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THE DISTRIBUTION OF VISUAL BINARIES WITH TWO BRIGHT COMPONENTS

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ABSTRACT

Among the 4908 stellar systems which have at least one resolvable component brighter than $V = 6.00$ are 115 systems with two or more resolvable components both brighter than $V = 6.00$. We model these bright stellar systems, both single and double, by a distribution convolving (*a*) formulae from theoretical models for stellar evolution, including giants as well as main-sequence stars, (*b*) an initial mass function and birth rate function, (*c*) a density distribution of stars as a function of distance from the Galactic plane, and (*d*) a distribution of mass ratios and orbital separations. One reasonably firm conclusion is that masses, even in wide binaries, are correlated: there are too many doubly bright visual binaries (DBVBs), by a factor of 3–5, to agree with the hypothesis that the component masses are selected independently from the same IMF or luminosity function. Another conclusion is that the number of DBVBs per decibel of separation a (absolute, not apparent) is not constant in the range $10 \leq a \leq 10^5$ AU, but instead decreases slowly with increasing a. We find distributions of mass ratio and of separation that give roughly the observed number (115) of DBVBs, as well as their distribution of angular separations; however, parameters in these distributions are quite sensitive to assumptions about stellar multiplicity. Modest discrepancies between the actual 4908 bright systems and our model suggest that the solar environment out to ~ 100 pc is deficient in stars above, and enhanced in stars below, a mass of $\sim 2.5\ M_{\odot}$, by factors of 2 or 3 compared with more distant regions.

Subject headings: stars: evolution — stars: stellar statistics — stars: visual multiples

I. INTRODUCTION

In this paper we attempt to model the distribution of bright stars, both evolved and unevolved, over spectral type, with emphasis on "doubly bright" systems, where two components are both bright. We investigate whether, in wide (i.e., visual) binaries, the masses of the two components are correlated or not; in other words, whether the two components have masses which can be generated independently from the same initial mass function (IMF). The assumption that the components of wide binaries are uncorrelated is sometimes made for convenience, when analyzing binary statistics by nearest neighbor analysis of substantial areas of the sky (Weinberg and Wasserman 1988). The data set in which we test this hypothesis is based on the Bright Star Catalogue (Hoffleit 1983), restricted to $V \leq 6.00$. Because we prefer to count *systems* rather than *stars*, we call this the Restricted Bright System Catalog or RBSC. However, in considering the visual binaries contained in the RBSC, we restrict ourselves much more severely, to only those visual binaries with *both* components satisfying $V \leq 6.00$. This is because as we go to fainter and fainter companions the data will be less and less complete, and because many of the listed visual companions are optical, not physical, companions. Halbwachs (1986) has considered the larger set, making allowance for the probability of optical doubles at larger separations and/or larger magnitude differences, but our restriction offers us the chance to be complete. In § II we discuss how we can sensibly define, and count, these "doubly bright" visual binaries (DBVBs) from the RBSC.

We describe in § III the input needed to set up a "Theoretical Bright System Catalog" or TBSC. We require (*a*) some simple interpolation formulae to describe stellar evolution, not just in the main-sequence (MS) band but in the Hertzsprung gap (HG) and on the giant branch (GB) as well; (*b*) an initial mass function (IMF) and a birth-rate function (BRF); (*c*) a density distribution for stars within ~ 1 kpc of the Sun; (*d*) a distribution of mass ratios and orbital separations in binaries.

Evidently there are large numbers of uncertain parameters which have to go into such a model, and for that reason we have striven for numerical simplicity in each of the four input areas above. Nevertheless we believe that our investigation of the specific hypothesis we set out to test its fairly robust. In § IV we describe how the elements above are combined to give a TBSC. In § V we give a discussion of the results, and our conclusions, the principal one being that there are substantially too many DBVBs to be reasonably accounted for on the hypothesis that the two component masses are chosen independently from the same IMF, or equivalent luminosity function. We suggest in § V that the method described here for dealing with stellar evolution and space distribution has several possible applications in related areas of astrophysics.

II. IDENTIFYING DOUBLY BRIGHT VISUAL BINARIES (DBVBs)

Our starting point for the observational data was the 5084 entries in the BSC (Hoffleit 1983) which have $V \leq 6.00$, augmented to 5089 by five more stars from the Supplement (Hoffleit, Saladyga, and Własuk 1983). Following Bahcall, Casertano, and Ratnatunga (1987), we take this sample to be complete. As always in counting multiple stars down to a particular magnitude level, we have to define carefully whether we are using a combined magnitude, or magnitudes of individual components. We also have to define carefully the concept of "resolvability," i.e., we need a criterion to define whether two stellar images are to be treated as separate or not. With the BSC, there is the problem that some DBVBs have two separate entries and others a single entry; and even when there are two separate entries, sometimes both are given the same (usually combined) magnitude. We identify 46 single-entry systems

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which are in fact resolvable and should be included in the statistics. Since they are mostly systems of rather small separation, they give more of a bias toward smaller angular separations than is apparent just from a nearest neighbor analysis. Where an orbit is known, we use the angular semimajor axis in place of an instantaneous value of the separation; in some cases the orbital period is so long (up to, and even over, 1000 yr) that one must suppose the semimajor axis is rather uncertain, but in these cases it seldom differs by much from the current value of separation. A further problem is that sometimes both components are listed with the same coordinates, despite the fact that they may be separated by up to 100 μrad (20″), as is the case for α Cen (HR 5459/60). Unlike Weinberg and Wasserman (1988), we will include these systems (17 with *both* components having $V \leq 6.00$) in our statistics. We assign them also a separation which is the angular semimajor axis of the orbit, since almost all are systems with known orbits. This further biases the distribution of angular separations to smaller values.

We adopted the following definitions.

1. A "system" is a set of stars which is believed to be gravitationally bound. A system may consist of one, two, or more "resolvable components," each of which might in principle contain one, two, or more "nonresolvable subcomponents." Although we would like a definitive criterion for gravitational boundedness, we have to accept that none is really possible: there will always be some weakly bound and weakly unbound systems whose binding energies will vary unpredictably through zero as a result of encounters with field stars, or interaction with the Galactic tidal field.

2. Two components are "resolvable" if and only if they satisfy the criterion

$$\log \Delta\theta > 0.22\Delta V - 0.05 , \quad (1)$$

where $\Delta\theta$ is the separation in μrad and ΔV is the difference in magnitudes. This criterion is based on the discussion of Heintz (1969), who showed that recognition of such systems among fairly bright stars is virtually complete. In practice, several stars have been resolved (for instance by speckle interferometry) that are "unresolvable" by the criterion (1); however we shall adhere to criterion (1) rigorously so as to be able to compare our synthetic catalogs with the real catalog in a consistent way.

3. Each entry in our "Restricted Bright System Catalog" is a *system*, in which one or more resolvable components has $V \leq 6.00$.

4. The magnitude of a resolvable component is the combined magnitude of all the subcomponents (by definition nonresolvable) that make up the component.

From the 5089 entries with $V \leq 6.00$ in the BSC and its Supplement we therefore have to reject (i) 107 entries which are given a combined magnitude ≤ 6.00 in the BSC, but which are resolvable according to criterion (1) and whose separate components all have $V > 6.00$; (ii) 74 entries which are secondaries (or in five cases tertiaries) of resolvable systems. We are left with 4908 *systems*, of which 115 have two or more resolvable components with $V \leq 6.00$ for each component. These are the DBVBs whose properties we will compare with theoretical models.

We have already indicated that we prefer to count *systems* rather than *components*. Of course, this is a luxury we could not grant ourselves if the only information we had on components was their position and magnitude, as will be the case in other circumstances which we intend to explore in the future (see § V). However, for almost all entries in the BSC some further information is available, and we feel that it is sensible to make use of it.

In order to identify binary systems in this catalog we have used a nearest neighbor analysis (see Bahcall and Soneira 1981; and Scott and Tout 1989 for a more detailed discussion). Basically one constructs the histogram of pairwise separations between each star and its nearest neighbor. If one believes that stars are isotropically distributed on the plane of the sky (i.e., no binaries) then the number of stars with nearest neighbors in the range $\theta, \theta + \delta\theta$ is given by

$$N(\theta, \theta + \delta\theta) = F_R(\theta)\delta\theta , \quad (2)$$

where

$$F_R(\theta) = 2\pi n^2 \Omega \theta e^{-n\pi\theta} . \quad (3)$$

Here n is the surface density of stars and Ω is the solid angle of the survey. In fact, since the surface density of stars $n(\lambda)$ will in general depend on Galactic latitude λ, then the appropriate distribution is

$$F_R(\theta) = 4\pi^2\theta \int_0^{\pi} n(\lambda)^2 \sin \lambda e^{-n(\lambda)\pi\theta^2} d\lambda . \quad (4)$$

Figure 1 shows the nearest neighbor distribution of the 5084 stars with $V \leq 6$ in the BSC. Dotted lines superposed on this histogram show the theoretically expected nearest neighbor distributions for a nonisotropic distribution with density varying with latitude (in Gould's Belt coordinates rather than Galactic coordinates). There is a fairly clear excess of 62 pairs

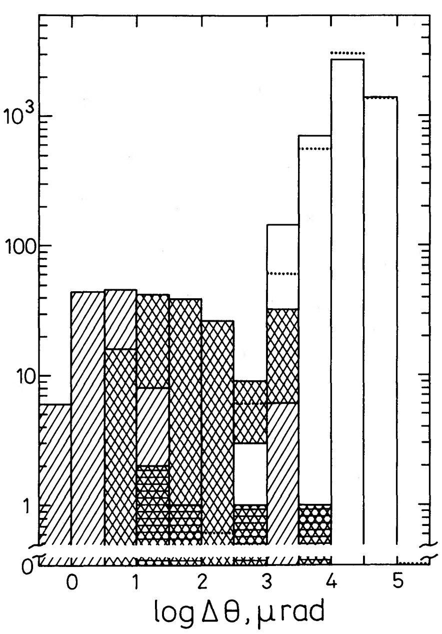

FIG. 1.—Distribution of angular separations among bright stars. The histogram includes, in white bars, separations which are optical, or probably optical, and in light cross-hatching, systems which are represented in the BSC by single entries. Moderately cross hatched regions are pairs believed to be physical; these include several whose components are listed in the BSC as having identical positions. Heavily crosshatched regions are tertiaries of triple systems and are not counted in our statistics of DBVBs. Note that each binary of Table 1 contributes two nearest neighbors to this figure. Dotted horizontal bars are the expected numbers from eq. (4).

with separations less than 1 mrad (∼200″). In this region one would expect three chance pairs. It is interesting to contrast this with the results of Bahcall and Soneira (1981). In their case they used the Weistrop (1972) data which cover a field of 13.5 square degrees. In this region, to $V = 12$ there were 244 stars and the nearest neighbor distribution showed a significant excess of pairs at $\Delta\theta \leq 0.6$ mrad. They found 19 pairs where one would expect eight chance superpositions on average. Thus further observations were required to determine the reality or otherwise of the candidate binaries. Latham *et al.* (1984) found that of the 19 candidate binaries, six were real and two were possibles. In our case we can be rather sure that essentially all of our candidate binaries are real and so detailed observations of individual candidates are not necessary. In fact, only one of the 62 nearest neighbor pairs closer than 1 mrad is, on the basis of radial velocity and proper motion discrepancies, accidental.

It appears that by using a larger sky coverage sample and a brighter limiting magnitude than Bahcall and Soneira we have managed to arrange that a large sample of real binaries can be easily separated from a population of "random" binaries. However, as a cautionary note, we should also point out that for 1 mrad $\leq \theta \leq$ 5 mrad there appears also to be a significant excess of pairs, as also noted by Weinberg and Wasserman (1988); to be precise, there are 273 observed as against 170 expected nearest neighbor separations in this range. This seems odd in view of the fact that there is an obvious tailoff in the distribution at $\Delta\theta \leq 1$ mrad. Although we do not address this problem in this paper we feel that it could be due to an inhomogeneously distributed sample of binaries which, if closer than the bulk of the other binaries, would lead to an excess of pairs at large separations. Alternatively, it may be because of clustering as in the Pleiades and Hyades, and may indicate, as Weinberg and Wasserman suggest, a quite substantial proportion of stars from disintegrating OB associations.

It appears that the nearest neighbor analysis applied to large sky coverage sampled to reasonably bright limiting magnitudes is a very fruitful means of detecting binaries. Taking a sample with a slightly fainter limiting magnitude and similar sky coverage should yield an even larger sample of wide binary systems. Even if one makes the limiting V fainter until the near gap ($0.3 \leq \Delta\theta \leq 1$ mrad) between the real binaries and the "random" pairs is eliminated, then one can still produce a large sample of almost certain binaries without need of further observations. Work is in progress on this point. For a substantially higher surface density of stars it might be desirable to consider not just nearest neighbors but near neighbors: see Scott and Tout (1989).

The BSC presents, as we have already indicated, a number of problems, which are presumably due to the historical development of the subject over about 80 years. There are several single entries in the Catalog that do in fact represent two resolvable stars. For these cases the Catalog usually contains a combined magnitude, a ΔV and the angular separation of the components; occasionally the magnitude quoted is that of the brighter component rather than the combination, even though the entry is single. We have searched the Catalog (by eye as well as by machine) for cases where, when these kinds of entries are split into their component stars, the individual components are bright enough to be included in our magnitude-limited sample, and also are resolvable with the criterion discussed above.

This procedure yields almost as many (46) DBVBs as the nearest neighbor analysis (62). One of the latter (against three as expected statistically) is known to be optical, and there are a further eight DBVBs with separations above 1 mrad that could not have been identified purely by nearest neighbor analysis, but which are almost certainly physical on the basis of other evidence. This gives us our total of 115 DBVBs. For a number of systems with known visual orbits we have used the angular semimajor axis rather than the separation. However, when constructing our theoretical comparisons we will always simply take the angular separation to be the semimajor axis divided by distance. We do not make a correction for a distribution of eccentricities, of inclinations, or of epochs. Since the range of angular separations covers three orders of magnitude, we believe these effects will be minor.

We have not included as DBVBs five pairs of stars which are described as common proper motion (cpm) pairs in the BSC, but which have the properties that (*a*) the separation is large (1–10 mrad), (*b*) both entries are intrinsically very bright stars, and therefore very distant (∼1 kpc). The "common proper motion" of these pairs is effectively zero, as one would expect, and the actual separations must be ∼1–10 pc. Possibly these pairs have a common origin, but we think it is unlikely that they represent bound systems. Among the 115 DBVBs we accept, the distribution of angular separations is shown in Table 1. The near cutoff at 1 μrad is due simply to the fact that the resolvability criterion (1) does not allow binaries to be deemed "resolvable" for $\Delta\theta < 0.9$ μrad. Although the largest angular separation is 12 mrad, comparable to the expected mean nearest neighbor separation of ∼25 mrad for ∼5000 systems distributed at random over the whole sphere, it appears to be real on the basis of radial velocities and proper motions, and corresponds to a physical separation of ∼0.5 pc. However the geometric mean angular separation for DBVBs is ∼30 μrad, and corresponds (from our modeling below) to ∼400 AU.

The uncertainty in the number of DBVBs is, we believe, of the order of 10%. As well as the five pairs described above, which we reject provisionally, there are several possible resolvable pairs where the magnitude of at least one component is close to 6.00 but rather uncertain. Specifically, if the separation is a few μrad, then usually a combined V is known fairly accurately, but the magnitude difference ΔV is not. A further problem is that there are five physical triples such that all three resolvable components have $V \leq 6.00$. We have simply rejected the faintest component (the most distant, in three cases). For a further handful of cases there are some reasons for doubting either that the system really is double (some bright stars having

TABLE 1

NUMBERS OF SYSTEMS IN THE RBSC[a]

log $\Delta\theta$	Number
<0.0	3
0.0–0.5	22
0.5–1.0	23
1.0–1.5	20
1.5–2.0	19
2.0–2.5	13
2.5–3.0	3
3.0–3.5	11
>3.5	1

[a] Divided into angular separation bins (μrad).

received remarkably little attention over the centuries) or that it is bound.

To check that our adopted stellar evolutionary approximations and IMF (below) are reasonable, we divided the RBSC into nine ranges of spectral types, corresponding (on the MS) to roughly equal intervals of log M. Because spectral type B covers a much greater range of masses than any other type (except M) we subdivided it into early B, mid B and lateB, as in Table 2. We have taken temperatures corresponding to spectral type boundaries from Popper (1980), and for early-type supergiants from Fitzpatrick (1988). The ratio of DBVBs to stars in each spectral-type bin decreases from $\sim 7\%$ at the earlier types to $\sim 1\%$ at the later. This decrease can be explained, as our model verifies, by the fact that ΔV is much more sensitive to mass ratio at later types than at earlier.

A list of HR numbers (i.e. of numbers in the Bright Star Catalogue) of DBVBs, of stars that we treat as single according to criterion (1), and of stars that we reject because, though each is a single entry in the BSC, they are resolvable and both components are fainter than $V = 6.00$, is available on request.

III. THE INPUT FOR A THEORETICAL BRIGHT SYSTEM CATALOG

In § I we identified the four main ingredients required for construction of a TBSC.

a) Stellar Evolution

We use a number of very elementary interpolation formulae (given in the Appendix) which allow us to compute very rapidly the visual luminosity L_V and effective temperature T_e of a star with a given mass and age. We do not believe that for the present analysis (or indeed for several other purposes) it is necessary to take into account every lump and bump on a theoretical track in the H-R diagram, and so we have contented ourselves with simple formulae.

Using a Monte Carlo approach, we generate 2×10^4 binary systems with a spectrum of masses and ages (§ III*b*), positions in the Galaxy (§ III*c*), and binary parameters (§ III*d*). For each binary we compute L_V and T_e for both components, and test for resolvability using condition (1). Then we determine which unresolvable binaries have combined apparent magnitude $V \leq 6.00$, which resolvable binaries have only one resolvable component with $V \leq 6.00$, and finally which resolvable binaries have both components with $V \leq 6.00$ (the theoretical DBVBs). Many of the 2×10^4 systems we generate are sufficiently faint intrinsically that even at the distance of the nearest stars (~ 1 pc) they do not contribute at all to the TBSC, but at the same time some of the most intrinsically luminous can contribute to the TBSC several times over. We have to normalize the space density so that the right total number (4908) of bright systems is obtained, although given the Monte Carlo nature of our calculation, this right total will only be approximately achieved in a particular TBSC.

TABLE 2

NUMBERS OF SYSTEMS IN THE RBSC, DIVIDED INTO TEMPERATURE BINS

Spectral Type Range	Approximate Mass (MS)	$T_e(10^3$ K) Dwarfs	$T_e(10^3$ K) Giants	Number in RBSC
O3–B0[a]	>16	>29	>23	54
B0.3–B2	8–16	22–29	17–23	217
B2.5–B5	4–8	15–22	13–17	273
B6–B9.5	2.2–4	10–15	10–13	499
A	1.5–2.2	7.0–10	7.4–10	951
F	1.0–1.5	6.0–7.0	5.7–7.4	622
G	0.8–1.0	5.2–6.0	4.75–5.7	682
K	0.5–0.8	3.9–5.2	3.9–4.75	1259
M[b]	<0.5	<3.9	<3.9	351
Total	...	...	...	4908

[a] Includes Wolf-Rayets.

[b] Includes S and C stars.

We believe our theoretical models for stars in the MS band are accurate to better than 10% in both L_V and T_e, but our interpolation formulae for Hertzsprung gap stars, red giants, and red supergiants are certainly less accurate. However the main parameter, from the point of view of imitating the RBSC, is the relative lifetime in the red giant (RG) and MS stages, and we believe we can approximate this to $\sim 10\%$. Apart from the fact that our interpolation formulae are too simple to represent theoretical evolutionary models to better than this, there are further uncertainties as follows.

1. We have used only a single initial composition, roughly corresponding to Population I (i.e., we take $X = 0.7$, $Y = 0.28$, $Z = 0.02$). But most bright stars appear to be Population I, so we believe this will be a minor source of error.

2. The theoretical evolution of massive stars ($\gtrsim 25\ M_\odot$) has long been a source of conflict with observations. This is usually attributed either to stellar wind mass loss, as in Of stars and Wolf-Rayets, or to convective overshooting, or to both. We avoid a detailed understanding of this conflict by simply insisting that stars over 25 $M_\odot$ restrict their evolution to the main-sequence band, followed by a "Wolf-Rayet" phase with a given L_V and T_e; and then they metamorphose directly into neutron stars or black holes.

3. Theoretical red supergiants, even of fairly low initial mass (1–2 $M_\odot$), can evolve briefly to very high luminosity ($\sim 10^5\ L_\odot$) as their degenerate cores approach the Chandrasekhar mass. Presumably mass loss by stellar winds prevents this in practice. We therefore impose artificially an upper limit to L (or, in effect, core mass) of the form $L \leq L_{max}(M)$, where M is the initial mass. Beyond that point we assume the star is a white dwarf. We do not at present allow a "hot subdwarf" phase, but will have to do so when extending the magnitude limit to $V = 10$ or more.

4. No allowance was made for interaction in close binaries, e.g., for Roche lobe overflow (RLOF). This is partly because our binary distribution (§ III*d*) mainly generates *wide* binaries, with $a \gtrsim 10$ AU, $P \gtrsim 30$ yr. Even so, some of the binaries we generate, will be expected to undergo RLOF, particularly at a late stage in evolution. However we suggest, following Tout and Eggleton (1988), that stellar winds, enhanced by tidal interaction prior to RLOF, may usually prevent RLOF in systems with $a \gtrsim 3$ AU. This would reduce the lifetimes and maximum luminosities of these binaries. In future investigations we intend to include binary interaction, if necesary.

b) The IMF and BRF

It is commonly assumed that the number of stars of a given mass and age can be separated into the product of an IMF (a distribution over masses) and a BRF (a distribution over ages). This is done more for convenience than from conviction, since it is all too easy to imagine that the IMF is actually time-dependent, or equivalently the BRF mass-dependent. Nevertheless we cling to this simplification for want of quantifiable models of greater complexity. Our algorithm could in fact easily incorporate a time-dependent IMF, at the expense of including further coefficients whose values would be hard to determine at least from the present investigation of the RBSC.

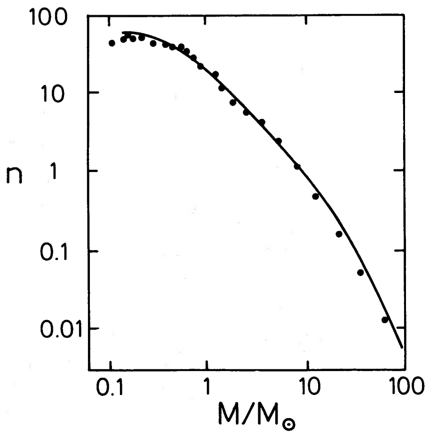

FIG. 2.—The IMF of Miller and Scalo (1979) is shown with dots, and our approximation to it, derived from eq. (5), is shown as a continuous curve.

Most bright systems contain moderately massive stars (late B, A main-sequence stars, and G, K giants which are presumably their descendants), and so come from a limited portion of the combined IMF/BRF, which may more reasonably be assumed to be separable.

We adopted a simple approximation to the IMF of Miller and Scalo (1979), which appears like a power law at moderately high masses ($\sim 1\ M_\odot$ to $20\ M_\odot$), steepening at still higher masses, and with a turnover at low masses (per unit interval of $\log M$, our IMF peaks at $M \sim 0.18\ M_\odot$). We find it simplest to express our IMF in the form we compute it. Let X be a random variable with uniform distribution in the range [0, 1]. Then we take a star to have mass (in solar units)

$$M = \frac{0.19X}{(1-X)^{0.75} + 0.032(1-X)^{0.25}} . \tag{5}$$

For $X \sim 0.7 - 0.999$, this gives a Salpeterish distribution

$$n(M) \equiv \frac{dX}{dM} \approx \text{const. } M^{-7/3} , \tag{6}$$

and for $X \ll 1$ it gives

$$n(M) \approx \text{const} . \tag{7}$$

For $X > 0.999$, the IMF slope in equation (6) steepens from 7/3 to 5. The median mass is $0.15\ M_\odot$ and the mean mass $0.43\ M_\odot$. The IMF for very low masses ($\lesssim 0.1\ M_\odot$) is of course not well-determined from observation, although constraints on the amount of dark matter in the Galaxy have been suggested. However our IMF, equation (5), means that the 50% of stars with $M \lesssim 0.15\ M_\odot$ contribute only 8% of the mass, and this is hardly excessive. Figure 2 compares our assumed IMF with Miller and Scalo's.

Although stars with $M \lesssim 0.8\ M_\odot$ (89% of stars, from eq. [5]) hardly figure in the RBSC, they are important for the hypothesis we are testing, since if two components of a binary are chosen at random from the same IMF or luminosity function, it matters whether there are many or few stars of low mass for each one of high mass.

The BRF is perhaps less problematic than the IMF, because although the rate of production of stars in the solar neighbourhood can hardly be supposed to have been constant since the origin of the Galaxy, most of the RBSC entries are moderately massive stars which must have been born in the last 10%–20% of the Galaxy's lifetime. We therefore content ourselves with a formula for the age t (in Myr) of a random star:

$$t = Y t_{\rm Gal} , \tag{8}$$

where Y, like X in equation (5), is a random number uniformly distributed in [0, 1]. We take $t_{\rm Gal}$, the age of the Galaxy, to be 1.2×10^4 Myr, but can easily vary this parameter.

c) The Galactic Distribution

On the assumption that most bright systems are well within 1 kpc of the Sun, we use a formula for the number density $n(z, t)$ of stars which is independent of R, the distance from the Galactic center, and dependent only on z (distance from the Galactic plane) and t (age), the latter by way of an age-dependent characteristic height $h(t)$. For considerable later numerical convenience, we adopt a z-dependence

$$n(z, t) = \frac{3}{4\pi d_0^2\, h t_{\rm Gal}} \left(1 + \frac{z^2}{h^2}\right)^{-5/2} , \tag{9}$$

where $h(t)$ is assumed to increase with age t as

$$h = h_0 (t/t_{\rm Gal})^{1/2} = 1200 (t/t_{\rm Gal})^{1/2} , \tag{10}$$

with $h(t)$ in pc and t, $t_{\rm Gal}$ in Myr. Note that h is not a scale height as usually defined, since n decreases by a considerably greater factor than e as z increases from h to $2h$, for example. Integrating equation (9) over z and then t gives the total number of stars per square parsec in the Galactic plane as $1/\pi d_0^2$, so that d_0 is the parameter which determines the projected number density of stars. Projected mass density is

$$\sigma = 0.43/\pi d_0^2\ M_\odot/{\rm pc}^2 . \tag{11}$$

Our choice of constant in equation (10) is based on the analysis of Kuijken and Gilmore (1989) of the density of K dwarfs at distances up to ~ 3 kpc from the plane. Since the K dwarfs will presumably represent a mixture of all ages, we have to integrate equation (9) with respect to age t, using equation (10), to get a distribution which we can compare with Kuijken and Gilmore (1989). The comparison, with $h_0 = 1200$ pc, is shown in Figure 3. Because equation (10) rather naively gives $h = 0$ when $t = 0$, the age-integrated distribution shown in Figure 3 has a cusp in it at $z = 0$, even though equation (9)

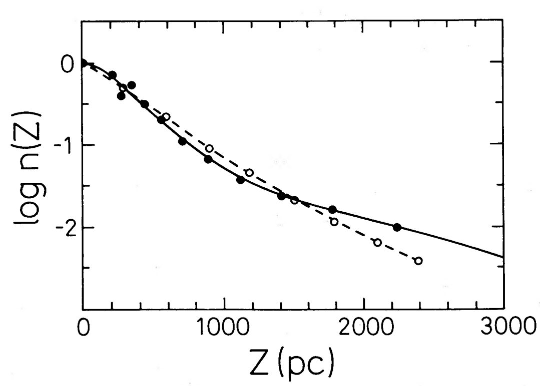

FIG. 3.—The numer density of K dwarfs as a function of distance from the Galactic plane. The dots and continuous curve are from Kuijken and Gilmore (1989). The open circles and broken line are our approximation obtained by integrating the distribution of eq. (9) over age, using eq. (10) for the time-dependent scale height and assuming a constant birth rate over the age of the Galaxy.

does not. According to G. Gilmore (private communication), the more distant K dwarfs in Figure 3 may have their distances somewhat overestimated, because lower metallicity at greater z has not been allowed for. This would make our approximation somewhat better than it appears.

From equation (9) the number of stars of age t within distance d of the Sun is

$$N(d, t) = \frac{1}{d_0^2 t_{\rm Gal}} \frac{d^3}{\sqrt{d^2 + h^2}}, \qquad (12)$$

a formula whose simplicity is the basis for adopting the particular z-dependence of equation (9).

d) *Distribution of Mass Ratios and Orbital Separations*

Since many stellar systems have multiplicity higher than two, it might be desirable to set up a model of stellar multiplicity in general. For some present purposes, however, it will be sufficient to assume that all stellar systems are exactly binary. We will find that one hypothesis we test produces far too few DBVBs, even when all systems are assumed to be wide binaries ($a \geq 10$ AU), and so *a fortiori* it will fail under a more reasonable distribution of separations.

It would not be difficult for us to model stellar multiplicity in substantially more detail by, for instance, allowing each component of the wide binary to have a certain probability of being a close binary, with a distribution of mass ratio, and of period, that would probably be quite different from wide binaries. We have such a possibility in hand for a future investigation of the statistics of interacting binaries. However, we hope it will be marginal for the present investigation.

Adopting for the present the assumption that all systems in a TBSC are to be binaries, we model the distribution of separations by

$$a = a_0 \left(\frac{a_1}{a_0}\right)^Z, \qquad (13)$$

Z being another random number with uniform probability in [0, 1]. This gives a uniform distribution of log a for $a_0 < a < a_1$, and we take $a_0 = 10$ AU and $a_1 = 10^5$ AU ~ 0.5 pc. As indicated above, our assumption that all systems are binary, with separation given by equation (13), is an extreme, since some stars will be single and others *close* binaries; but we shall find that even this extreme does not yield enough DBVBs, thus strengthening our case that the two masses in wide binaries are to some extent correlated. We refer to the model above, with component masses chosen independently from the same IMF, equation (5), and separations from the distribution (13), as our model 2 (model 1 being single stars only, which we do not discuss here).

Anticipating that the above assumptions will not adequately model the DBVBs of § II, we define an alternative model 3 as follows. First, the *total* mass of a binary is determined from equation (5); second, the mass ratio $Q (Q \geq 1)$ is taken to satisfy the normalized distribution

$$Qn(Q) = \frac{4\alpha/\pi}{Q^\alpha + Q^{-\alpha}} \qquad (14)$$

which is obtained if Q is given by

$$Q^\alpha = \sec \frac{\pi X'}{2} + \tan \frac{\pi X'}{2}, \qquad (15)$$

with X' another random variable uniformly distributed in [0, 1]. The constant α will be chosen to give about the right number of DBVBs. The form of equation (14), and consequently also equation (15), may seem ad hoc but we use it because, although it is little different from the traditional power law in Q at large Q, it is more sensible in that it gives the symmetry one would expect physically between $Q < 1$ and $Q > 1$; an exact power law would give a cusp at $Q = 1$. This is independent of the fact that we adhere for consistency to $Q > 1$ only: presumably nature does not.

The separation, in our alternative model 3, is taken to satisfy a distribution rather similar to equation (14), with Q replaced by a/a_0:

$$an(a) \propto \frac{\beta}{(a/a_0)^\beta + (a_0/a)^\beta}, \qquad (16)$$

although the normalization is slightly different so that a does not range from unity to infinity, as does Q, but from $10^{-3}a_0$ to $10^3 a_0$. This means that a/a_0 is generated by a formula similar to equation (15), except that the constant in it is not quite $\pi/2$, and $X' \in [-1, 1]$. Note that whereas there is a slight rationale behind the form of equation (14) for Q, there is less behind equation (16) for a; we simply find it convenient to assume a power law for large a, with necessarily some turnover at small a. Following Heintz (1969), we anticipate that we can adopt $\beta = 0.33$, and $a_0 = 30$ AU. In fact we shall not attempt to vary these quantities to get a better fit.

IV. SETTING UP A TBSC

According to the input described in § III, we need four random numbers to generate a system: one for each mass, one for the age, and one for the separation. Thus even with $N_* = 2 \times 10^4$ systems, the value we use most often, parameter space is not very well covered, and this is especially true of high-luminosity systems, whose lifetimes are short compared with the age of the Galaxy, $t_{\rm Gal}$. We therefore modify equations (5), (8), and (12) in a complementary way, as follows. For a single star, choose two random numbers U, $V \in [0, 1]$, and use a transformation $X = X(U)$, $Y = Y(U, V)$ to map the unit square on to itself, crowding points toward higher masses, and for each mass toward that age at which the luminosity is greatest. This transformation is listed in the Appendix. We then have to modify equation (12), which determines N, the number of systems to be expected within a sphere of radius d, thus:

$$N = \frac{\partial(X, Y)}{\partial(U, V)} \frac{1}{d_0^2 N_*} \frac{d^3}{\sqrt{d^2 + h^2}}. \qquad (17)$$

The area element in U, V space is of course $1/N_*$. For a binary with independently chosen masses we need two applications of $X(U)$ and hence a slightly more elaborate Jacobian in equation (17); but if instead the secondary mass is generated from equation (15) there does not appear to be any need to bias this distribution in order to achieve reasonably uniform sampling of the three-parameter space. Whichever model is used for generating the two masses, it is the *larger* mass which is used to bias the choice of age.

We now have two component masses M_j, M'_j as well as an age t_j for the jth system out of N_*. We can now use the stellar evolution algorithms of the Appendix to work out the visual luminosities of each component and, from their sum, compute d_j, the distance out to which the system (if nonresolvable) would be visible down to a given limit, $V = 6.00$ in our case.

Equation (17), with h a function of t_j given by equation (10), can then be used with $d = d_j$ to find $N_j = N(d_j)$, the total number of such systems which should be visible; reddening will be allowed for later.

Round N_j down to the nearest integer, $[N_j]$; this may of course be zero, a case we deal with shortly. If nonzero, we now generate $[N_j]$ systems, each consisting of the same two stars, but with various separations selected at random using equation (12). Their distances are also selected at random using equation (17) in reverse: it is to be solved for d with the left-hand side replaced by $[N_j]W$, with W uniformly distributed in $[0, 1]$. Once the distance is determined we can allow for reddening, which for the present we take to be isotropic and homogeneous, at 1.9 mag per kpc (Allen 1973). Further, the distance determines the angular separation, and hence via criterion (1) whether the system is resolvable or not. We then accept or reject the system for the TBSC on the same principle as for the RBSC (see § II): accept if either unresolvable with combined $V \leq 6.00$, or if resolvable and one or other $V \leq 6.00$ (or both); and otherwise reject. Those with two resolvable bright components are the theoretical DBVBs.

If $[N_j]$ is zero, we do not just reject the system out of hand. This would mean, for example, that we would never get any K dwarfs into the TBSC. What we must do is accumulate the fractional parts, $N_j - [N_j]$, in a large, but not too large, collection of boxes in the (L_V, T_e)-plane of the primary (the brighter in L_V); and when the accumulation exceeds unity, generate an extra model of the type just computed. We do this even if $[N_j] > 0$, though it is less important there. We use 9×21 boxes, with critical T_e as given in Table 2, and equal intervals of $\log L_V$ in the range 10^{-3}–$10^{6.5}$ $L_\odot$.

V. RESULTS AND CONCLUSIONS

To set up a TBSC for comparison with the observed RBSC a number of parameters have to be specified, or alternatively they might be determined by attempting a least-squares fit. For our model 2 of § III*d* we allow only one parameter to be free, i.e., d_0 in equation (9), which determines the surface density σ of stars projected on the Galactic plane at the solar neighborhood. We take the Galactic age $t_{\rm Gal}$ to be 12,000 Myr, following Miller and Scalo (1979), and then determine d_0 by trial and error to give the right number (4908) of systems. Since we use a Monte Carlo approach, we will of course not get this number exactly right in any one TBSC. In our alternative model 3 of § III*d* we treat two parameters as free: we add α, determining the distribution of mass ratios, to d_0.

Tables 3 and 4 show the distribution of stars in spectral type bins, and of DBVBs in angular separation bins, for the RBSC, our standard model 2, and our alternative model 3 with $\alpha = 0.8$, $a_0 = 30$ AU, and $\beta = 0.33$). For each of our model results we averaged 20 Monte Carlo runs, and present an rms scatter as well as the average. It seems clear that model 2 gives only a quarter as many DBVBs as required, a discrepancy of $\sim 7\,\sigma$ if we estimate the rms error of the RBSC number to be ~ 12 (see § II). Before attaching too much weight to this discrepancy, however, we should note that several other discrepancies exist, some of them potentially more significant. The most significant appears to us to be that we obtain too many late-B stars and too few A + F stars, both by factors of ~ 1.5–2. However the total for all three bins together is about right. So also is the total of evolved giants (mainly G, K, and M). The late-B/(A + F) discrepancy cannot, we believe, be just an error of the IMF, since the range of mass is quite limited. The increase in slope of the IMF that would be required would be enormous. Nor does it seem reasonable that the stellar models are in error by such a large and rapidly changing amount, particularly since the problem is mainly in main-sequence stars. The stellar models we have used are in remarkably good agreement with Popper's (1980) compilation of masses, luminosities, and effective temperatures (see Fig. 4, in the Appendix). A further discrepancy in our models is that we never achieve the near-equality of numbers of early-B and mid-B stars evident in the RBSC. However, we are fairly sure that this represents a statistical quirk in the observed distribution rather than an important effect that we should seriously endeavor to model. We do not have any clear concept of the standard deviations in the observed numbers (and are unlikely to unless we are able to draw up a similar RBSC centered on different stars than the Sun), but we suspect the inherent standard deviations in the *observed* data are large enough to accommodate the transfer of even 50 to 100 stars from the early-B to the mid-B bin.

We believe that the most likely cause of the late-B/(A + F)

TABLE 3

DISTRIBUTION OF BRIGHT STARS OVER SPECTRAL TYPE OF PRIMARY

Spectral Type	RBSC	Model 2	Model 3	Model 4
O3–B0	54	56 ± 3	43 ± 4	42 ± 5
B0.3–B2	217	115 ± 8	104 ± 9	102 ± 5
B2.5–B5	273	433 ± 22	399 ± 25	336 ± 29
B6–B9.5	499	960 ± 50	1005 ± 65	567 ± 35
A	951	658 ± 36	687 ± 23	1097 ± 35
F	622	316 ± 12	335 ± 35	591 ± 33
G	682	586 ± 33	611 ± 49	768 ± 22
K	1259	1265 ± 46	1312 ± 111	1190 ± 94
M	351	444 ± 41	434 ± 62	330 ± 76
$V \leq 0.0$	4	11 ± 2	9 ± 3	2 ± 2
DBVBs	115	27 ± 4	119 ± 12	113 ± 15
d_0 (pc)	...	0.088	0.055	0.054
σ ($M_\odot$/pc^2)	...	35	45	...

TABLE 4

NUMBERS OF SYSTEMS DIVIDED INTO ANGULAR SEPARATION BINS (μrad)

Model	log $\Delta\theta$; RBSC								
	<0.0 3	0.0–0.5 22	0.5–1.0 23	1.0–1.5 20	1.5–2.0 19	2.0–2.5 13	2.5–3.0 3	3.0–3.5 11	>3.5 1
Model 2	0 ± 0	2 ± 1	3 ± 1	4 ± 2	4 ± 2	3 ± 2	3 ± 2	4 ± 2	4 ± 2
Model 3	0 ± 1	19 ± 5	28 ± 4	26 ± 4	19 ± 4	13 ± 3	10 ± 2	4 ± 2	1 ± 1
Model 4	0 ± 1	20 ± 7	24 ± 5	23 ± 6	19 ± 3	12 ± 3	9 ± 2	4 ± 2	2 ± 1

NOTE.—All entries have been rounded to integers, so row totals need not agree with Table 3.

discrepancy is nonuniformity of the stellar population over length scales of ~ 100 pc, or time scales of ~ 100 Myr, or both. Thus if the solar neighborhood out to ~ 100 pc contains fewer massive ($\gtrsim 2.5\ M_{\odot}$) stars by perhaps a factor of 3, while containing more less-massive stars by perhaps a factor of 3, the discrepancy can be avoided. Recently, Gilmore and Roberts (1988) have emphasized the nonuniversality of the IMF, and of the stellar luminosity function on which the IMF is based.

Another reason for accepting that the solar neighborhood out to ~ 100 pc is short of massive stars is the fact (also shown in Table 3) that our TBSCs tend to have too many *very* bright stars, $V < 0.00$: typically 11 ± 2 as against four observed. The excess very bright stars are generally massive, and nearby in the sense of the previous paragraph–often one or two will be O stars at perhaps 50 pc, though they are more commonly G or K supergiants of 5–10 $M_{\odot}$ at a similar distance. Yet our TBSC clearly does not contain too many massive stars altogether. Down to $V = 6.00$, O to mid-B stars are about right in total number. We feel that this also is consistent with our tentative picture that the solar neighborhood out to ~ 100 pc is deficient in massive stars.

To test this very crude picture, a TBSC (model 4 in Tables 3 and 4) was set up to resemble model 3 except for the following modifications: first, all stars more massive than 2.5 $M_{\odot}$ and closer than 100 pc were rejected; second, stars less massive than 2 $M_{\odot}$ were increased in number by a factor of 2.5, except that those more distant than 80 pc were rejected. This gave excellent agreement with the RBSC, solving both the late-B/(A+F) anomaly and the $V < 0.00$ anomaly while keeping the good agreement between late giants and early dwarfs; although little can be read into this since the alterations were so ad hoc. It did not solve the early-B problem, but we have argued that this is less significant. Evidently a roughly equivalent stratagem would be to multiply the IMF by a factor of ~ 4 for masses below $\sim 2.5\ M_{\odot}$. But we feel it is much less likely that there should be a "universal" IMF with such an abrupt jump in it, than that there should be variations in the IMF on length scales of ~ 100 pc, or (more-or-less equivalently) on times scales of ~ 100 Myr.

The values of surface density σ of the Galactic disk in the solar neighborhood given in Table 3 are roughly in agreement with Kuijken and Gilmore (1989), who obtained 46 ± 9 $M_{\odot}/\text{pc}^2$. For model 2 the average mass of a *system* is of course twice that of a single star, or of a single system in model 3. The larger surface density of model 3 reflects the fact that putting mass into binaries with components of comparable mass reduces the light-to-mass ratio substantially. For model 4, σ is not definable since equation (11) relating σ to d_0 presupposes a universal IMF.

If provisionally we accept that our late-B/(A+F) discrepancy is accounted for in the manner suggested above, then we believe our conclusion regarding DBVBs can still be sustained. In particular, the number of DBVBs and their distribution over angular separations was found to be barely altered from model 2 when we computed a variant of model 4 which contains the same assumptions about mass ratios and separations as model 2. Thus we reject the model 2, with all the more conviction because it assumes *all* stars are binaries with $a > 10$ AU. Our alternative model 3 with $\alpha = 0.8$ gives much better agreement, both in terms of total number of DBVBs and their distribution over angular separations. It is noteworthy that the flat distribution (13) over log a used in model 2 transforms into a similarly flat distribution over log $\Delta\theta$ (Table 4), so that the actual distribution of log $\Delta\theta$ in the RBSC clearly requires a falling off of probability with log a at separations large enough to produce resolvable visual binaries as determined by criterion (1). A number of tests with our Monte Carlo procedure have established that the minimum separation for producing visual binaries in significant numbers in a TBSC is about 10–100 AU. In trying to model both the total number and the angular separation of DBVBs better than model 2 we found model 3 satisfactory, but we have not attempted a least-squares solution. It is clear, for instance, that the value of α cannot be well-defined, at least from our investigation alone. If we assumed that only 50% of all systems were binaries, we would need a value for α larger by a factor of about 2 to obtain the same total of DBVBs. We emphasize that by restricting ourselves to that rather limited but carefully definable class of visual binaries that are DBVBs we can only claim to be exploring a range of mass ratios $1 \leq Q \lesssim 3$, so we cannot determine whether the distribution (14) extrapolates to much larger values of Q. However, we do not believe it would be helpful to include all the companions fainter than $V = 6.00$ that are noted in the BSC since it is difficult to assess completeness, and also since a high proportion of them are in fact optical. Models 3 and 4 show a decreasing proportion of DBVBs at later types, from $\sim 5\%$ at the earlier to $\sim 1\%$ at the later, very much in line with the RBSC. We also emphasize that we can say nothing about the applicability of our distributions to masses less than solar, since such stars contribute very little to the RBSC, but we have little doubt that there exists a great predominance by number of these low-mass stars, as Miller and Scalo's (1979) IMF shows.

The strength of our approach is, we believe, that we are making a direct confrontation between an entirely theoretical model on the one hand and an entirely observational set of data on the other hand, a set which has not been truncated somewhat arbitrarily by, for example, excluding stars bluer than $B-V = 0.0$ on the grounds that they may be Gould's Belt stars (Bahcall, Casertano, and Ratnatunga 1987). Provided we apply to our theoretical "catalog" exactly the same criteria as are used for admission to the observed catalog, we can make a *direct* comparison; thus all of our assumptions are included on an equal footing, and their influence individually can be assessed. Admittedly, in our present study we had to "truncate" the observational data set rather significantly, especially as regards DBVBs. A preliminary assessment suggested to us that there would be a rather larger number of these, perhaps 250, but then we found we had to reject several (107, § II) on the basis that though the combined magnitude was less than 6.00, the systems were resolvable and had separate magnitudes, both greater than 6.00. Also, the historical development of the BSC has meant that some double stars had one entry and others two, in a very unsystematic way: some double-entry systems have separations less than 10 μrad, and some single-entry systems separations greater than 1 mrad. Thus our "truncation" of the BSC to the RBSC, in § II, was an attempt to convert the Catalog to what would have been listed if every resolvable component had been listed to $V = 6.00$.

We believe an approach like ours could be very fruitfully applied to larger data sets, such as Schmidt plates down to $V = 20.0$, or the Guide Star Catalog (Lasker *et al.* 1988) which should be complete to $V = 14.0$ for latitudes greater than 30°. For such deeper surveys we should however have to incorporate a more sophisticated Galactic model, as well as evolution of Population II (and intermediate-metallicity) stars. Another circumstance in which we believe that the simple kinds of approximation that we make to stellar evolution would pay off is in combining stellar evolution with an N-body gravitational calculation so that the evolution of clusters of stars such as M67 can be followed more physically than by assuming that stars are points of constant mass.

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APPENDIX

THE EVOLUTION FORMULAE

Interpolation formulae describing the radius and luminosity of a star given its mass and age are based on models evolved using the stellar evolution program developed by Eggleton (1971, 1972). Figure 4 shows an H-R diagram on which are plotted theoretical evolutionary tracks for 1, 2, 4, 8, 16, and 32 $M_\odot$ stars. All stars were started on the zero-age main sequence (ZAMS) with a uniform composition of hydrogen, $X = 0.7$, helium, $Y = 0.28$, and metallicity, $Z = 0.02$. Also plotted are observational data points taken from the review of stellar masses by Popper (1980), except for some revisions by Popper (1982). All these points have fairly well-determined masses and are grouped according to mass in the ranges between our theoretical models. The data appear to indicate that apart from a few small discrepancies our modeling of stellar evolution across the main sequence is fairly good. Unfortunately there is no such wealth of comparably good data on the masses of red giants and we have just to trust our models. Phases of evolution that take place rapidly compared with the usual nuclear evolution of the stars are shown as dotted lines on our tracks. We do not take as much care over modeling these stages since they are less likely to contribute much to our catalogues.

We regard the ZAMS as particularly important, since it establishes the position in the Hertzsprung-Russell diagram where the star spends most of its lifetime (i.e., the MS band). We therefore use 38 models with masses varying from 0.1 to 100 $M_\odot$ to obtain the

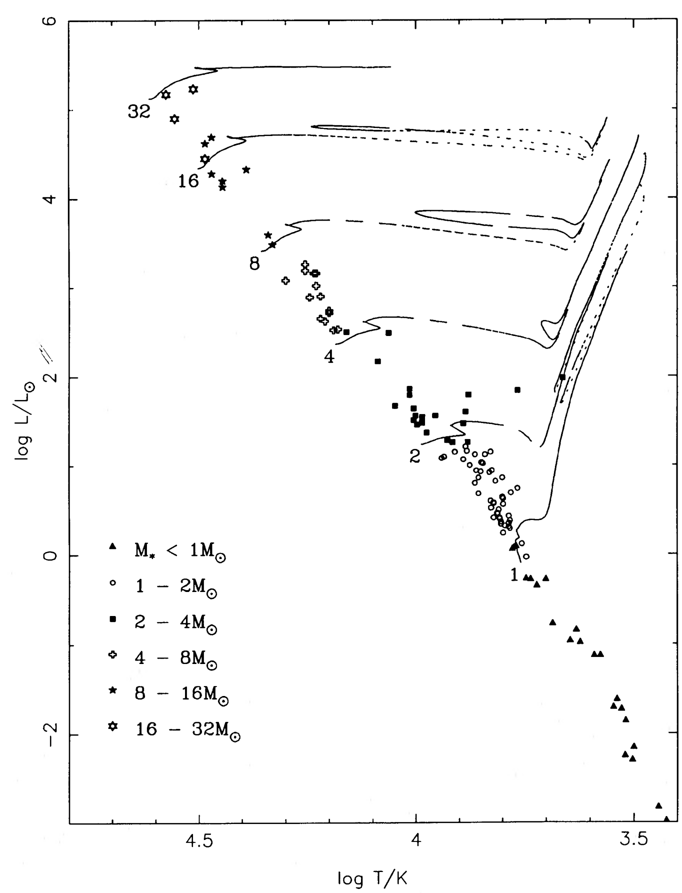

FIG. 4.—H-R diagram showing data from Popper (1980, 1982) as individual symbols identified by mass range, and our computed models as continuous (nuclear evolution) and broken (faster evolution) curves.

approximations

$$L_0 = \begin{cases} \dfrac{1.107M^3 + 240.7M^9}{1 + 281.9M^4} , & \text{if } M \leq 1.093 ; \\ \dfrac{13{,}990M^5}{M^4 + 2151M^2 + 3908M + 9536} , & \text{if } M \geq 1.093 ; \end{cases} \tag{A1}$$

and

$$R_0 = \begin{cases} \dfrac{0.1148M^{1.25} + 0.8604M^{3.25}}{0.04651 + M^2} , & \text{if } M \leq 1.334 ; \\ \dfrac{1.968M^{2.887} - 0.7388M^{1.679}}{1.821M^{2.337} - 1} , & \text{if } M \geq 1.334 . \end{cases} \tag{A2}$$

The most important evolutionary time scale for a star is its main-sequence lifetime. Rather than take the point at which central hydrogen is exhausted as the end of the main sequence we prefer to use the more well-defined point of maximum luminosity reached after the hook at the end of the main sequence. A further reason for including stars somewhat beyond the hook is that the amount of time spent in early shell-burning on the left-hand side of the Hertzsprung gap, especially in the mass range 2–4 $M_\odot$, can be quite considerable. For stars of mass greater than 32 $M_\odot$ the maximum luminosity does not exist but the evolution is so rapid at this point that the point of maximum temperature on the hook suffices with no loss of accuracy. For stars less than 1 $M_\odot$ there is again no maximum but at this point the end of the main sequence is so ill-defined (the main sequence continuing without any significant gap into the giant branch) that we prefer just to extrapolate our fit for higher masses. We obtain a main-sequence lifetime by fitting stars of masses 1, 1.3, 1.6, 2, 4, 8, 16, 32, 64, and 80 $M_\odot$, from which we obtain

$$t_{\rm MS} = \frac{2550 + 669M^{2.5} + M^{4.5}}{0.0327M^{1.5} + 0.346M^{4.5}} . \tag{A3}$$

We take some care in modeling the evolution during this interval, since it is most of the star's nuclear burning life. We use the same 10 models together with one of 0.8 $M_\odot$ and find that it is adequate to fit the luminosity by a quadratic in fractional time $\tau_{\rm MS} = t/t_{\rm MS}$, t being the age of the star:

$$\log_{10} L = \log_{10} L_0 + \alpha\tau_{\rm MS} + \beta\tau_{\rm MS}^2 , \tag{A4}$$

where L_0 is the ZAMS luminosity and α and β are given by

$$\alpha = \begin{cases} 0.2594 + 0.1348 \log_{10} M , & \text{if } M \leq 1.334 ; \\ 0.09209 + 0.05934 \log_{10} M , & \text{if } M > 1.334 ; \end{cases} \tag{A5}$$

and

$$\beta = \begin{cases} 0.144 - 0.833 \log_{10} M , & \text{if } M \leq 1.334 ; \\ 0.3756 \log_{10} M - 0.1744(\log_{10} M)^2 , & \text{if } M > 1.334 . \end{cases} \tag{A6}$$

In a similar way we fit the radius using a cubic of the form

$$\log_{10} R = \log_{10} R_0 + \alpha'\tau_{\rm MS} + \beta'\tau_{\rm MS}^2 + \gamma'\tau_{\rm MS}^3 , \tag{A7}$$

where R_0 is the ZAMS radius and α', β', and γ' are given by

$$\alpha' = \begin{cases} 0 , & \text{if } M \leq 1.334; \\ 0.1509 + 0.1709 \log_{10} M , & \text{if } M > 1.334 ; \end{cases} \tag{A8}$$

$$\beta' = \begin{cases} 0.2226 \log_{10} M , & \text{if } M \leq 1.334 ; \\ -0.4805 \log_{10} M , & \text{if } M > 1.334 ; \end{cases} \tag{A9}$$

and

$$\gamma' = \begin{cases} 0.1151 , & \text{if } M \leq 1.334 ; \\ 0.5083 \log_{10} M , & \text{if } M > 1.334 . \end{cases} \tag{A10}$$

Equations (A4)–(A10) are used only while $0 \leq t \leq t_{\rm MS}$. After the end of the main sequence stars grow rapidly to become red giants. For the more massive stars there is a distinct rapid phase of evolution as the stars cross the Hertzsprung gap (HG) whereas for the lower masses there is a gradual change with no distinct HG phase. For those stars that show a maximum luminosity at the end of the main sequence there is a corresponding minimum at the base of the giant branch (BGB). We define the Hertzsprung gap time as the time between these two turning points and find that it can be fitted by

$$\frac{t_{\rm HG}}{t_{\rm MS}} = \frac{0.543}{M^2 - 2.1M + 23.3} \tag{A11}$$

which we extrapolate for the lower masses where there is no distinct HG. We model the rapid evolution across the gap by two very simple formulae. With a new fractional time

$$\tau_{\rm HG} = \frac{t - t_{\rm MS}}{t_{\rm HG}} , \tag{A12}$$

we use

$$L = L_{\rm TMS}\left(\frac{L_{\rm BGB}}{L_{\rm TMS}}\right)^{\tau_{\rm HG}} \tag{A13}$$

and

$$R = R_{\rm TMS}\left(\frac{R_{\rm BGB}}{T_{\rm TMS}}\right)^{\tau_{\rm HG}} , \tag{A14}$$

where TMS refers to the terminal main sequence (obtained by substituting $\tau_{\rm MS} = 1$ into eqs. [A4] and [A7]) and BGB refers to the base of the giant branch. The luminosity at the base of the giant branch can be approximated fairly well by

$$L_{\rm BGB} = \frac{2.15M^2 + 0.22M^5}{5 \times 10^{-6}M^{1.5} + 0.35M^{4.5}} . \tag{A15}$$

Since giants are confined to Hayashi tracks the radius over the whole of the giant evolution can be found very well as a function only of luminosity and mass. We use a Hayashi track radius

$$R_{\rm HT} = (0.25L^{0.4} + 0.8L^{0.67})M^{-0.27} , \tag{A16}$$

and hence we obtain $R_{\rm BGB}$. Formulae (A12)–(A14) are only used if $t_{\rm MS} < t \le t_{\rm MS} + t_{\rm HG}$.

Evolution up the giant branch takes place rapidly and with increasing rapidity as luminosity increases, so we seek a formula for the luminosity that fits especially well at the base but is still representative higher up. We use an approximation based on the fact that the giant luminosity is proportional to the burnt core mass to the sixth power and that the rate of change of core mass is proportional to the luminosity. This leads to

$$L = \frac{L_{\rm BGB}}{\tau_G^{7/6}} , \tag{A17}$$

where

$$\tau_G = \frac{t_G + t_{\rm MS} + t_{\rm HG} - t}{t_G} \tag{A18}$$

and t_G is the total giant lifetime. We take

$$t_G = 0.15t_{\rm MS} , \tag{A19}$$

which gives a reasonable fit for all masses. Real stars never reach the infinite luminosity at $t = t_{\rm MS} + t_{\rm HG} + t_G$ because mass is lost from the envelope through stellar winds. We prevent artificially high luminosities by adopting a maximum luminosity of

$$L_{\rm max} = 4000M + 500M^2 , \tag{A20}$$

which fits in well with our theoretical tracks for lower masses and (because of the very rapid evolution at high luminosity) makes little difference to the actual giant lifetime. In addition for high masses, $M > 25\ M_\odot$, we ignore giant evolution altogether treating any star beyond the main sequence as a Wolf-Rayet star with luminosity $L = 10^5\ L_\odot$ and radius $R = 5\ R_\odot$.

The giant evolution is interrupted when helium ignites in the core and we find that this occurs at a luminosity of about

$$L_{\rm IG} = L_{\rm BGB} + 2000 . \tag{A21}$$

Using this value of L in equation (A17) gives the age, $t_{\rm IG}$, at which the helium ignition occurs relative to the BGB. During core helium burning the luminosity and the radius of the star are temporarily reduced before the star moves on to the asymptotic giant branch when shell helium burning begins. Core helium burning lasts for a time

$$t_{\rm He} = \frac{0.54t_{\rm MS}}{M^2 - 2.1M + 23.3} \tag{A22}$$

and takes place at an almost constant luminosity given by

$$L_{\rm He} = 0.763M^{0.46}L_0 + 50M^{-0.1} . \tag{A23}$$

Stars of higher mass than about 3 $M_\odot$ make an excursion across the H-R diagram in a blue loop. Figure 4 indicates that this evolution is concentrated toward the high-temperature (low-radius) end so we model it as a one way trip with

$$R = R_{\rm HT}\left(\frac{25}{R_{\rm HT}}\right)^{\tau_{\rm He}} , \tag{A24}$$

where

$$\tau_{\rm He} = \frac{t - t_{\rm MS} - t_{\rm HG} - t_{\rm IG}}{t_{\rm He}} \quad \text{(A25)}$$

and $R_{\rm HT}$ is the Hayashi track radius corresponding to $L_{\rm He}$, given by equation (A16). If $R_{\rm HT} < 25\ R_\odot$, the minimum radius, then the star is assumed not to have a blue excursion (as for the 1 and 2 $M_\odot$ in Fig. 4). Thus the giant evolution is split into a red giant phase, before helium ignition, and a supergiant phase, after core helium has been exhausted. Both phases are governed by equations (A17)–(A19), except that for the supergiant phase we replace (A18) by

$$\tau'_G = \frac{t_{\rm He} + t_G + t_{\rm HG} + t_{\rm MS} - t}{t_G} . \quad \text{(A26)}$$

Using $L_{\rm max}$ from equation (A20) in equation (A17) with equation (A26) we find $t_{\rm WD}$, the age at which we assume nuclear burning ceases. For $t > t_{\rm WD}$ we model the star by a cooling white dwarf of mass 1 $M_\odot$ with radius $R = 0.01\ R_\odot$ and luminosity given by

$$L = \frac{40}{(t - t_{\rm WD})^{1.4}} . \quad \text{(A27)}$$

However, we do not expect these to feature often in our catalogs for $V \leq 6.0$.

We can obtain the effective temperature of a star from its luminosity and radius from the simple relation

$$T_e = \left(1130 \frac{L}{R^2}\right)^{1/4} , \quad \text{(A28)}$$

where T_e is measured in kK. To compare our theoretical catalogs with the real RBSC we must convert our luminosities to absolute magnitudes using

$$M_V = -2.5 \log_{10} L + 4.75 - {\rm BC} , \quad \text{(A29)}$$

where BC is the bolometric correction. To obtain a formula for BC we interpolate in Popper's (1980) Table 1. For main-sequence stars we use

$$10^{0.4{\rm BC}} = \begin{cases} \dfrac{6.859 \times 10^{-6} T_e^8 + 9.316 \times 10^{-3}}{1 + 5.975 \times 10^{-10} T_e^{14}} , & \text{if } T_e \leq 4.452 \text{ kK} ; \\ \dfrac{3.407 \times 10^{-2} T_e^2}{1 + 1.043 \times 10^{-4} T_e^{4.5}} , & \text{if } 4.452 \text{ kK} \leq T_e \leq 10.84 \text{ kK} ; \\ \dfrac{2728 T_e^{-3.5} + 1.878 \times 10^{-2} T_e}{1 + 5.362 \times 10^{-5} T_e^{3.5}} , & \text{if } T_e \geq 10.84 \text{ kK} ; \end{cases} \quad \text{(A30)}$$

and for the more evolved stars (mostly red giants)

$$10^{0.4{\rm BC}} = \begin{cases} \dfrac{1.724 \times 10^{-7} T_e^{11} + 1.925 \times 10^{-2}}{1 + 1.884 \times 10^{-9} T_e^{14}} , & \text{if } T_e \leq 4.195 \text{ kK} ; \\ \dfrac{7.56 \times 10^{-2} T_e^{1.5}}{1 + 6.358 \times 10^{-5} T_e^{4.5}} , & \text{if } 4.195 \text{ kK} \leq T_e \leq 10.89 \text{ kK} ; \\ \dfrac{2728 T_e^{-3.5} + 1.878 \times 10^{-2} T_e}{1 + 5.362 \times 10^{-5} T_e^{3.5}} , & \text{if } T_e \geq 10.89 \text{ kK} . \end{cases} \quad \text{(A31)}$$

Figure 5 is essentially the same as Figure 4 except that the detailed theoretical tracks have been replaced by tracks corresponding to our simple formulae. It can be seen that these formulae do indeed give a reasonable representation of the evolution that we wish to include. We have also included in this figure a zero-age main sequence line corresponding to equations (A1) and (A2) and it is pleasing to note that all Popper's stars of higher masses lie to the evolved side of this line. We would expect all the low-mass stars to actually lie on this line and so are not too perturbed by the spread putting it down to observational noise and our use of a fixed metallicity.

In § IV we referred to a transformation that we make from the X, Y unit square (which is a mapping of M, t space such that stars should be distributed over it with uniform probability per unit area) to a U, V unit square in which a much greater fraction of the area is devoted to stars of intrinsically high luminosity, i.e., either massive and young, or of moderate mass and just old enough to be giants or supergiants. The transformation we use is

$$X = 2U - U^2 , \quad \text{(A32)}$$

$$Y = \frac{t_N}{t_{\rm Gal}} f(\kappa V) , \quad \text{(A33)}$$

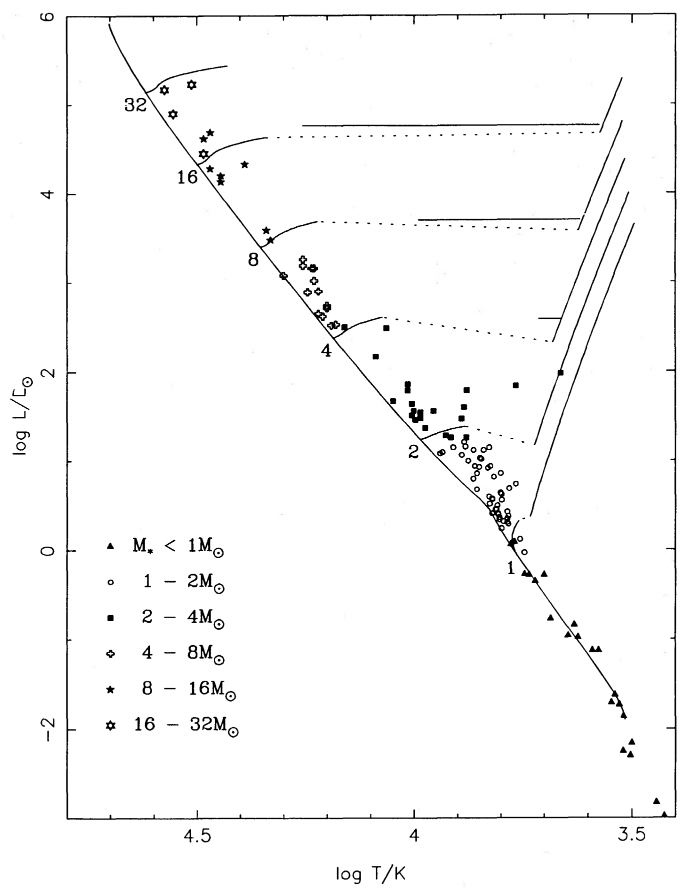

FIG. 5.—H-R diagram similar to Fig. 4, but the theoretical models have been replaced by the approximations discussed in this Appendix

where

$$f(v) = \begin{cases} 2v - v^2 & \text{if } 0 \le v \le 1 \, , \\ \dfrac{3 - 3v + v^2}{2 - v} & \text{if } 1 \le v \le 2 \, , \end{cases} \tag{A34}$$

and κ is determined by

$$1 = \frac{t_N}{t_{\rm Gal}} f(\kappa) \, . \tag{A35}$$

Here $t_{\rm Gal}$ is the assumed Galactic age (12,000 Myr), a constant, and t_N is an indicator of the nuclear lifetime of the star:

$$t_N = t_{\rm MS} + t_G = 1.15 t_{\rm MS}(M) \, . \tag{A36}$$

Since this is a function of M only, which is in turn a function of X only, via equations (A3), (A19), and (A5), and hence of U only via equation (A32), it follows that equation (A33) does indeed determine Y as a function of U, V only. Further, since X is independent of V, the Jacobian required in equation (17) is just

$$\frac{\partial(X, Y)}{\partial(U, V)} = \frac{dX}{dU} \frac{\partial Y}{\partial V} = 2(1 - U) \frac{t_N}{t_{\rm Gal}} \kappa f'(\kappa V) \, . \tag{A37}$$

Note that by our definition of κ, equation (A35), the two functional forms of equation (A34) apply respectively to the two cases $t_N < t_{\rm Gal}$ and $t_N > t_{\rm Gal}$. The (just) monotonic function $f(v)$ was chosen so that

$$f(0) = 0 \, , \quad f(1) = 1 \, , \quad f'(1) = 0 \quad \text{and} \quad f(2) = \infty \, . \tag{A38}$$

It therefore biases Y so that, for any star with $t_N < t_{\rm Gal}$, t is much more likely to be close to t_N than if we simply used the original prescription of equation (8).

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