THE ASTROPHYSICAL JOURNAL, 344:341–349, 1989 September 1

FINITE SOURCE DEPOLARIZATION FACTORS FOR CIRCUMSTELLAR SCATTERING

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Received 1988 November 16; accepted 1989 February 22

ABSTRACT

We consider the extent to which it is possible to generalize the simple depolarization factor $D = \mu_*$ obtained by Cassinelli, Nordsieck, and Murison for the reduction in circumstellar Thomson scattering polarization due to the finite angular radius $\cos^{-1} \mu_*$ of a uniform spherical light source as seen from the scattering point. By formulating the equations for the total and polarized scattered fluxes for an arbitrary illuminating radiation field, and for generalized spherical scatterers, we consider a series of special cases and reach the following conclusions:

1. For general (limb-darkened) spherical sources a simple $D(\mu_*)$ factor can still be used for Rayleigh/Thomson scattering but with the functional form of $D(\mu_*)$ depending on the limb darkening law.
2. The $D(\mu_*)$ factor is applicable, in Rayleigh/Thomson scattering, for *any* spatial distribution of the scatterers.
3. A similar $D(\mu_*)$ factor also applies for a disk light source for Rayleigh scatterers near the disk axis (e.g., in the jets of SS-433).

A correction factor C is also derived for the total intensity of Rayleigh scattered light. This depends only on the observer's direction i, and on μ_*, for any localized scattering volume, but varies with the distribution of the scatterers and of the light source for extended scattering volumes.

Non-Rayleigh scatterers are then considered, and it is shown that in general no C and D factors can be obtained which depend only on μ_* and i. Rather, these cases depend on the scatterer and the light source distributions, and on the form of the coefficients in the Fourier expansion of the scattering functions.

Subject headings: polarization — radiation transfer — stars: circumstellar shells

I. INTRODUCTION

There has been considerable recent interest in utilizing the polarization of light scattered off circumstellar matter as a diagnostic of the geometry of this matter and of the illuminating stars (e.g., Brown and McLean 1977; Brown, McLean, and Emslie 1978; Rudy and Kemp 1978; Daniel 1981; Simmons 1982, 1983; Dolan 1984; Friend and Cassinelli 1986; Drissen *et al.* 1986*b*; Clarke and McGale 1986, 1987). Most of the papers cited utilize a single scattering approximation in their theoretical analysis, the adequacy of which appears to be borne out by comparison with the multiple scattering analyses (e.g., Daniel 1981; Dolan 1984). However, most of these calculations also utilize a point light source treatment which overestimates the degree of polarization (see Brown *et al.* 1978) because it does not allow for the differing polarimetric position angles of light scattered in the same locality but incident from different parts of the light source. Rudy and Kemp (1978) do incorporate a polarimetric diminution factor of 1 or less to allow for this effect but do not evaluate its dependence on scattering location or angle.

Cassinelli, Nordsieck, and Murison (1987, hereafter CNM) have addressed this problem analytically, for the case of a uniformly bright spherical star illuminating an electron scattering envelope with axial symmetry, by starting with the radiative transfer formulation of Chandrasekhar (1960) and going to the optically thin limit above the stellar surface (radius R_*). They find that the polarization for scattering off electrons at distance r from the stellar center has the same direction as in the case of a concentric point light source but that the degree of polarization is reduced by a factor $D = \cos \theta_*(r) = (1 - R_*^2/r^2)^{1/2}$ where θ_* is the angular stellar radius seen from distance r.

In this paper we use single scattering theory directly to derive the $D(r)$ result of CNM and to investigate the extent to which it can be generalized to the following: nonuniform spherical light sources (e.g., limb-darkened stars); nonspherical light sources (e.g., accretion disks); arbitrary spatial distributions of scattering electrons; scattering particles other than electrons. We also obtain expressions for the factor by which the total scattered (as well as polarized) intensity is modified by finite size light sources. This can be of particular interest when the much stronger direct unpolarized starlight is eclipsed.

II. STOKES INTENSITIES, FOR EXTENDED SOURCES OF LIGHT, SCATTERED OFF LOCALIZED SPHERICAL SCATTERERS

We consider first a small volume containing N spherical scattering particles, with scattering functions $i_1(\chi)$, $i_2(\chi)$ in the terminology of van de Hulst (1957) and with χ the scattering angle, located at the origin O of a Cartesian coordinate system $Oxyz$. Axis Oz will eventually be taken to be a symmetry (or other convenient) axis, but for the moment is arbitrary, and plane Oxz is chosen to

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contain the distant observer's direction E at polar angle inclination i to the Oz axis–see Figure 1a. Unpolarized light will be taken to be incident on these scatterers with a specific intensity $I'(\theta', \phi')$ in direction (θ', ϕ'). Then the emergent (scattered) radiation arising from radiation incident in solid angle $d\omega' = \sin\theta'\, d\theta'\, d\phi'$ is characterized by (see, e.g., Simmons 1983)

$$\left.\begin{matrix} dF_S \\[2ex] dF_Q \\[2ex] dF_U \end{matrix}\right\} = \frac{NI'(\theta', \phi')d\omega'}{d^2k^2} \times \left\{\begin{matrix} \dfrac{(i_1 + i_2)}{2} \\[2ex] \dfrac{(i_1 - i_2)}{2}\cos 2\psi\,, \\[2ex] \dfrac{(i_1 - i_2)}{2}\sin 2\psi \end{matrix}\right. \tag{1}$$

where k is the wavenumber, d is the distance to the observer, and ψ is the angle of the polarization direction (maximum $\boldsymbol{E}$ vector) to the direction on the sky chosen for the Stokes Q-axis. Then dF_S, dF_Q, and dF_U are the elementary contributions from $d\omega'$ to, respectively, the scattered flux at the Earth and the unnormalized Stokes parameters along directions Q, U on the sky which we will refer to as "Stokes fluxes." For a finite light source, we must integrate equation (1) over ω'.

For spherical scatterers, the polarization direction $\hat{\boldsymbol{p}}$ can be taken as normal to the scattering plane as shown in Figure 1. [In cases where it lies in the scattering plane, this appears as a change of sign in the scattering function $(i_1 - i_2)$, equivalent to $\pi/2$ rotation in ψ.] Then if we choose to define the Q-axis as the projection of Oz on the plane normal to OE we can express ψ, as well as χ, in terms of θ', ϕ', i from the spherical geometry of Figure 1b. Here these are in fact expressed most usefully in the form

$$\cos\chi = \cos i \cos\theta' + \sin i \sin\theta' \cos\phi'\,, \tag{2a}$$

$$\sin\chi \cos\psi = \sin\theta' \sin\phi'\,, \tag{2b}$$

$$\cos\theta' = \cos\chi \cos i + \sin\chi \sin i \sin\psi\,, \tag{2c}$$

so that, using equation (2a) in equation (2c)

$$\sin\chi \sin\psi = \sin i \cos\theta' - \cos i \sin\theta' \cos\phi'\,. \tag{2d}$$

We then obtain, using equations (2b) and (2d),

$$\sin^2\chi \sin 2\psi = \sin i \sin 2\theta' \sin\phi' - \cos i \sin^2\theta' \sin 2\phi'\,, \tag{3}$$

and, using equations (2a) and (2b),

$$\sin^2\chi \cos 2\psi = \tfrac{1}{2}[\sin^2 i(1 - 3\cos^2\theta') - (1 + \cos^2 i)\sin^2\theta' \cos 2\phi' + 2\sin 2i \sin\theta' \cos\theta' \cos\phi']\,, \tag{4}$$

while by equation (2a)

$$1 + \cos^2\chi = \tfrac{1}{2}[(3 - \cos^2 i) - (1 - 3\cos^2 i)\cos^2\theta' + \sin^2 i \sin^2\theta' \cos 2\phi' + 2\sin 2i \sin\theta' \cos\theta' \cos\phi']\,. \tag{5}$$

We have obtained these forms because for Rayleigh scattering $(i_1 + i_2) \sim 1 + \cos^2\chi$ and $(i_1 - i_2) \sim \sin^2\chi$ (the constant of proportionality being $3\sigma_T k^2/8\pi$) while for general scatterers the first terms in the Fourier expansions of $(i_1 + i_2)$ and $(i_1 - i_2)$ are of

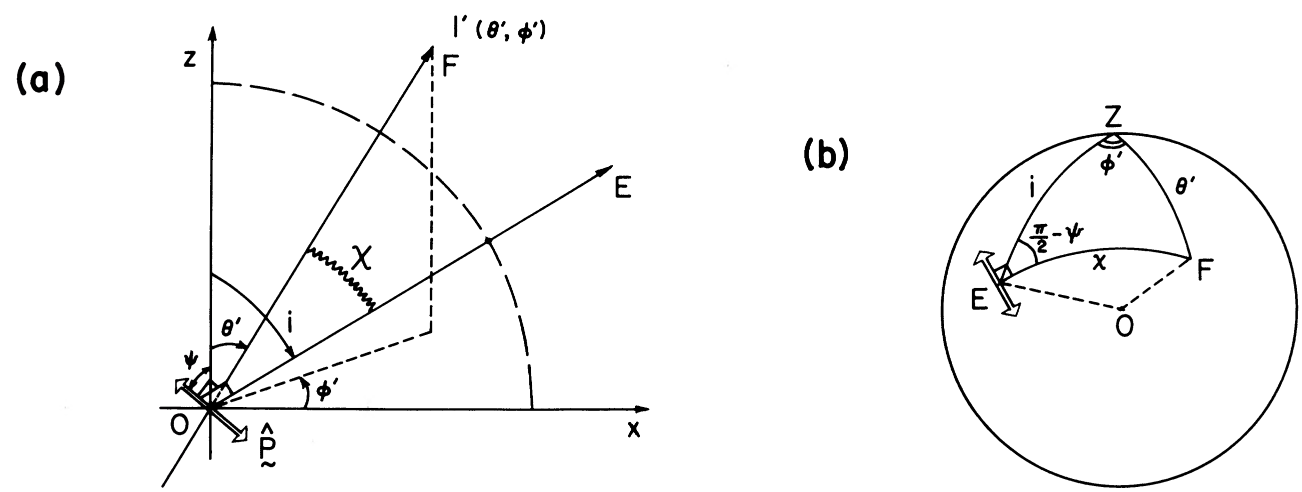

FIG. 1.—Geometry and terminology for scattering of light from an extended source off of a localized scattering region, (a) is centered on a scatterer at O with incident light of intensity I' from direction θ', ϕ'. To clarify the scattering geometry, (b) shows the angular variables of the problem in terms of a spherical triangle ZEF centered on the scattering point O, *directions* in (a) appearing as *points* on a sphere in (b).

the Rayleigh form (Simmons 1982). It is then convenient to redefine the scattering functions in terms of

$$f_k^{+}(\chi) = \frac{i_1 + i_2}{2(1 + \cos^2 \chi)} ; \qquad f_k^{-}(\chi) = \frac{i_1 - i_2}{2 \sin^2 \chi} , \tag{6}$$

so that f_k^{+}, f_k^{-} are independent of χ for Rayleigh scatterers. Then setting $d\omega' = \sin \theta' \, d\theta' \, d\phi'$, $\mu' = \cos \theta'$, and integrating equation (1) over all incident light directions ω' we obtain, using equations (3)–(6), the completely general expressions

$$\left.\begin{matrix} F_S(i, k) \\ F_Q(i, k) \\ F_U(i, k) \end{matrix}\right\} = \frac{N}{d^2 k^2} \int_{-1}^{1} \int_0^{2\pi} I'(\mu', \phi') \, d\phi' \, d\mu'$$

$$\times \begin{cases} \dfrac{f_k^{+}(\chi)}{2} [(3 - \cos^2 i) - (1 - 3 \cos^2 i)\mu'^2 + \sin^2 i(1 - \mu'^2) \cos 2\phi' + 2 \sin 2i\mu'\sqrt{1 - \mu'^2} \cos \phi'] , & (7a) \\ \dfrac{f_k^{-}(\chi)}{2} [\sin^2 i(1 - 3\mu'^2) - (1 + \cos^2 i)(1 - \mu'^2) \cos 2\phi' + 2 \sin 2i\mu'\sqrt{1 - \mu'^2} \cos \phi'] , & (7b) \\ f_k^{-}(\chi)[2 \sin i\mu'(1 - \mu'^2)^{1/2} \sin \phi' - \cos i(1 - \mu'^2) \sin 2\phi'] , & (7c) \end{cases}$$

where χ is given, in terms of θ', ϕ', by equation (2a).

To evaluate the normalized Stokes Parameters we use $Q = F_Q/F_{\rm tot}$; $U = F_U/F_{\rm tot}$, where $F_{\rm tot} = F_S + F_*$ and F_* is the flux at the observer of the total direct light from the primary source. This latter will depend on the brightness distribution of the source as seen from E, which for a general source, will not be expressible in terms of $I'(\theta', \phi')$ at the scattering volume 0. Therefore, to find Q, U for a finite light source, we will have to obtain F_* for each specific source geometry.

We now consider the special form taken by equations (7a)–(7c) for some particular scattering functions and light source geometries.

III. DEPOLARIZATION FACTORS FOR RAYLEIGH AND THOMSON SCATTERING

For these two cases we can write

$$f_k^{+} = f_k^{-} = Ak^6 , \tag{8}$$

where A is independent of both k and χ for the Rayleigh case (depending only on particle size) while for the Thomson case $A = (3/16\pi)\sigma_{\rm T}/k^4$ (k thus canceling entirely in expression [1] for the F's) where $\sigma_{\rm T}$ is the Thomson cross-section. Because all χ-dependence in these cases is therefore already incorporated in the brackets factors in equations (7a)–(7c), we can see that F_S, F_Q, F_U are expressible in terms of integral moments of $I'(\mu', \phi')$ which we now evaluate in some special cases of interest.

a) General Axisymmetric Light Sources

In cases where the small scattering volume lies on the axis of rotational symmetry we take Oz to be collinear with this axis, so that $I'(\mu', \theta') = I'(\mu')$ only (see § IV). Then all the ϕ' dependent terms in expression (7) integrate to zero, resulting in the greatly simplified expressions

$$\left.\begin{matrix} F_S \\ F_Q \\ F_U \end{matrix}\right\} = \frac{2\pi N A k^4}{d^2} \times \begin{cases} (3 - \cos^2 i)J' - (1 - 3 \cos^2 i)K' & (9a) \\ - \sin^2 i(3K' - J') \qquad , & (9b) \\ 0 & (9c) \end{cases}$$

where

$$H' = \tfrac{1}{2} \int_{-1}^{1} \mu' I'(\mu') d\mu'; \qquad J' = \tfrac{1}{2} \int_{-1}^{1} I'(\mu') d\mu'; \qquad K' = \tfrac{1}{2} \int_{-1}^{1} I'(\mu')\mu'^2 \, d\mu' \tag{10}$$

have their usual meanings.

With the appropriate value of A, equations (9b) and (9c) are identical to those obtained by CNM for Thomson scattering when expressed in terms of the moments J', K' of I'. Their results were in fact obtained for spatially extended scatterers with axial symmetry of which our localized scattering "point" is a special case. However, we will show shortly (see § IV) that starting from the point scattering results (9) allows easier demonstration of when and how results can be extended to arbitrary distributions of scatterers.

As noted by CNM, in the case of the light source shrinking to a point, $J' = K' = H'$ from which we can obtain the correction factors C, D needed for a point source treatment to give the proper finite source values, viz., for the polarized and scattered fluxes,

$$\frac{F_Q}{F_Q{}^p} = \frac{F_U}{F_U{}^p} = D; \qquad \frac{F_S}{F_S{}^p} = C , \tag{11}$$

where

$$C = \frac{(3J' - K') + (3K' - J') \cos^2 i}{2(1 + \cos^2 i)H'} \tag{12}$$

and

$$D = \frac{3K' - J'}{2H'} , \tag{13}$$

where J', K', H' are values at the scattering site and superscript p indicates point source values. The actual fluxes F for an *isotropic point* source, at distance r from the scatterers along the axis of symmetry, with the same total luminosity L as the source, can be written directly, or from equations (9) with $J' = K' = H' = L/(4\pi r)^2$ as

$$\left.\begin{matrix} F_S{}^p \\ F_Q{}^p \\ F_U{}^p \end{matrix}\right\} = \frac{NLAk^4}{4\pi d^2 r^2} \begin{cases} (1 + \cos^2 i) \\ -\sin^2 i \\ 0 \end{cases} . \tag{14}$$

Result (13) for the depolarization factor is identical to that obtained by CNM in their equation (18) for the case of a spherical stellar light source. In fact, in the form (13) it applies equally to any light source with axisymmetry through the scattering point—see below.

Result (12) was not derived by CNM, and we show below how it can be incorporated in their expressions. An important difference between the C and D factors is that D is independent of the observer's direction i, while C is not, which will be important in our discussion of special cases and of extended scattering volumes.

b) *Specific Axisymmetric Light Sources*

i) *Uniform Spherical Star of Radius R, Luminosity L*

In this case $I'(r, \mu') = L/4\pi^2 R^2$ for all μ' in $\mu_* < \mu' < 1$, where $\mu_* = \cos \theta_* = (1 - R^2/r^2)^{1/2}$ and θ_* is the stellar angular radius seen from the scattering point at distance r from its center. Then evaluation of J', K', H' and substitution in equations (12) and (13) gives, by comparison with a point source at the *center* of the sphere

$$C = \frac{8 - \mu_*(1 + \mu_*)(1 - 3 \cos^2 i)}{3(1 + \mu_*)(1 + \cos^2 i)} \tag{15}$$

and

$$D = \mu_* = (1 - R^2/r^2)^{1/2} , \tag{16}$$

the latter being precisely the CNM result.

The Stokes fluxes are, therefore, by equations (11)–(14)

$$\left.\begin{matrix} F_S \\ F_Q \\ F_U \end{matrix}\right\} = \frac{NLAk^4}{4\pi r^2 d^2} \times \begin{cases} C(\mu_*, i)(1 + \cos^2 i) \\ -D(\mu_*) \sin^2 i \\ 0 \end{cases} . \tag{17}$$

We are also interested in the normalized Stokes parameters $Q = F_Q/F_*(1 + \delta)$ and $U = F_U/F_*(1 + \delta)$ where $\delta = F_S/F_*$ and F_* is the flux of direct starlight at the Earth, namely $F_* = L/4\pi d^2$ in this case, so that $F_*(1 + \delta)$ is the total light flux at the Earth (direct plus scattered). Thus

$$\delta(r, i) = \frac{NAk^4}{r^2} C(\mu_*, i)(1 + \cos^2 i) , \tag{18}$$

$$Q(r, i) = -\frac{NAk^4}{r^2} \frac{D(\mu_*)}{1 + \delta} \sin^2 i ; \qquad U \equiv 0 . \tag{19}$$

Since there is no U component of polarization, the degree of polarization $P = -Q$.

The appearance of an i dependence in δ and hence in Q means that the finite source polarization correction cannot strictly be treated as a function of r only when dealing with Q, as supposed by CNM, although it can for F_Q. In most cases δ is small, i.e., the degree of scattering polarization is small, and δ can to a first approximation be neglected in equation (19). In cases of substantial polarization, however, the factor $1 + \delta$ in equation (19) can become significant, especially as the factor C can be substantial as seen in Figure 2 which shows C to vary with μ_* and i up to almost 3.

The physical reasons for the i and μ_* dependence of C can be understood in terms of two factors. The dominant one is that as R_* increases (μ_* decreases) the surface of the star approaches the scatterer and the inverse square law tends to increase the intensity of scattered light toward the value for scattering above a plane parallel atmosphere ($\mu_* \to 0$). The second factor, and the one which

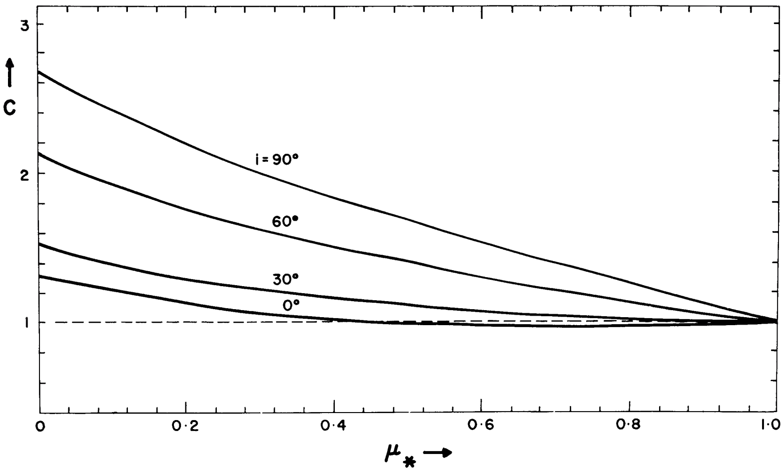

FIG. 2.—Correction factor $C(\mu_*, i)$ for the modification of scattered light intensity off of a localized scattering region due to finite angular radius $\cos^{-1} \mu_*$ of a uniform spherical light source, for observational direction i.

introduces an i dependence into C, is that increase of R_* introduces a spread in χ values about the value $\chi = i$ relevant to a point source. In particular, for $i = 90°$ the $1 + \cos^2 \bar{\chi}$ intensity factor takes the value 1 for a point source ($\bar{\chi} = \chi = i = 90°$) but a value >1 for a nonpoint source since $\chi < 90°$ scatterings are introduced. This adds to the inverse square law effect. On the other hand, at $i = 0°$, the $1 + \cos^2 \bar{\chi}$ factor is 2 for a point source ($\bar{\chi} = \chi = i = 0°$) but <2 for a non-point source. This acts in the opposite direction to the inverse square factor, overcoming it for μ_* not too small, and producing a shallow minimum in $C(\mu_*, i)$ with respect to μ_*, for small i.

ii) *Spherical Star with Limb Darkening*

For a star of radius R with limb-darkening law $I'(\mu')$ we see from equation (12) that the correction factor for F_S is

$$C = \frac{(3 - \cos^2 i) \int_{\mu_*}^1 I'(\mu')d\mu' - (1 - 3\cos^2 i) \int_{\mu_*}^1 I'(\mu')\mu'^2 \, d\mu'}{(1 + \cos^2 i) \int_{\mu_*}^1 I'(\mu')\mu' \, d\mu'} , \tag{20}$$

while the depolarization factor is

$$D = \frac{\int_{\mu_*}^1 I'(\mu')(3\mu'^2 - 1)d\mu'}{2 \int_{\mu_*}^1 \mu' I(\mu')d\mu'} . \tag{21}$$

It is evident from equation (21) that the depolarization factor for the Stokes flux F_Q in this case is again a function only of μ_*—i.e., of R/r—but a more complicated function than in the uniform star case (i) alone. For example, the simplest case of a limb-darkening law $I'(\mu') \sim 1 + \beta\mu'$ (limb brightening for $\beta < 0$) gives, by equation (21)

$$D(\mu_*, \beta) = \mu_*[1 + \beta(1 + 3\mu_*^2)/(4\mu_*)]/\{1 + 2\beta(1 + \mu_* + \mu_*^2)/[3(1 + \mu_*)]\} , \tag{22}$$

which goes to μ_* (see eq. [16]) as $\beta \rightarrow 0$ and goes to 1 as $\mu_* \rightarrow 1$ as expected. Results for $D(\mu_*, \beta)$ are shown in Figures 3*a* and 3*b*. For $\beta \geq 0$ (Fig. 3*a*) these results show that limb darkening reduces the effective angular radius of a star and so increases D relative to a uniform star of the same geometric radius. For $-1 \leq \beta \leq 0$ (Fig. 3*b*) limb brightening increases the effective stellar radius and, for small radii, decreases D compared to a uniform star (i.e., the depolarization is not so great).

However, for stars of large enough geometric radius (small enough μ_*) limb brightening results in *negative depolarization factors* D. This is to be interpreted as meaning that scattering material close to the star mainly scatters radiation coming from the bright stellar limb, this light having a polarization plane orthogonal to that from a point source. In this (admittedly unusual) case, an equatorial scattering disk would result in an equatorial rather than polar polarization plane! Limb-brightening laws steeper than this linear one (limited to $\beta \geq -1$) can given even more negative D.

For this same $I'(\mu')$ law we find also

$$C = \frac{8 - \mu_*(1 + \mu_*)(1 - 3\cos^2 i) + [3\beta(1 + \mu_*)/4][(1 + 3\mu_*^2)\cos^2 i + 5 - \mu_*^2]}{3(1 + \mu_*)(1 + \cos^2 i)\{1 + (2\beta/3)[(1 + \mu_* + \mu_*^2)/(1 + \mu_*)]\}} , \tag{23}$$

which again reduces to equation (15) when $\beta \rightarrow 0$ and to the point case as $\mu_* \rightarrow 1$.

In this case, the point source expressions are the same as in case (i) since by spherical symmetry $dL/d\omega' = L/4\pi r^2$ in all directions for any limb-darkened sphere. Thus equations (17)–(19) with the values of C, D given in equation (22) and (23) describe F_S, F_Q, F_U, and δ, Q.

iii) *Accretion Disks*

In some close binaries an accretion disk is an important source of light. For such a source $I'(\mu')$ is not axisymmetric at general scattering points but is axisymmetric for scattering sites close to the disk axis. Precisely this combination of circumstances is

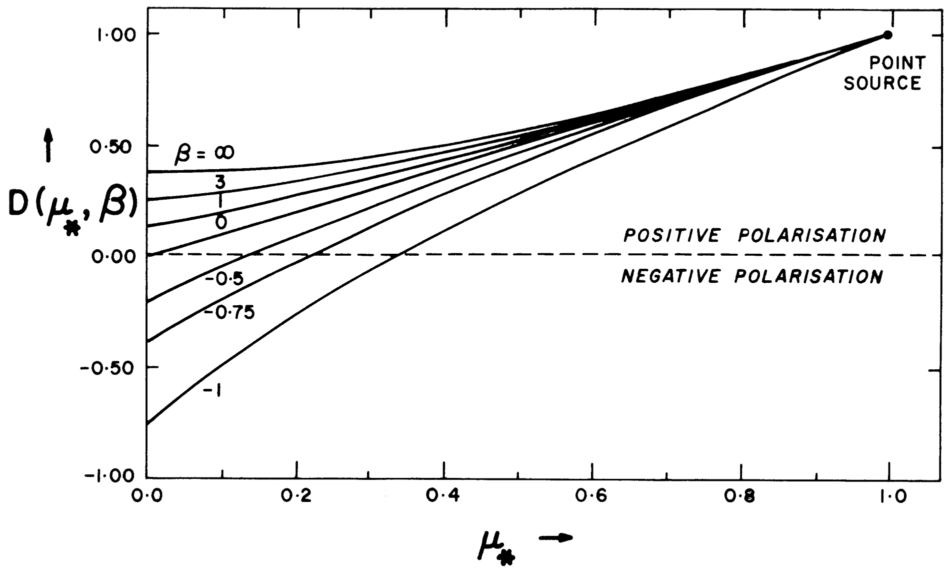

FIG. 3.—Modification of the depolarization factor for a spherical light source of angular radius μ_* but with a limb-darkening law $I'(\mu') \sim 1 + \beta\mu'$

relevant to the scattering of light from the accretion disk of SS 433 off the narrow coaxial jet material, which contributes to the interpretation of the variable polarization of SS 433 (McLean and Tapia 1980; Carlaw and Brown 1988). Light from the primary star also contributes to the scattering polarization in SS 433 but this can be treated as case (i) or (ii) above (Carlaw 1988; Carlaw and Brown 1988).

For a flat disk, the radiation field is not isotropic, the radial flux being maximal along the disk axis and zero in the plane of a thin disk. Thus the specific intensity I_d' from a uniformly bright thin disk along its axis is related to its luminosity L by (with superscript or subscript d denoting disk)

$$L^d = 2\pi R_d^2 \int_\Omega I_d' d\Omega = 4\pi^2 R_d^2 \int_0^1 I_d' \mu \, d\mu = 2\pi^2 R_d^2 I_d' ,$$

so that

$$I_d' = L/(2\pi^2 R_d^2) \tag{24}$$

where R_d is the linear disk radius, as compared to $I' = L/(4\pi^2 R^2)$ for a spherical source. Thus, for a scattering volume on the axis of symmetry, expressions (11)–(13) will give the fluxes from a finite disk compared to those of an arbitrarily small disk, but if we want correction factors for a finite disk compared to an *isotropic* point source at its center *and of the same total luminosity*, factors (12) and (13) have to be multiplied by 2. Thus if we denote by superscript *ip* values for an isotropic point source and define

$$\frac{F_{Q,U}{}^d}{F_{Q,U}{}^{ip}} = D_d \quad \text{and} \quad \frac{F_S{}^d}{F_S{}^{ip}} = C_d , \tag{25}$$

then along the axis of a uniform disk, analogously to equations (15) and (16)

$$D_d = 2\mu_d \tag{26}$$

and

$$C_d = \frac{2[8 - \mu_d(1 + \mu_d)(1 - 3\cos^2 i)]}{3(1 + \mu_d)(1 + \cos^2 i)} \tag{27}$$

where

$$\mu_d = (1 + R_d^2/r^2)^{-1/2} = \cos\theta_d \tag{28}$$

with θ_d the angular size of the disk seen from distance r on the axis.

In the more realistic case of a nonuniform disk, hottest and brightest at its center, expressions (20) and (21) for a limb-darkened sphere can be similarly generalized along the above lines.

The anisotropy of disk radiation results in a further modification when we consider the normalized quantities Q, δ since the disk flux at the Earth will now be $F_d = L \cos i/(2\pi d^2)$ instead of $L/4\pi d^2$ for a spherical source of the same L. Consequently equation (18) and (19) are replaced for points along the disk axis by

$$\delta_d = \frac{NAk^4}{2r^2} C_d \frac{(1 + \cos^2 i)}{\cos i} \tag{29}$$

and

$$Q_d = -\frac{NAk^4}{2r^2}\frac{D_d}{1+\delta_d}\frac{\sin^2 i}{\cos i}. \tag{30}$$

IV. SPATIALLY EXTENDED RAYLEIGH SCATTERING REGIONS

We now consider the case of an extended light source centered on an origin O the light from which is scattered off an extended spatial distribution of scatterers having number density $n(r, \theta, \phi)$ at the point with spherical polar coordinates (r, θ, ϕ) centered on O and with ϕ measured about the axis of symmetry of the light source from the plane containing the observer whose direction is at angle i_0 to Oz (Fig. 4). The equations of the previous sections then apply to an elementary scattering volume when we set $N = n\,dV$ and measure Q in the plane OSE and perpendicular to OE, the direction to the observer (Fig. 3a). When we come to integrate over all V, three complicating factors arise.

1. Some of the scattering matter will be occulted by the finite light source and the corresponding ΔV should be omitted from the integral (see Milgrom 1978). Following CNM, we will not consider this further here as the material concerned is that which backscatters light and so contributes little to the polarization (and indeed little to the scattered light in the case of dust scattering (but see Brown and Fox 1989).

2. In cases where the light source does not look axisymmetric from dV, the simplifications obtained between equations (7a)–(7c) and equations (9a)–(9b), even in the Rayleigh case, will not occur; i.e., D cannot then be written as a function of μ_* only and C as a function of μ_* and i only. For general spatially extended scattering volumes, the light source can only look symmetric from all dV if it is spherically symmetric, although possibly limb darkened, or if the scattering volume extends only along an axis of symmetry such as the disk/jet case discussed in § IIIb(iii). In general cases, therefore, there is no alternative but to resort to equations (7a)–(7c) and carry out simultaneously the integrations over ω' and over V. Here we will restrict ourselves to those cases where the light source symmetry condition does hold, which will cover most practical cases.

3. The direction (Q) of the scattered polarization will vary with the direction $\hat{r}$ of the scattering element dV at P, being normal (or parallel) to the line OP projected on the sky, as also will the value of i. To obtain the Stokes fluxes from the entire volume V we must therefore integrate the components of the local contributions ΔQ along a common polarimetric reference (Q_0, U_0), which we will choose to have Q_0 in the plane OEZ, and with the local i value. The contributions to F_S, F_{Q_o}, F_{U_o} from volume dV are, by equation (17)

$$\left.\begin{matrix} dF_S \\ dF_{Q_o} \\ dF_{U_o} \end{matrix}\right\} = \frac{LAk^4}{4\pi d^2}\frac{n(r)dV}{r^2}\left\{\begin{matrix} C(r, i)(1+\cos^2 i) \\ D(r)\sin^2 i \cos 2\Omega \\ D(r)\sin^2 i \sin 2\Omega \end{matrix}\right. \tag{31}$$

where Ω is the rotation of the Q-axis with respect to the Q_0 axis. Here i and Ω depend on i_0, θ, ϕ in a way determined by the geometry of Figures 3a and 3b. In fact the geometry involved, and the identities we need, are essentially identical to those of § II, apart from the factors C, D, because we are now integrating over emergent ray paths in precisely the same way as we did before over incident ray paths. Thus we utilize the identities (3)–(5) with i_o replacing i, i replacing χ, Ω replacing ψ, and θ, ϕ replacing θ', ϕ'. That is,

$$D(r)\sin^2 i \cos 2\Omega = \frac{D(r)}{2}[\sin^2 i_o(1-3\mu^2) - (1+\cos^2 i_o)(1-\mu^2)\cos 2\phi + 2\sin 2i_o(1-\mu^2)^{1/2}\mu\cos\phi], \tag{32}$$

$$D(r)\sin^2 i \sin 2\Omega = D(r)[2\sin i_o\,\mu(1-\mu^2)^{1/2}\sin\phi - \cos i_o(1-\mu^2)\sin 2\phi], \tag{33}$$

$$C(r, i)(1+\cos^2 i) = C_o[(3-\cos^2 i_o) - (1-3\cos^2 i_o)\mu^2 + \sin^2 i_o(1-\mu^2)\cos 2\phi + 2\sin 2i_o(1-\mu^2)^{1/2}\mu\cos\phi], \tag{34}$$

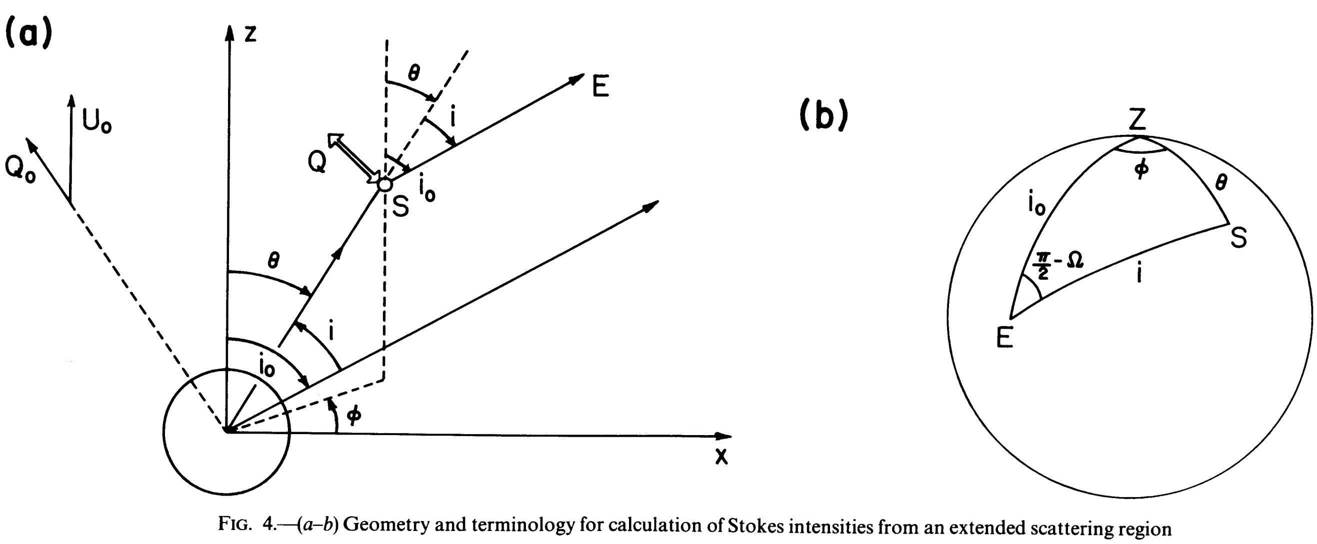

FIG. 4.—(a–b) Geometry and terminology for calculation of Stokes intensities from an extended scattering region

where $D(r)$ is given by equation (13) and C_o by the new equation

$$C_o(i_o, r, \theta, \phi) = \frac{3J - K}{2H} + \frac{(3K - J)}{2H}\left[\cos^2 i_o + \frac{1}{2}(1 - 3\cos^2 i_o)(1 - \mu^2) + \frac{1}{2}\sin^2 i_o(1 - \mu^2)\cos 2\phi + \sin 2i_o(1 - \mu^2)^{1/2}\mu\cos\phi\right], \tag{35}$$

and on integration over V with $dV = r^2\, dr\, d\mu\, d\phi$ we get

$$\begin{Bmatrix} F_S \\ F_Q \\ F_U \end{Bmatrix} = \frac{LAk^4}{4\pi d^2}\int_{-1}^{1}\int_0^{2\pi}\int_{r_{\min}}^{\infty} n(r, \theta, \phi)d\mu\, d\phi\, dr \times \begin{Bmatrix} C_o f_S(i_o, \mu, \phi) \\ Df_Q(i_o, \mu, \phi) \\ Df_U(i_o, \mu, \phi) \end{Bmatrix}, \tag{36}$$

where the functions f_S, f_Q, f_U are given in brackets in equation (32), (33), and (34).

Functions f_Q, f_U are identical to those for an isotropic point light source at the center (with the appropriate factor in the disk case). Therefore, provided we are dealing with a situation where the source looks axisymmetric from all scattering points (i.e., a spherical source or a disk source with scattering along its axis) then D is solely a function of r. Because the same factor D appears in the equations F_Q and F_U, we can then calculate these quantities for *any* spatial distribution of scatterers using the point light source equations if we replace the real $n(r, \theta, \phi)$ by a weighted function $n_{\text{eff}} = D(r)n(r, \theta, \phi)$. The second and third of equations (36) may alternatively be expressed in terms of an effective optical depth function (see Simmons 1983)

$$\tau_{\text{eff}}(\theta, \phi) = \int_r D(r)n(r, \theta, \phi)dr\,. \tag{37}$$

All the usual equations for stars with axisymmetric Rayleigh scattering envelopes (Brown and McLean 1977) and for rotating envelopes in binary systems (Brown *et al.* 1978) then carry over to the extended light source case.

Unfortunately, this simple representation does not apply also to the total scattered flux F_S because C_o depends on i_o, θ and ϕ as well as on r. In cases where F_S is important and where the light source is comparable in size to that of the scattering region, the rather complicated function (35) will have to be incorporated in the integrations (36); i.e., when the light source is large, the scattered flux F_S will involve higher integral moments of the function $n(r, \theta, \phi)$ than do the polarized fluxes F_Q, F_U.

V. NON-RAYLEIGH SCATTERERS

We have seen that for Rayleigh scatterers the depolarization factor $D(r)$ derived by CNM for uniform spherical sources illuminating axisymmetric scatterer distributions can in fact be generalized to arbitrary distributions of scatterers and to more general light sources, including disks when the scatterers are along its axis. The finite source correction C to the scattered light flux F_S, however, does not have such a simple form and depends also on the direction of observation and in a more complex way on the source intensity distribution. Fortunately, F_S is of less interest than the polarized flux in most cases, and the simple $D(r)$ factor will prove to be a very useful result.

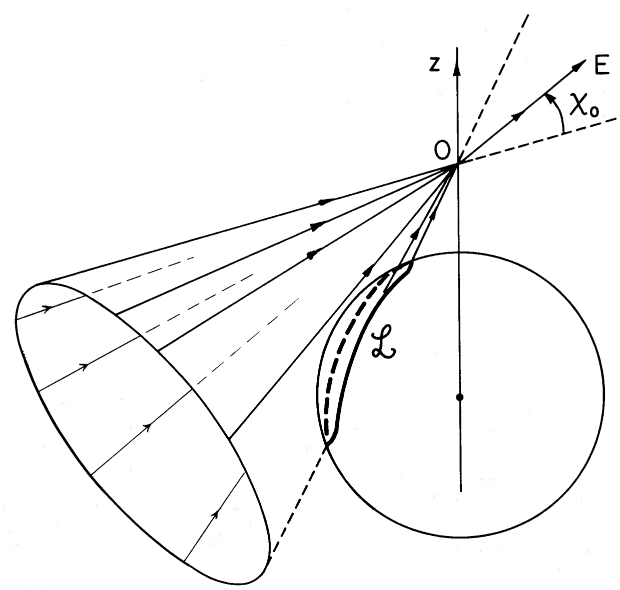

FIG. 5.—Illustrative example of non-Rayleigh scattering. Here the scattering function $f_k^{-}(\chi)$ is assumed to be a delta-function in χ so that all contributions to the polarized intensity must originate on the surface of the cone of half-angle χ, hence on the locus $\mathscr{L}$ of intersection of this cone with the stellar surface.

Finally, it is of interest to see whether such a factor can be obtained for non-Rayleigh scattering. To do so we revert to the simplest possible case of a uniform spherical source on the grounds that if this is not amenable to obtaining an analog to $D(r)$, more complex sources will not either.

We therefore return to equations (7a)–(7c) with $I'(\mu', \phi') = I'(\mu')$ only. For non-Rayleigh scatterers f_k^- is no longer independent of χ so that it depends on (eq. [2a]) $\chi = \cos^{-1}(\cos i \cos \theta' + \sin i \sin \theta' \cos \phi')$. Inspection of equation (7) then shows that the ϕ' dependent integrand terms will not in general integrate to zero because in the Fourier expansion of $f_k^-(\chi)$ there will occur terms of the form $\cos j\phi'$ and $\sin j\phi'$ which, when multiplied by $\sin \phi'$ or $\cos \phi'$ in equation (7) will give terms like $\cos \{[(j+1)/2]\, \phi'\}$ which will not all integrate to zero. Physically this just means that $f_k^-(\chi) \neq f_k(\chi + \pi)$, in general, since non-Rayleigh scattering functions are commonly strongly peaked for forward scattering. This means that expressions (7a)–(7c) will involve higher moments of I' than in the Rayleigh case, including moments over ϕ' as well as μ', which when divided by the results for a point source will lead to finite source correction factors depending not only on the stellar size but also in detail on the form of the Fourier expansion of f_k^-. Perhaps this is most easily seen by considering an extreme geometrical case of strong forward scattering with $f_k^-(\chi) \sim \delta(\chi - \chi_o)$, where δ is the delta function. Then light scattered at O in Figure 1*a* can only reach the observer if it originates on the surface of the cone of half-angle χ_o shown in Figure 5. From this figure it is clear that only light from the locus $\mathscr{L}$ on the stellar surface is relevant to determining F_Q, F_U at the observer, and the rest of the star can be removed without changing the observed fluxes. Clearly then the spherical star can never be replaced by any equivalent point source at its center for any scale factor analogous to $D(r)$ in the Rayleigh case.

It appears, therefore, that the simple D factor treatment of Rayleigh scattering of light from finite sources is fortuitous, arising from the simple periodic ϕ' dependence of expressions (3) and (4) in the Rayleigh problem.

We gratefully acknowledge the support of a Glasgow University Scholarship (V. A. C.), of the Nuffield Foundation, a University of Wisconsin Brittingham Visiting Professorship, and NASA Grant NAGW-422 (J. C. B.).

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