the solar surface can be scanned by a real magnetograph. This observational constraint suggests that we pose the extrapolation problem differently. The Neumann problem makes no use of the transverse field components at the boundary. Is there an advantage in not posing the problem over the infinite boundary but instead using all three field components of the field as boundary conditions on a limited part of the plane? We would be treating a problem involving an open boundary in this case. Such an approach was adopted by Wu *et al.* (1990) and Cuperman, Ofman, and Semel (1989) for the general force-free field. In the case of a potential field, if the three components of the field are given exactly, a Taylor expansion about the open boundary should recover the true potential field in the neighborhood of the boundary. But this problem is ill posed, as first shown by Hadamard (1932), in that this extrapolation does not tolerate any kind of deviation of the three field components from the exact potential values. A small change of the boundary values results in divergent changes in the extrapolated field.

This behavior is simple to understand physically. Consider a smooth, bounded potential field \mathbf{B}^* in $z > 0$. This field may be regarded as being due to virtual discrete sources located outside the domain of interest, i.e., in $z < 0$. Take \mathbf{B}^* on $z = 0$ and use it as boundary condition over some part of the plane

to pose an open-boundary problem. One can show from direct calculation that these boundary values can recover the potential field in the neighborhood of the open boundary. This is just a matter of self-consistency. Now allow the boundary values to deviate by small amounts from \mathbf{B}^* evaluated on $z = 0$. To establish our point, we in particular keep B_z unchanged and allow B_x and B_y to deviate from B_x^* and B_y^* , at the boundary $z = 0$. The potential field consistent with the modified boundary values must then be due to the original sources in $z < 0$ superposed linearly with additional sources in both $z > 0$ and $z < 0$ arranged in pairs of image sources about the boundary plane so that the normal field B_z on $z = 0$ is unchanged by their presence. Hence, the potential field consistent with the modified boundary values is necessarily singular, right at the locations of the additional sources, or is unbounded in $z > 0$. For arbitrary deviations of B_x and B_y at $z = 0$, these singularities arise in an arbitrary manner and can be infinitesimally close to the open boundary. Simple examples are well known, showing the potential field diverging to infinity even in the neighborhood of the boundary $z = 0$ (see, e.g., Carrier and Pearson 1988). Thus, the extrapolated field is not smoothly dependent on the boundary data.

This mathematical property is basic to elliptic partial differential equations and should be expected also for force-free fields with $\alpha \neq 0$. The case of a constant α is governed by a Helmholtz equation (Chandrasekhar and Kendal 1957), and the matter is straightforward as in the case of the potential field. For a nonconstant α , it is less trivial to be explicit because the governing equations are nonlinear. To see that the same basic obstacle is present, let us recall the virial result of Chandrasekhar (1961):

$$\int_V \mathbf{r} \cdot [(\nabla \times \mathbf{B}) \times \mathbf{B}] dV = \int_V \frac{1}{2} B^2 dV + \int_{\partial V} [(\mathbf{B} \cdot \mathbf{r})\mathbf{B} - \frac{1}{2} B^2 \mathbf{r}] \cdot d\mathbf{s}, \quad (22)$$

given in standard notation. If the field \mathbf{B} is force-free in a volume V , the left-hand side vanishes and the total energy is determined uniquely by the surface vector field, a result exploited in equation (16). If we take V to cover all space, the surface integral in equation (22) vanishes as the boundary ∂V recedes to infinity for a localized current system. The magnetic field in that case cannot be force-free everywhere in the infinite space, or else the contradiction of zero total energy arises. Hence, a force-free field in the infinite space cannot have a localized current system (e.g., a constant- α force-free field), or it must possess singularities where the Lorentz force is not zero and is, in fact, divergent, so that the left-hand side of equation (22) has a finite contribution. In this important respect, the general force-free field is similar to the subset of potential fields. Any force-free field (or potential field) extrapolated smoothly from a surface to fill all space must either diverge or become singular somewhere in space. The locations of these singularities or divergencies do not depend continuously on the value of the vector field at the surface. Small departures from a surface vector field can introduce singularities in the extrapolated field in the physical domain where none are present if these departures are precisely zero. This is the nature of the ill-posed character of the extrapolation problem.

There is a property in the case of a variable α that presents difficulties not found when α is a constant. Since α is constant along a field line (see eq. [2]), it can be used as a label for

magnetic lines of force. Consider the situation where a particular line of force originates and returns to a given open boundary. The value of α of this field line must be realized at least twice on that boundary. Now the boundary field determines the value of α on $z = 0$:

$$\alpha = \frac{1}{B_z} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right), \quad (23)$$

as follows from equation (1). This is a highly nonlinear relation. Any deviation of the boundary field from the true force-free value can result in changes in the boundary values of α , so that the force-free equations would give a field of quite different connectivity in $z > 0$. For instance, a line of force may connect two points on the boundary where α takes on a local minimum value of α_0 , say. Suppose that α_0 is also the absolute minimum of α on the boundary. A deviation from the exact boundary values that leaves one minimum unchanged but reduces the value of the second minimum to some value below α_0 would not allow the calculated field line originating from the latter to connect with the former.

We have described some of the obstacles associated with the ill-posed nature of the extrapolation problem. We do not at present have any general recommendation for removing them (but see Sakurai 1981). It seems clear that further study of this important problem is needed, and we have much to learn from trials and tests. In this connection, the closed-form force-free fields presented in this paper may prove useful. An important test to perform is to take one of our solutions to prescribe a boundary distribution of the vector field, superpose on the latter a controlled amount of noise, and evaluate how well, if at all, a given extrapolation procedure is able to recover the solution in the physical domain.

This work was done during the tenure of one of us (Y. Q. L.) as Visiting Scientist at the High Altitude Observatory and the Advanced Study Program, National Center for Atmospheric Research. We thank Bruce Lites for commenting on the paper, and an anonymous referee for suggesting an investigation of the solutions for noninteger n .

APPENDIX

To keep track of physical dimensions, we introduce a constant length r_0 and field intensity B_0 so that the separable force-free field for $n = 1, m = 1$ takes the form

$$\mathbf{B} = \frac{B_0}{R^3} \left[-\frac{dP_{1,1}}{d\mu} \hat{r} + \frac{P_{1,1}}{(1-\mu^2)^{1/2}} \hat{\theta} + \frac{\eta P_{1,1}^2}{(1-\mu^2)^{1/2}} \hat{\phi} \right], \quad (A1)$$

in spherical coordinates, where $R = r/r_0$, and we have introduced a dimensionless form of the eigenvalue $\eta = a_{1,1} B_0 r_0^3$. Using equations (13)–(15), the energy integral (16) can be put in the form

$$E = \frac{B_0^2 r_0^3}{4\pi} \int_{-1}^1 d\mu \left(\int_{\infty}^{R_*} dR J_- + \int_{R_*}^{\infty} dR J_+ \right), \quad (A2)$$

where

$$J_{\pm} = \frac{1}{(1-\mu^2)^2} \left\{ \mp L\mu W_1 W_3 \frac{1-\mu^2}{R^4 [(1-\mu^2)R^2 - L^2]^{1/2}} \right. \\ \left. \pm L(W_1^2 - W_2^2) \frac{[(1-\mu^2)R^2 - L^2]^{1/2}}{R^6} \right. \\ \left. + (W_1 - \mu W_3) W_2 \frac{1-\mu^2}{R^4} - \frac{2L^2 W_1 W_2}{R^6} \right\}, \quad (A3)$$

here we have introduced

$$W_1 = -(1-\mu^2) \frac{dP_{1,1}}{d\mu} + \mu P_{1,1}, \quad (A4)$$

$$W_2 = \eta P_{1,1}^2, \quad (A5)$$

$$W_3 = \mu \frac{dP_{1,1}}{d\mu} + P_{1,1}. \quad (A6)$$

Equation (A2) has been obtained by a change of variables from x, y to r, μ on the plane $z = 0$, using $x = r\mu$ and $y = \pm [(1-\mu^2)r^2 - l^2]^{1/2}$. The integration range as indicated is evident from noting that on the plane $z = 0$, curves of constant μ are parabolae symmetric about the x -axis, and on each of these parabolae r varies from a minimum value of $r_* = l/(1-\mu^2)^{1/2}$ to infinity. In equations (A2) and (W3), $R_* = r_*/r_0$ and $L = l/r_0$. Performing the integration with respect to R analytically yields the expression given by equation (17).

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