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SPONTANEOUS SUPERFLUID UNPINNING AND THE INHOMOGENEOUS DISTRIBUTION OF VORTEX LINES IN NEUTRON STARS

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ABSTRACT

We study the equation of motion of the pinned superfluid which couples to the crust of neutron stars via thermal vortex creep. An analytical solution can be obtained even when $\omega_{cr}(r)$, the maximum value of the lag $\Omega_s - \Omega_c$ that can be supported by pinning forces, has large spatial variations. Substantial inhomogeneous distributions of vortex lines will result from this kind of spatial variation; vortex lines can be either accumulated or depleted from these regions, depending upon the sign of $d\omega_{cr}/dr$. At a boundary between a vortex accumulating region with $d\omega_{cr}/dr > 2\Omega_c/r$, and a neighboring depletion region, a single free vortex line can initiate the unpinning of a whole layer of pinned vortex lines. In vortex depletion regions with $d\omega_{cr}/dr < -2\Omega_c/r$(vortex-free regions), the Magnus force is larger than the pinning force, so that vortices cannot be pinned. The conditions for the occurrence of these two situations are discussed. We suggest that such structural inhomogeneities in the crust of neutron stars may be responsible for frequent microglitches which lead to pulsar timing noise. Model calculations are compared with observations. A generalization of the model shows promise for explaining the origin of the giant glitches in pulsars.

Subject headings: dense matter — pulsars — stars: neutron

I. INTRODUCTION

Neutron stars have (1) a solid outer crust of increasing mass density (7×10^6 g cm$^{-3} \lesssim \rho \leq 4 \times 10^{11}$ g cm^{-2}), which contains a solid array of fully ionized nuclei and highly degenerate relativistic electron plasma (Ruderman 1969); (2) an inner crust containing a lattice of increasingly neutron-rich nuclei and relativistic electrons at density 4.3×10^{11} g cm$^{-3} \lesssim \rho \lesssim 2.4 \times 10^{14}$ g cm^{-3}, which coexist with a highly degenerate neutron superfluid liquid likely in a $^1S_0$ pairing state, and (3) a quantum liquid interior, where the neutron-rich crustal nuclei have dissolved into free neutrons and protons which form a $^3P_2$ paired superfluid and a $^1S_0$ paired superconducting fluid, respectively. If the basic hadron equation of state is comparatively stiff (Pines 1980), the crust may be a kilometer thick.

The irregularities in the rotation period of pulsars can be understood as a dynamical consequence of the existence of superfluid in the interior of neutron stars. In particular, the observed relaxation of the rotation rate $\Omega_c(t)$ and the spin-down rate $\dot{\Omega}_c(t)$ after sudden jumps (glitches) in these quantities detected in the Crab and Vela pulsars can be successfully explained by the vortex creep theory, in which the crust and the crustal neutron superfluid (Alpar *et al.* 1984*a*, *b*; Alpar, Nandkumar, and Pines 1985) are assumed to be coupled primarily via the thermal creep of superfluid vortex lines which are pinned to the crustal nuclei and thus corotate with the neutron star with angular velocity Ω_c.

Baym, Pethick, and Pines (1969) showed that the electrons and protons must corotate with the crust by electromagnetic coupling, irrespective of whether or not the protons are superfluid, since any appreciable differential rotation will give rise to an inordinately large magnetic field. Easson (1979) has carried out a model calculation which shows that the core plasma responds to sudden changes in the crustal angular velocity with a characteristic spin-up time of the order of seconds. Recently, Alpar, Langer, and Sauls (1984) have examined the coupling of the core neutron superfluid to the crust which results from the interaction between the neutron and proton condensates in the quantum liquid interior. They show that superfluid drag induces a proton charge current around each neutron vortex, which in turn gives rise to a huge magnetic field. The scattering of charges off such magnetic flux lines equilibrates the core superfluid to the plasma and crust on a time scale of the order of minutes. Since such times are short compared with characteristic postglitch relaxation times, the liquid core of the neutron star can be considered to rotate rigidly with the crust in responding to sudden jumps of rotation rate, whether the liquid core is superfluid or not. However, the distinction between superfluid and normal fluid in the core of the neutron star becomes possible if the perturbation time scale of an external torque applied to the crust is less than or comparable to the coupling time between superfluid core and the crust (Cheng 1987*a*).

Pulsar glitches have been attributed to a wide variety of causes, e.g., crust quakes, core quakes and the catastrophic unpinning of vortices in the crustal neutron superfluid. Ruderman (1969), before the first glitch observation, had already pointed out that since the neutron star is covered by a solid crust of nuclei in a crystalline lattice when the star is spinning comparatively fast in an early epoch, it will be relatively oblate. Because of the rigidity, the solid crust is more oblate than it would be in equilibrium at current rotational velocity; therefore, spin-down can lead to events in which the crustal matter moves inward to relieve the stresses resulting from the excessive oblateness. Such a "starquake" mechanism was explored in some detail by Baym and Pines (1971). However, this model cannot explain the interval between glitches in the Vela pulsar.

Anderson and Itoh (1975) suggested, by analogy to the phenomenon of flux pinning in type II superconductors, that the entire

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dynamics of the glitch process was related to the neutron superfluid vortex pinning and unpinning in the crustal lattice of a neutron star. Following this idea, Ruderman (1976) argued that such pinning by the Vela crust lattice would indeed strain it enough to break the lattice periodically and that the crust spin-up came from the sudden sharing of angular momentum between the neutron superfluid and the more slowly rotating parts of the neutron star. However, Anderson *et al.* (1982) concluded that such a crust-breaking mechanism was not dynamically plausible because the global pinning forces are not contained by elasticity but by the much stronger gravitational force.

Anderson *et al.* (1982) thus conclude that catastrophic unpinning events, in which a large number of vortex lines suddenly unpin from the lattice, represent a reasonable model for glitches. Alpar *et al.* (1984*a*, *b*) have worked out a general theory of postglitch behavior, in which the superfluid follows the crust's spin-down by means of the thermal creep of vortex lines against the pinning energy barriers. This theory provides a quantitative explanation of the postglitch behavior observed in the Vela and Crab pulsars and in PSR 0525+21, and makes it possible to infer their internal temperature. In the present paper, we examine a possible mechanism for the initiation of glitches: spontaneous unpinning at locations characterized by a very inhomogeneous distribution of vortex lines.

II. VORTEX CREEP THEORY IN A REGION WITH LARGE SPATIAL VARIATION

The distribution of vortex lines in a rotating superfluid is related to the rotation rate:

$$\kappa n(r, t) = 2\Omega_s(r, t) + r\,\frac{\partial \Omega_s(r, t)}{\partial r}\,. \tag{1}$$

Here κ [$= h/2m_n$] is the vorticity quantum carried by each vortex line, and r is the distance to the rotation axis. Ω_s will change with time according to the following equation:

$$\frac{\partial \Omega_s(r, t)}{\partial t} = -\left[2\Omega_s(r, t) + r\,\frac{\partial \Omega_s(r, t)}{\partial r}\right]\frac{v_r(r, t)}{r} \tag{2}$$

if vortex lines move radially with velocity v_r. Equation (2) is a direct consequence of conservation of vorticity and equation (1). In the neutron star crust, the neutron superfluid coexists with a lattice, and vortex lines pin to the nuclei forming the lattice (Anderson *et al.* 1982; Alpar *et al.* 1984*b*). As the pulsar spins down, a relative angular velocity ω ($\equiv \Omega_s - \Omega_c$) builds up between the superfluid and pinned vortex lines which move with the star's crust. This relative velocity is sustained by pinning forces on the vortex line, through the Magnus equation of motion:

$$\boldsymbol{f} = \rho \boldsymbol{\kappa} \times (\boldsymbol{v}_s - \boldsymbol{v}_L)\,. \tag{3}$$

The largest value of the lag, $\Omega_s - \Omega_c$, is given by the maximum pinning force f_p;

$$\omega_{\rm cr}(r) \equiv (\Omega_s - \Omega_c)_{\rm max} = \frac{f_p(\rho(r))}{\rho \kappa r} = \frac{E_p(\rho(r))}{\rho \kappa r \xi b}\,, \tag{4}$$

where ρ is the nucleon density, E_p is the pinning energy, ξ is the coherence length of the neutron superfluid, and b is the spacing between successive pinning centers. Vortex creep is quantum tunneling between adjacent geometric possibilities for pinning. The mean radial velocity v_r of vortex creep in a medium characterized by a temperature T and pinning energy barriers of size E_p can be estimated by means of a simple statistical argument (Alpar *et al.* 1984*a*) which gives

$$v_r = v_0 \exp\left[-\frac{E_p}{kT}\left(\frac{\omega_{\rm cr} - \omega}{\omega_{\rm cr}}\right)\right]. \tag{5}$$

T is the internal temperature of the neutron star, and v_0 is about 10^7 cm s^{-1}. Combining equations (2) and (5), we obtain the equation of motion for the pinned neutron superfluid coupled to the normal component. At values of the lag $\omega \gtrsim \omega_{\rm cr}$, a condition under which vortex lines cannot pin, the expression for v_r becomes

$$v_r = v_0 \exp\left[-\frac{E_p}{kT}\left(\frac{\omega_{\rm cr} - \omega}{\omega_{\rm cr}}\right)\right] \Big/ \left\{\exp\left[-\frac{E_p}{kT}\left(\frac{\omega_{\rm cr} - \omega}{\omega_{\rm cr}}\right)\right] + 1\right\}. \tag{6}$$

The equation of motion of the normal component of the star is

$$I_c \dot{\Omega}_c = N_{\rm ext} - \int dI_p \dot{\Omega}_s(r, t)\,. \tag{7}$$

Here $N_{\rm ext}$ is the external torque, which we take to be constant because the time scale of the pulsar torque is much longer than the other time scales in the problem. The second term is the internal torque from the pinned superfluid. We can rewrite equations (1), (2), and (6) in terms of ω and Ω_c:

$$n(r, t) = \frac{1}{\kappa}\left[2\Omega_c(t) + 2\omega(r, t) + r\,\frac{\partial \omega(r, t)}{\partial r}\right], \tag{8}$$

$$\frac{\partial \omega}{\partial t} = -\frac{1}{r}\left(2\Omega_c + 2\omega + r\,\frac{\partial \omega}{\partial r}\right) v_0 \exp\left[-\frac{E_p}{kT}\left(\frac{\omega_{\rm cr} - \omega}{\omega_{\rm cr}}\right)\right] \Big/ \left\{1 + \exp\left[-\frac{E_p}{kT}\left(\frac{\omega_{\rm cr} - \omega}{\omega_{\rm cr}}\right)\right]\right\} - \dot{\Omega}_c(t)\,, \tag{9}$$

and

$$\dot{\Omega}_c(t) = \dot{\Omega}_\infty - \frac{1}{I}\int dI_p \frac{\partial\omega(r, t)}{\partial t}, \tag{10}$$

where

$$\dot{\Omega}_\infty = N_{\text{ext}}/I \quad \text{and} \quad I = I_c + I_p .$$

a) Accumulation and Depletion Regions of Vortex Lines

The vortex creep process has a steady state characterized by a specific value of the lag $\Omega_s - \Omega_c$. In steady state, the superfluid and the crust decelerate at the same rate $\dot{\Omega}_\infty$ [in other words, $(\partial\omega_\infty/\partial t) = 0$], so that they share the effect of the external torque in proportion to their respective moments of inertia. The subscript ∞ indicates the steady state value of these quantities. Previous applications (Alpar *et al.* 1984*a*, *b*) to postglitch relaxation indicate that vortex creep is in the regime $\omega < \omega_{\text{cr}}$, where equation (5) holds. As we show in Appendix A, one does not need to work with the more general expression, equation (6), since using the latter gives rise only to minute changes in the steady state solution ω_∞ in most regions of interest. From equations (1), (2), (5), and (8) we obtain

$$\frac{\omega_{\text{cr}} - \omega_\infty}{\omega_{\text{cr}}} = \frac{kT}{E_p} \ln\left(\frac{v_0}{v_\infty}\right), \tag{11}$$

where

$$v_\infty = \frac{|\dot{\Omega}_\infty| r}{2\Omega_c + 2\omega_\infty + r(\partial\omega_\infty/\partial r)} = \frac{|\dot{\Omega}_\infty| r}{\kappa n_\infty}. \tag{12}$$

Even for a hot young pulsar such as the Crab pulsar, the internal temperature is somewhat smaller than E_p ($\sim$100 keV); for typical older pulsars, with internal temperatures $\sim 10^5$ K, one finds $E_p \sim 10^4\, kT$. Therefore, ω_∞ should have a spatial variation similar to that of $\omega_{\text{cr}}(r)$, and we have

$$n_\infty(r) \approx \frac{1}{\kappa}\left(2\Omega_c + r\frac{d\omega_{\text{cr}}}{dr}\right), \tag{13}$$

where we have neglected ω_{cr} ($\sim$0.1 radians s^{-1}) compared with Ω_c ($\sim$10 radians s^{-1}). It is rather interesting to note that the local vortex density can be extremely high (a vortex accumulation region) or low (a depletion region), depending upon the sign of $d\omega_{\text{cr}}/dr$, if the spatial variation is very large. In the accumulation regions, the vortex density could be further approximated by

$$n_\infty(r) \approx \frac{r}{\kappa}\frac{d\omega_{\text{cr}}}{dr} \equiv n_a \quad \text{if} \quad \frac{d\omega_{\text{cr}}}{dr} \gg \frac{2\Omega_c}{r}. \tag{14}$$

However, equation (13) becomes incorrect in the regions with $-d\omega_{\text{cr}}/dr > 2\Omega_c/r$, since the vortex density must remain positive. In other words, $d\omega_\infty/dr$ cannot be approximated by $d\omega_{\text{cr}}/dr$; moreover, ω_∞ will be bigger than ω_{cr} in these regions as a consequence of ω_{cr} falling off more rapidly than ω_∞. Therefore, an extremely small vortex density will obtain; ω_∞ will tend to follow ω_{cr} as closely as possible in order to reduce the Magnus force on the vortex lines. The steepest negative slope in these regions is easily shown to be

$$-\left(\frac{d\omega_\infty}{dr}\right)_{\max} \approx \frac{2\Omega_c}{r} \tag{15}$$

such that

$$n_\infty \approx \frac{1}{\kappa}\left(2\Omega_c + r\frac{d\omega_\infty}{dr}\right) \gtrsim 0 .$$

There are some interesting features of regions in which $-d\omega_{\text{cr}}/dr > 2\Omega_c/r$. First, these are vortex depletion regions with extremely low n_∞, of the order of zero; hence we may consider them to be "vortex-free regions." In order to maintain steady state, the radial velocity of the vortex line must be very high, basically $v_r \approx v_0$, which implies a huge Magnus force. No vortex line can be pinned in these regions, since $\omega > \omega_{\text{cr}}$. In Appendix B we choose a simple form of ω_{cr} and use the general expression for v_r (eq. [6]) to give an explicit demonstration of this argument. Second, such regions will extend far beyond the region where ω_{cr} varies, which we shall call the "fluctuation region" (Fig. 1). It is easy to estimate the radial size of the depletion region ("vortex-free" region) by using equation (15):

$$\delta r_d \approx \frac{[\omega_\infty - (\omega_\infty)_0]r_0}{2\Omega_c} \approx \frac{\delta\omega_0 r_0}{2\Omega_c}. \tag{16}$$

Here r_0 is the distance of the fluctuation region from the rotation axis, $\delta\omega_0$ is the maximum value of $\omega_{\text{cr}}(r) - \omega_0(r)$, ω_0 is the smooth part of ω_{cr}, and $(\omega_\infty)_0$ is the steady state value of ω outside the region of spatial variation.

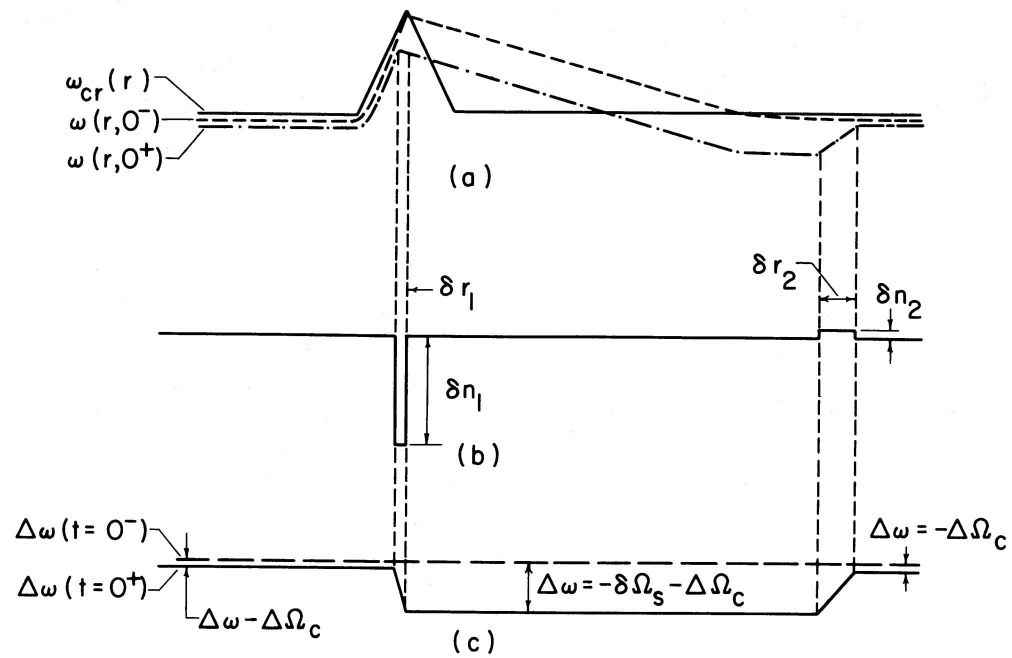

FIG. 1.—(*a*) Model for the fluctuation site. The vortex accumulation region ($d\omega_\infty/dr > 0$) and vortex depletion region are adjacent to each other. We have assumed that the pinned superfluid is in steady state before the glitch, i.e., $\omega(r, 0^-) = \omega_\infty(r)$. (*b*) Change of vortex density at the glitch. A single layer of vortex lines is unpinned from the accumulation region, passing through the depletion region and repinned right after this region, where $\delta n_1 \approx n_{\max}$, $\delta n_2 \approx n_0 \approx 2\Omega_c/\kappa$, $\delta r_1 \approx n_{\max}^{-1/2} \approx 10^{-4}$ cm, and $\delta r_2 \approx n_{\max}\,\delta r_1/n_0 \approx 1$ cm. (*c*) Change of ω in different regions.

b) Basic Time Scales for Reaching Steady State

Let us consider the spatial variation of $\omega_{\rm cr}$ in the following form:

$$\begin{aligned} \omega_{\rm cr}(r) &= \omega_0(r) + \delta\omega_{\rm cr}(r)\,, \qquad r_0 - \delta r_0 < r < r_0 + \delta r_0\,. \\ &= \omega_0(r)\,, \qquad \text{otherwise} \end{aligned} \tag{17}$$

Here $\omega_0(r)$ is a spatially smooth function which can be treated as a constant in the range in which we are interested; $\delta\omega_{\rm cr}(r)$ is the size of the small local deviation from $\omega_0(r)$, and δr_0 is the radial distance over which this deviation exists. The spatial derivative of $\delta\omega_{\rm cr}$ can be bigger than $2\Omega_c/r$ in absolute value. Equations (8) and (9) can be further simplified by the following approximations: (*a*) we can neglect $\omega \approx \omega_{\rm cr}$ (~ 0.1 radians s^{-1}) in equations (8) and (9) in comparison with Ω_c (~ 10 radians s^{-1}); (*b*) $\dot\Omega_c(t)$ in equation (9) can be replaced by $\dot\Omega_\infty$, since the inertial moment of the pinning region ($I_p \lesssim 10^{-2} I$) is so small; (*c*) r can be approximated by r_0; and (*d*) we neglect higher order terms in $(\delta\omega_{\rm cr}/\omega_0)$. Equations (8) and (9) become

$$n(r, t) = \frac{1}{\kappa}\left[2\Omega_c + r_0 \frac{\partial\omega(r, t)}{\partial r}\right], \tag{18}$$

$$\frac{\partial\omega(r, t)}{\partial t} = -\frac{v_0}{r_0}\left[2\Omega_c + r_0\frac{\partial\omega(r, t)}{\partial r}\right] \exp\left[\frac{E_p}{kT}\left(\frac{\omega_0 - \omega}{\omega_0}\frac{\delta\omega_{\rm cr}}{\omega_0}\right)\right] \bigg/ \left\{1 + \exp\left[-\frac{E_p}{kT}\left(\frac{\omega_0 - \omega}{\omega_0} + \frac{\delta\omega_{\rm cr}}{\omega_0}\right)\right]\right\} - \dot\Omega_\infty\,. \tag{19}$$

The general solutions are presented in Appendix A. The internal torque can be obtained from equation (10).

It is interesting to note that there are only two important time scales to reach the steady state in this superfluid-crust coupling system (the time scale in the depletion region is too short to be of interest). The first is the thermal relaxation time,

$$\tau = \frac{kT\omega_0}{E_p|\dot\Omega_\infty|}, \tag{20}$$

which characterizes the thermal creep process. By using equation (A10), one can see that this is the only relevant time scale in the case of microglitches, for which $(E_p/kT)(\delta\omega_{\max}/\omega_0) \ll 1$, where $\delta\omega_{\max}$ is the maximum value of $\omega(0, r)) - \omega_\infty(r)$. A second time scale,

$$t_0 = \frac{\delta\omega_{\max}}{|\dot\Omega_\infty|} = \frac{E_p}{kT}\frac{\delta\omega_{\max}}{\omega_0}\,\tau\,, \tag{21}$$

characterizes the relaxation if $t_0 \gg \tau$. These two time scales are independent of the functional form of $\delta\omega_{\rm cr}(r)$.

c) Dynamic Response of the Pinned Superfluid in the Accumulation and Depletion Regions

In Appendix A we show that the whole depletion region responds to any change from the steady state almost coherently. This can be understood as follows. Suppose that any part of the depletion region responds differently from the rest of the region, because the spatial derivative of $\omega(r, t)$ is not close to $-2\Omega_c/r$. However, $-d\omega/dr > 2\Omega_c/r$ is impossible because $2\Omega_c/r$ is the steepest negative slope of $d\omega/dr$. Similarly, if the vortex density $2\Omega_c + r(d\omega_{cr}/dr)$ is greater than zero, it must be very small, because vortices cannot be pinned in this region. Therefore, the entire depletion region must respond. In order to change the superfluid velocity in the depletion region, some vortex lines must enter from the accumulation region. The superfluid velocity of the accumulation region is then also changed as a result of the migration of vortex lines. The vortex lines released from the accumulation region immediately traverse the entire depletion region where they cannot pin. Therefore, the depletion region as a whole partakes in the response of one accumulation region. The response from these two regions can be expressed as

$$\frac{\Delta\dot{\Omega}_c(t)}{\dot{\Omega}_\infty} = \frac{I_d + \delta I_A}{I} \frac{(e^{t_0/\tau} - 1)e^{-t/\tau}}{1 + (e^{t_0/\tau} - 1)e^{-t/\tau}} , \tag{22a}$$

where I_d and δI_A are respectively the moments of inertia in the depletion region and in that part of the accumulation region involved in vortex unpinning (see Fig. 1), and $t_0 = \delta\Omega_s/|\dot{\Omega}_\infty|$. We will show that $I_d \gg \delta I_A$ and $t_0 \ll \tau$ in the case of microglitches. Equation (22a) can then be approximated by

$$\Delta\dot{\Omega}_c(t) = \frac{\Delta\Omega_c}{\tau} e^{-t/\tau} , \tag{22b}$$

where $\Delta\Omega_c = I_d \delta\Omega_s/I$ is the initial frequency jump experienced by the crust as a result of the unpinning event.

III. APPLICATIONS AND DISCUSSION

a) A Naive Spontaneous Unpinning Scenario

In the last section, we saw that the spatial variations in $\omega_{cr}(r)$ can result in vortex accumulation (depletion) regions if $d\omega_{cr}/dr$ is positive (negative). In the accumulation region, the Magnus force on a particular vortex line required to balance the local superfluid velocity set up by a neighboring vortex is

$$\delta f_m \approx \rho\kappa^2 n_a^{1/2} . \tag{23}$$

The total force in steady state, $f = \rho\kappa r\omega_\infty$, is below the critical value for unpinning, by an amount

$$\delta f \equiv f_p - f = \rho\kappa r(\omega_{cr} - \omega_\infty)$$
$$= \frac{kT\rho\kappa r_0 \omega_{cr}}{E_p} \ln\left(\frac{v_0}{v_\infty}\right) . \tag{24}$$

Suppose that a few free vortex lines migrate into the vortex accumulation region. The source of these vortex lines will be discussed later. They will rotate with the superfluid, which is moving faster than the crust. These free vortex lines will circulate around the spin axis of the neutron star and drift out slowly, contributing additional local superfluid velocity to the pinned vortex lines; an extra pinning force of order δf_m, given in equation (23), is then required if these lines are to remain pinned. If δf, or, equivalently, $\rho\kappa^2 n_a^{1/2}$, exceeds

$$\frac{kT\rho\kappa r_0 \omega_{cr}}{E_p} \ln\left(\frac{v_0}{v_\infty}\right) , \tag{25}$$

then some vortex lines will be forced out of their pinning sites. Since the pinning sites are the local energy minima for the vortex lines, unpinned vortex lines will try to return to these positions. However, for certain fluctuations in ω_{cr}, the accumulation region is adjacent to a depletion region as the calculation in Appendix A indicates (see Fig. 1). If that is the case, the pinned vortex lines at the boundary between the accumulation and depletion regions, which are expelled from their pinning sites, will immediately pass through the "vortex-free" region without being repinned. They will be repinned once they are past the depletion region. It is likely that because of its fast azimuthal velocity a free vortex line will complete many cycles around the star's rotation axis, without moving significantly (more than a single vortex layer) in the radial direction. This will persist until almost all vortices in the layer are unpinned and scattered through the depletion region. The initial free vortex line that triggered this unpinning is then in an unbalanced vortex environment, with accumulated vorticity to one side and no vorticity to the other. Any further scattering will then move it into the depletion region, where it will shoot radially outward. The total number of vortices unpinned is then

$$N_u \approx 2\pi r_0 n_a^{1/2} . \tag{26}$$

We call this the "single-layer unpinning" mechanism, It can be shown from equation (25) that $n_a^{-1/2} \sim 10^{-4}$ cm for this mechanism, so that $\delta I_A/I \approx n_a^{-1/2}/r_0 \sim 10^{-10}$. The resulting change of angular velocity of the superfluid in the depletion region is then

$$\delta\Omega_s = \frac{\kappa N_u}{2\pi r_0^2} . \tag{27}$$

As we shall see, the inertial moment of the "vortex-free" region is much bigger than the inertial moments of the accumulation or depletion regions. Hence the inertial moment associated with the glitch is that of the depletion region and is

$$\frac{I_d}{I} \approx \frac{\delta r_d}{r_0} = \frac{\delta\omega_a}{2\Omega_c} , \tag{28}$$

where we have used equation (16) to express the width δr_d of one depletion region in terms of the excess rotation $\delta\omega_a$ built up in the accumulation region. Because of the conservation of angular momentum in the glitch, the crust will be suddenly spun up by an amount

$$\Delta\Omega_c = \frac{I_d}{I}\,\delta\Omega \approx \frac{\delta r_d}{r_0}\frac{\kappa}{r_0}\,n_a^{1/2}\,. \tag{29}$$

We now apply these considerations to pulsar glitches and noise. The local vortex density in the accumulation region is given by

$$n_a\kappa = r\,\frac{\delta\omega_a}{\delta r_a} = \frac{\delta r_d}{\delta r_a}\,2\Omega\,, \tag{30}$$

so that

$$\frac{\delta r_d}{\delta r_a} = \frac{n_a/n_0}{2\Omega} = n_a\,l_v^2\,, \tag{31}$$

where we have introduced the intervortex spacing l_v and vortex density $n_0 = 2\Omega/k$ corresponding to uniform rotation at the rate Ω. Using this expression in equation (29), we obtain

$$\frac{\Delta\Omega_c}{\Omega} = \frac{2\delta r_d}{r}\frac{l_v^2}{r}\,n_a^{1/2} = 2\,\frac{\delta r_d}{r}\frac{l_v}{r}\left(\frac{\delta r_d}{\delta r_a}\right)^{1/2}; \tag{32}$$

this expression determines the magnitude $\Delta\Omega_c/\Omega_c$ of events resulting from unpinning in a single-layer accumulation region. For a succession of N adjacent accumulation-depletion regions of similar widths, one could conceive a generalization of this unpinning mechanism in which vortices moving through one depletion region start the unpinning process in the next accumulation region, resulting in a cascade involving the N regions. In this case,

$$\frac{\Delta\Omega_c}{\Omega_c} = 2N\,\frac{\delta r_d}{r}\frac{l_r}{r}\left(\frac{\delta r_d}{\delta r_a}\right)^{1/2}. \tag{33}$$

The factor $N\,\delta r_d/r$ reflects the fractional moment of inertia in the vortex depletion regions, I_d/I (eq. [28]). This should be less than, but comparable to, the fractional moment of inertia I_p/I in the pinned superfluid, or in the regions involved in postglitch relaxation. Application of vortex creep theory to postglitch relaxation data has yielded $I_p/I \sim 10^{-4}$–10^{-2} (Alpar *et al.* 1984*b*; Alpar, Nandkumar, and Pines 1985). In terms of the fluctuation $\delta\omega_a$ in critical frequency in the vortex accumulation regions, equation (33) can be written as

$$\frac{\Delta\Omega_c}{\Omega_c} = \frac{N\,\delta\omega_a}{2\Omega}\frac{1}{\Omega}\left(\frac{\kappa}{r}\frac{\delta\omega_a}{\delta r_a}\right)^{1/2}. \tag{34}$$

Generalizing equation (28) to

$$\frac{I_d}{I} = \frac{N\,\delta\omega_a}{2\Omega} \tag{35}$$

and using the assumption $I_d/I \sim I_p/I$, one obtains

$$\delta\omega_a \approx \frac{2\Omega}{N}\frac{I_p}{I}\,, \tag{36}$$

$$\frac{\delta\omega_a}{\delta r_a} = \left[\left(\frac{\Delta\Omega_c/\Omega_c}{I_p/I}\right)\Omega\right]^2\frac{r}{\kappa}\,, \tag{37}$$

$$\frac{\delta r_d}{\delta r_a} = \frac{n_a}{n_0} = \left[\left(\frac{\Delta\Omega/\Omega}{I_p/I}\right)\frac{r}{2l}\right]^2. \tag{38}$$

Substituting observed values of $\Delta\Omega/\Omega$ and values for I_p/I referred from postglitch fits, one finds that parameters that may seem reasonable for *individual* glitches are obtained. There is also a suggestive correspondence between the values of $\delta\omega_a/\delta r_a$ obtained through equation (37) (which, incidentally, does not involve N) and the values obtained by using equation (30) and setting n_a to its critical value in equation (25):

$$\frac{\delta\omega_a}{\delta r_a} = \frac{r}{\kappa}\frac{\omega_{cr}^2}{E_p^2}\ln\left(\frac{v_0}{v_a}\right)^2(kT)^2\,. \tag{39}$$

Taking $\omega_{cr} \approx 10^{-4}$, $E_p \approx 0.1$ MeV, $\ln(v_0/v_\infty) \approx 35$, and using T inferred from postglitch timing fits or from the rate of energy dissipation in vortex creep (Alpar, Nandkumar, and Pines 1985; Alpar *et al.* 1987), we find agreement between equation (37) and equation (39), within an order of magnitude or better, for the Crab pulsar, PSR 0525+21, and PSR 0355+54. For the Vela pulsar glitches, however, the estimate given by equation (39) is 4 orders of magnitude smaller than the value inferred from the observed glitches through equation (37). Thus, the Vela pulsar differs from the other pulsars in that its glitches seem to involve a much higher vortex accumulation of vortices than the critical density required for spontaneous unpinning. More generally, a consistent picture in terms of similar values of $\delta\omega_a$, δr_a, δr_d, and N for glitches of similar magnitude $\Delta\Omega/\Omega$ does not follow. Equation (36) serves to illustrate this: glitches of the same magnitude and similar I_p/I—say, in the Vela pulsar and in PSR 0355+54 ($\Delta\Omega/\Omega \sim 10^{-6}$) or in the Crab pulsar and in PSR 0525+21 ($\Delta\Omega/\Omega \sim 10^{-9}$)—require quite different values of the product $N\,\delta\omega_a$. This implies that if $\delta\omega_a$ were to be a typical fluctuation size—say, comparable to $\omega_{\rm cr}$ in the weakest pinning layers, with similar values in all neutron stars—then the number N of accumulation-depletion regions required cannot be a parameter of neutron star structure, determined solely by microscopic pinning structures, but scales with the Ω of the pulsar. Thus, to apply the above arguments of spontaneous unpinning to macroscopic glitches as a class, one must understand the parameters of accumulation and depletion regions, i.e., the effective distribution of pinning inhomogeneities, as functions of the rotation rate of the star (and possibly its age and thermal history). A related question is the relationship of the regions involved in spontaneous unpinning (I_d/I) to the regions involved in postglitch response (I_p/I). A discussion of this set of problems is beyond the scope of the present work.

b) Pulsar Timing Noise and Microglitches

The fluctuations in the pulsar period, known as timing noise, were first recognized in the Crab pulsar by Boynton *et al.* (1972). They found a large phase residual remaining after removal of the low-order polynomial and noted that this could result from a random walk in the rotation frequency. Cordes and Helfand (1980) found that noise was a characteristic feature of the timing behavior of 50 pulsars they examined using JPL data (Cordes and Downs 1985). Alpar, Nandkumar, and Pines (1986) used vortex creep theory to construct model noise power spectra to compare with the power spectra of 25 pulsars analyzed by Boynton and Deeter (1986). The results show no clear evidence of the kind of structure which would indicate that the source of the noise lies in the pinned superfluid alone. Cordes and Downs (1985) analyzed the arrival time of 24 pulsars and concluded that simple random walk processes composed solely of step functions in rotational phase or in one of its derivatives are generally not consistent with the data. Rather, it appears that most activity is due to a mixture of discrete changes in the phase frequency (or frequency derivative). Cheng (1987*b*) has presented a modified timing noise model in which the variations in rotation period are contributed by two related mechanisms, microglitches and the fluctuating magnetospheric torques induced by microglitches.

On the basis of the observational upper limits on $\Delta\Omega_c$ of the *individual* (unresolved) microglitches, one concludes that the time scale t_0 (eq. [21]) is too small to be relevant to structures in the power spectra, and τ is the only time scale left in this model. It is natural to assume that the rate of microglitches R will be of the order of $1/\tau$. In the present single-layer unpinning model, $\Delta\Omega_c/\Omega_c$ is indeed very small, and gives a negligible t_0/τ; we need at least one free vortex line to initiate the whole mechanism. Such free vortex lines could be created as the pinned vortex lines creep outward and encounter a spatially irregular region with missing pinning sites. This situation can happen anytime, but the mean time interval for such events must be the relaxation time scale τ for excursions from the steady state of thermal creep.

By fitting the data analyzed by Boynton and Deeter (1986), one can obtain the random walk parameter S_{pN}, which is related to $\Delta\Omega_c$ through (Cheng 1987*b*)

$$\Delta\Omega_c^{\rm obs} = \left(\frac{S_{pN}}{R}\right)^{1/2} \frac{1}{2\pi\tau}\,, \tag{40}$$

where the superscript "obs" represents the observed value of $\Delta\Omega_c$. Using the above argument for R, equation (40) can be expressed as

$$\Delta\Omega_c^{\rm obs} = \frac{1}{2\pi}\left(\frac{S_{pN}}{\tau}\right)^{1/2} \propto \left(\frac{S_{pN}\dot{\Omega}_c}{T}\right)^{1/2}, \tag{41}$$

for the typical event size inferred from the noise strength. To obtain a theoretical estimate based on our picture of spontaneous vortex unpinning, once again we postulate that the vortex density in accumulation regions does not increase beyond the critical density for unpinning obtained from expression (25),

$$n_a^{1/2} = \frac{r_0}{\kappa}\frac{\omega_{\rm cr}}{E_p} kT \ln\left(\frac{v_0}{v_\infty}\right). \tag{42}$$

On combining equations (28), (29), and (42), we have

$$\Delta\Omega_c^{\rm th} = \frac{\delta\omega_a}{2\Omega_c}\frac{\omega_{\rm cr}}{E_p} kT \ln\left(\frac{v_0}{v_\infty}\right) \propto \frac{T}{\Omega_c}\,, \tag{43}$$

where the superscript "th" refers to the theoretical value of $\Delta\Omega_c$. The internal temperature is calculated by using the model of Gudmundsson, Pethick, and Epstein (1982) and assuming that vortex creep is the ultimate heating mechanism of neutron stars, (Alpar *et al.* 1984*a*; Alpar, Nandkumar, and Pines 1985; Cheng 1987*a*):

$$T_6 = 2.3(I_{p,43}\,\bar{\omega}_{\rm cr}\,\dot{\Omega}_{-14})^{0.455}\,. \tag{44}$$

In Table 1 we summarize our model results by using the data analyzed by Boynton and Deeter (1986). In order to eliminate the

TABLE 1

NOISE LEVELS OF 14 PULSARS, NORMALIZED TO RESULTS OBTAINED FOR PSR 2021+51

Pulsar	$S_{pN}^{\rm obs}/(S_{pN}^{\rm obs})_{2021}$	$[\Delta\Omega_c/(\Delta\Omega_c)_{2021}]^{\rm obs}$	$\left[\frac{\Delta\Omega_c}{(\Delta\Omega_c)_{2021}}\right]^{\rm th}$
0329+54..........	$(3.91 \pm 0.12) \times 10^{-1}$	$(4.59 \pm 0.07) \times 10^{-1}$	9.0×10^{-1}
0355+54..........	$\leq(5.39 \pm 1.6) \times 10^{-1}$	$\leq(4.77 \pm 0.12)$	1.13
0628−28..........	$\leq(5.34 \pm 0.17)$	$\leq(1.76 \pm 0.03)$	1.71
0736−40..........	$\leq(8.80 \pm 0.83) \times 10^{-1}$	$\leq(9.17 \pm 0.43) \times 10^{-1}$	7.83×10^{-1}
0833−45..........	$\leq(2.22 \pm 0.24) \times 10^{-1}$	$\leq(2.93 \pm 0.13) \times 10^{1}$	7.0
1133+16..........	$(1.83 \pm 0.13) \times 10^{-1}$	$(2.82 \pm 0.09) \times 10^{-1}$	1.2
1237+25..........	$(2.61 \pm 0.15) \times 10^{-1}$	$(2.08 \pm 0.06) \times 10^{-1}$	7.1×10^{-1}
1642−03..........	(1.83 ± 1.43)	(1.31 ± 0.51)	0.85
1706−16..........	$(5.30 \pm 0.21) \times 10^{-1}$	$(7.81 \pm 1.53) \times 10^{-1}$	1.47
1749−28..........	(2.19 ± 0.27)	(1.80 ± 0.11)	1.7
1818−04..........	$\leq(6.80 \pm 0.46)$	$\leq(2.81 \pm 0.10)$	1.57
1911−04..........	(0.65 ± 1.68)	(0.82 ± 1.68)	1.3
2021+51..........	(1.0 ± 0.08)	(1.0 ± 0.04)	1.0
2045−16..........	$(1.43 \pm 0.05) \times 10^{-1}$	$(2.54 \pm 0.04) \times 10^{-1}$	2.1
2217+47..........	$\leq(4.0 \pm 0.27) \times 10^{1}$	$\leq(5.90 \pm 0.20)$	1.0

unknown parameter $\delta\omega_a$, we choose PSR 2021+51, which has the smallest variances in $S_{pN}^{\rm obs}$ and $S_{SN}^{\rm obs}$ (Cheng 1987*b*), as a reference. For the Vela pulsar, which is too young for equation (44) to apply, we have taken $T = 10^8$ K. In expression (44), $I_{p,43}\,\bar{\omega}_{\rm cr} \lesssim 1$ (cf. Alpar, Nandkumar, and Pines 1986) refers to the strongest pinning regions in the star, which dominate the energy dissipation and set the temperature in old pulsars. The observed values and theoretical values of $\Delta\Omega_c$ are consistent with each other, suggesting that the noise process has the T, Ω, $\dot{\Omega}$ dependence predicted by the spontaneous unpinning model, with the unpinning of a single vortex accumulation region at critical density responsible for each event. We now employ the data on PSR 2021+51, used as the reference for the normalization in Table 1, to infer the values of $\delta\omega_a$, $\delta\omega_a/\delta r_a$, and δr_a. Setting $\Delta\Omega_a^{\rm obs} = \Delta\Omega_c^{\rm th}$ for this pulsar, and using equations (41), (42), (43), and (44),

$$\delta\omega_a \approx 1.7 \times 10^{-3}\left(\frac{\omega_{{\rm cr},\,-4}}{E_p/0.1\ {\rm MeV}}\right)^{-3/2}\left[\frac{\ln\,(v_0/v_\infty)}{35}\right]^{-1}, \tag{45}$$

$$\frac{\delta\omega_a}{\delta r_a} \approx 0.14\left(\frac{\omega_{{\rm cr},\,-4}}{E_p/0.1\ {\rm MeV}}\right)^{2}\left[\frac{\ln\,(v_0/v_\infty)}{35}\right]^{2}, \tag{46}$$

$$\delta r_a = 1.2 \times 10^{-2}\left(\frac{\omega_{{\rm cr},\,-4}}{E_p/0.1\ {\rm MeV}}\right)^{-7/2}\left[\frac{\ln\,(v_0/v_\infty)}{35}\right]^{-3}\ {\rm cm}\,. \tag{47}$$

These values are entirely reasonable. For the superweak regions where the spontaneous unpinning events are likely to originate, E_p could actually be significantly lower than 0.1 MeV, which would lead to $\delta\omega_a \approx \omega_{\rm cr}$ and δr_a much smaller than 10^{-2} cm.

IV. DISCUSSION

The above arguments show that for the sample of pulsars exhibiting phase noise, an evaluation of the observed noise strengths in terms of the spontaneous vortex unpinning events as modeled in the present work yields encouraging results, both in terms of the expecting scaling of noise strengths as a function of the Ω, $\dot{\Omega}$, and T of the pulsars (assuming that T is related to $\dot{\Omega}$ through the energy dissipation rate in the neutron star) and in terms of the microscopic parameters of the spontaneous unpinning regions.

An application of the same ideas to the glitches in the Crab pulsar, PSR 0525+21, and PSR 0355+54, which requires a cascade of unpinning from some large number N of accumulation regions, also yields the expected scaling; however, the Vela pulsar glitches do not fit the same relationship, and there are further problems in solving the relation of the spontaneous unpinning process to vortex creep and relaxation processes. This set of problems concerning the macroglitches will be addressed in future work.

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APPENDIX A

A GENERAL SOLUTION OF THE EQUATION OF MOTION FOR PINNED SUPERFLUID

In this appendix we present the general solution of the equation of motion of the pinned superfluid when $\omega_{\rm cr}$ has a large spatial variation. Let us define the dimensionless quantities

$$x = \frac{r - r_0}{\delta r_0}\,, \quad \gamma = \frac{E_p}{kT}\,, \quad \tau = \frac{\omega_0}{\gamma|\dot{\Omega}_\infty|}\,, \quad x_1 = \frac{t}{\tau}\,, \quad U(x_1, x) = \frac{\omega(t, r) - \omega_0}{\omega_0}\,, \quad \delta U_{\rm cr}(x) = \frac{\delta\omega_{\rm cr}(r)}{\omega_0}\,.$$

Here r_0 is the mean distance of the spatial variation region from the rotation axis, δr_0 is the radial dimension of this region, and ω_0 and $\delta\omega_{cr}(r)$ are the smooth part and spatial fluctuation part of $\omega_{cr}(r)$, respectively. Then the equation of motion of the pinned superfluid in terms of these dimensionless quantities is

$$\frac{\partial U(x_1, x)}{\partial x_1} = \frac{1}{\gamma} - \frac{\alpha_1}{\gamma}\left(\alpha_2 + \frac{\partial U}{\partial x}\right) \exp \frac{[\gamma(U - \delta U_{cr})]}{[\exp \gamma(U - \delta U_{cr}) + 1]} , \tag{A1}$$

where $\alpha_1 = \omega_0 v_0/\delta r_0 |\dot{\Omega}_\infty|$ and $\alpha_2 = 2\Omega_c \delta r_0/\omega_0 r_0$. The condition for steady state discussed in § II simply means $\partial U(x_1, x)/\partial x_1 = 0$. Equation (A1) becomes

$$\frac{dU_\infty(x)}{dx} = \frac{\exp\{-\gamma[U_\infty - \delta U_{cr}(x)]\}}{\alpha_1} - \left(\alpha_2 - \frac{1}{\alpha_1}\right), \tag{A2}$$

where the subscript ∞ represents the steady state value. It is easy to show that α_2 is much larger than $1/\alpha_1$. The solution is

$$\exp[\gamma U_\infty(x)] = \frac{\gamma}{\alpha_1} \exp(-\gamma\alpha_2' x) \int_{-\infty}^{x} \exp[\gamma(\alpha_2' x\, \delta U_{cr})]dx , \tag{A3}$$

where $\alpha_2' \equiv \alpha_2 - 1/\alpha_1$. The vortex density distribution and mean radial velocity of vortex lines in steady state are

$$n_\infty(x) = \frac{2\Omega_c}{\kappa}\left(1 + \frac{1}{\alpha_2}\frac{dU_\infty}{dx}\right) = n_0\left(1 + \frac{1}{\alpha_2}\frac{dU_\infty}{dx}\right) \tag{A4}$$

and

$$v_\infty(x) = \frac{|\dot{\Omega}_\infty| r_0}{n_\infty \kappa} , \tag{A5}$$

respectively.

To solve the time-dependent equation (A1), we first introduce the quantities

$$\delta U(x_1, x) = U(x_1, x) - U_\infty(x) , \tag{A6}$$

$$y(x) = \exp[\gamma(U_\infty - \delta U_{cr})] , \tag{A7}$$

$$z(x_1, x) = \exp[\gamma\, \delta U(x_1, x)] . \tag{A8}$$

Equation (A1) becomes

$$\left(\frac{1}{z} + y\right)\frac{\partial z}{\partial x_1} = (1 - z) - \frac{\alpha_1}{\gamma} y \frac{\partial z}{\partial x} . \tag{A9}$$

The solution of equation (A9) is obtained by using the method of characteristic curves (John 1986), as an implicit function of two parametric functions S_1 and S_2:

$$z(x_1, x) = 1 + \{\exp[\gamma\, \delta U(0, S_1)] - 1\} \exp(-S_1) , \tag{A10}$$

$$S_1(x_1, x) = \frac{\gamma}{\alpha_1}[(\alpha_1\alpha_2 - 1)x + \alpha U_\infty(x)] - \frac{\gamma}{\alpha_1}[(\alpha_1\alpha_2 - 1)S_2 + \alpha_1 U_\infty(S_2)] , \tag{A11}$$

$$S_2(x_1, x) = x - \frac{\alpha_1}{\gamma}\left\{x_1 + \frac{\gamma\, \delta U(0, S_2) - \ln z}{\exp[\gamma\, \delta U(0, S_2)] - 1}\right\}^4 , \tag{A12}$$

with initial conditions

$$x_1(S_1 = 0, S_2) = 0 , \tag{A13}$$

$$x(S_1 = 0, S_2) = S_2 , \tag{A14}$$

$$z(S_1 = 0, S_2) = \exp[\gamma\, \delta U(0, x)] . \tag{A15}$$

This full solution is somewhat inconvenient. We concentrate, therefore, on the solutions in the two types of regions of interest, namely, in vortex accumulation and depletion regions. In Figure 1, we plot the steady state solution of U. One can see that $U_\infty - \delta U_{cr} < 0$ in the accumulation region and $U_\infty - \delta U_{cr} > 0$ in the depletion region. Since γ is a large parameter ($\sim 10^5$ for a typical pulsar), y is much smaller (larger) than unity in the accumulation (depletion) regions. Let us now solve equation (A9) in these two regions separately.

I. IN THE ACCUMULATION REGION

Since vortex lines only unpin near the boundary between the accumulation region and the depletion region, we have $\delta U(0, x) = \Delta\Omega_c/\omega_0 < |U_\infty - \delta U_{cr}|$ in most of this region. Equation (A9) can be approximated by

$$\frac{\partial z_A}{\partial x_1} = (1 - z_A)z_A - y z_A \frac{\partial z_A}{\partial \tilde{x}} , \tag{A16}$$

where $\tilde{x} = \gamma x/\alpha_1$ and we use the subscript A to represent the quantities in the accumulation region. The solution of equation (A16) is an implicit function of the parameter S; it is

$$z_A(x_1, \tilde{x}) = [1 + \{\exp[-\gamma \delta U_A(0, S)] - 1\} \exp(-x_1)]^{-1} , \tag{A17}$$

where

$$S = \tilde{x} + \frac{1}{\alpha_1 \alpha_2} \{-x_1 + \gamma[U_A(x_1, x) - U_A(0, S)]\} \approx \tilde{x} . \tag{A18}$$

Here we use the fact that $\alpha_1 \alpha_2 = v_0 2\Omega_c/r_0 |\dot{\Omega}_\infty| \approx 10^{13}$ is much bigger than all the other quantities. Equation (A17) is identical to the solution obtained by Alpar *et al.* (1984*a*).

II. IN THE DEPLETION REGION

It is easy to show that

$$\delta U(0, x) = \frac{\Delta\Omega_c + \delta\Omega_s}{\omega_0} \approx \frac{\delta\Omega_s}{\omega_0} < |U_\infty - \delta U_{\rm cr}| \approx \delta U_{\rm cr}$$

in most of the accumulation region. We can approximate equation (A9) by

$$y_d \frac{\partial z_d}{\partial x_1} = (1 - z_d) - y \frac{\partial z_d}{\partial \tilde{x}} , \tag{A19}$$

where the subscript d represents the quantities in the depletion region. The solution is

$$z_d(x_1, \tilde{x}) = 1 - \frac{1 - \exp[\gamma \delta U_d(0, \tilde{x} - x_1)]}{\exp\{(\alpha_1\alpha_2 - 1)x_1 + \gamma[U_\infty(\tilde{x}) - U_\infty(\tilde{x} - x_1)]\}} . \tag{A20}$$

Equation (A20) is subject to a boundary condition,

$$\delta U_d(x_1, \tilde{x}_B) = \delta U_A(x_1, \tilde{x}_B) , \tag{A21}$$

where $\tilde{x}_B$ is the position of the boundary between the accumulation region and the depletion region, $\delta U_A(x_1, \tilde{x}_B)$ is the value of $\delta U_d(x_1, \tilde{x})$ at the boundary, and $\delta U_A(x_1, x_B)$ is described by equation (A17). The reason for such matching is that the vortex lines are always flowing from the accumulation region to the depletion region (creeping radially outward); therefore the boundary value of $\delta U_d(x_1, \tilde{x})$ is controlled by the activity of the vortex lines in the accumulation region. We are going to show that the pinned superfluid in the depletion region responds almost coherently. Consider $x_1 < \tilde{x}$, and expand the exponential factor in the denominator of equation (A20) for large y_d:

$$1 - \exp[\gamma \delta U(x_1, \tilde{x})] = \{1 - \exp[\gamma \delta U_d(0, \tilde{x} - x_1)]\}/\exp(x_1/x_d) \approx \{1 - \exp[\gamma \delta U_d(0, \tilde{x} - x_1)]\} ; \tag{A22}$$

here we have used equation (A2). Equation (A22) is a solution of a wave equation (one can obtain this result by omitting the first term on the right-hand side of eq. [A19]), which has a characteristic velocity $dr/dt = v_0$. Physically, this can be understood because the vortex lines in this region are moving with this microscopic velocity (see Appendix B). Therefore, the typical time scale in this region is $\delta r_d/v_0 \approx 10^{-1}(\delta r_d)_6$ s. For later time we have

$$\delta U_d(x_1, \tilde{x}) = \delta U_d(0, \tilde{x}_B) , \tag{A23}$$

where $\delta U_d(0, \tilde{x}_B)$ is the value of $\delta U_d(x_1, x)$ at the boundary. Since this value is not constant (see eq. [21]) equation (A23) should read

$$\delta U_d(x_1, \tilde{x}) = \delta U_A(x_1, \tilde{x}_B) . \tag{A24}$$

Equation (A24) tells us that the whole depletion region responds to the accumulation region almost simultaneously.

The vortex density distribution, the mean radial velocity of vortex lines, and the internal torque acting on the crust due to the reaction of the superfluid are

$$n(x_1, x) = n_0\left(1 + \frac{1}{\alpha_2}\frac{\partial U}{\partial x}\right) = n_\infty(x) + \frac{n_0}{\alpha_2 \gamma z}\frac{\partial z(x_1, x)}{\partial x} , \tag{A25}$$

$$\begin{aligned} v_r(x_1, x) &= v_0 \exp\{\gamma[U(x_1, x) - \delta U_{\rm cr}]\} \\ &= v_\infty(x) z(x_1, x) , \end{aligned} \tag{A26}$$

and

$$\frac{N_{\rm int}(t)}{N} = -\frac{1}{I|\dot{\Omega}_\infty|}\int_0^{\Delta r_0} dI_p \frac{\kappa}{r} n(t, r)v_r(t, r) \approx -\frac{\delta r_0 \kappa \Delta r_0}{|\dot{\Omega}_\infty| r_0^2}\int_0^{x_0} dx\, n(x_1, x)v_r(x_1, x) , \tag{A27}$$

where Δr_0 is the radial size of the pinned superfluid, which is about $\delta\omega_0 r_0/2\Omega_c + \delta r_0$, and x_0 equals $\Delta r_0/\delta r_0$. The most interesting quantity is the relaxation of the jump in the observed spin-down rate of the crust, which is

$$\Delta\dot{\Omega}_c(t) = \frac{N_{\rm int}(t) - N_{\rm int}(0^-)}{I_c}$$
$$\approx -\frac{\delta r_0 \kappa}{r_0^2}\int_0^{x_0} dx[n(x_1, x)v_r(x_1, x) - n_\infty(x)v_\infty(x)]$$
$$= \gamma\dot{\Omega}_\infty \frac{\Delta r_0 \delta r_0}{r_0^2}\int_0^{x_0} dx \frac{\partial}{\partial x_1}[\delta U(x_1, x)] . \tag{A28}$$

Here, we assume that the superfluid is in steady state before the glitch. If $\delta U(0, x)) = -\Delta\omega/\omega_0$ is independent of x, equations (A28) and (A24) yield

$$\frac{\Delta\dot{\Omega}_c}{\dot{\Omega}_\infty} = \frac{I_d + \delta I_A}{I}\frac{[\exp(t_0/\tau) - 1]\exp(-t/\tau)}{1 + [\exp(t_0/\tau) - 1]\exp(-t/\tau)} , \tag{A29}$$

where

$$t_0 = \frac{\Delta\omega}{|\dot{\Omega}_\infty|} \quad \text{and} \quad \frac{\delta I_A + I_d}{I} \approx \frac{\delta\omega_0}{2\Omega_c} ,$$

which is the combined moment of inertia in the accumulation and depletion region affected by the unpinning event.

APPENDIX B

A NAIVE MODEL FOR THE SPATIAL VARIATION OF $\omega_{\rm cr}$

The associated distribution of vortex lines is not manifest in the general solution of the equation of motion of pinned superfluid with arbitrary form of $\delta\omega_{\rm cr}$ (Appendix A). In the text we argue that there exist two types of regions, vortex accumulation regions ($d\omega_{\rm cr}/dr > 0$) and vortex depletion regions ($d\omega_{\rm cr}/dr < 0$) in which the vortex density roughly equals $(r_0/\kappa)\, d(\delta\omega_{\rm cr})/dr$ and zero, respectively, when $|d(\delta\omega_{\rm cr})/dr|$ is much bigger than $2\Omega_c/r_0$. In this section, we would like to use a simple functional form of $\delta\omega_{\rm cr}(r)$ to illustrate this. Let us assume that $\delta\omega_{\rm cr}$ has a triangular shape,

$$\delta\omega_{\rm cr}(r) = \frac{\delta\omega_0}{2}(1 - |1 - x|) , \qquad 0 < x < 1 ,$$
$$= 0 , \qquad \text{otherwise} , \tag{B1}$$

where

$$x = \frac{r - r_0}{\delta r_0} ; \tag{B2}$$

$\delta\omega_0$ and δr_0 are the height and the base of this triangle, respectively. Hence, the spatial derivative of $\delta\omega_{\rm cr}$ is a constant,

$$\frac{d(\delta\omega_{\rm cr})}{dr} = \begin{cases} \dfrac{\delta\omega_0}{\delta r_0} , & 0 < x < \frac{1}{2} , \\ -\dfrac{\delta\omega_0}{\delta r_0} , & \frac{1}{2} < x < 1 . \end{cases} \tag{B3}$$

Substituting equation (B3) in equation (A3), we obtain

(1) $x \leq 0$:

$$\exp(\gamma U_\infty) = \frac{1}{\alpha_1 \alpha_2'} = \text{constant} , \qquad n_\infty = n_0 \tag{B4}$$

where n_0 equals $2\Omega_c/k$;

(2) $\frac{1}{2} \geq x > 0$:

$$\exp(\gamma U_\infty) = \frac{\exp(-\gamma\alpha_2' x)}{\alpha_1}\left\{\frac{1}{\alpha_2'} + \frac{\exp[\gamma(\delta U_0 + \alpha_2')x] - 1}{\delta U_0 + \alpha'}\right\} ,$$
$$\approx \frac{\exp(-\gamma\,\delta U_0 x)}{(\delta U_0 + \alpha_2')\alpha_1} \qquad \text{if} \quad \delta U_0 \gg \alpha_2' \tag{B5}$$

$$n_\infty \approx n_0 \frac{\delta U_0}{\alpha_2'} \approx \frac{r_0}{\kappa}\frac{\delta\omega_0}{\delta r_0} \qquad \text{if} \quad \delta U_0 \gg \alpha_2' , \tag{B6}$$

which agrees with the result obtained in the text by general argument;

(3) $1 > x > \frac{1}{2}$:

$$\exp(\gamma U_\infty) = \frac{\exp(-\gamma\alpha_2' x)}{\alpha_1}\left[\frac{1}{\alpha_2'} + \frac{\exp[\gamma(\delta U_0 + \alpha_2')/2] - 1}{\delta U_0 + \alpha_2'} + \frac{\exp(\gamma\,\delta U_0)\{\exp[\gamma(\alpha_2' - \delta U_0)x] - \exp[\gamma(\alpha_2' - \delta U_0)/2]\}}{\alpha_2' - \delta U_0}\right]$$

$$= \frac{2\exp(-\gamma\alpha_2' x + \gamma\,\delta U_0/2)}{\alpha_1 \delta U_0}\left\{1 - \exp\left[-\gamma\,\delta U_0\left(x - \frac{1}{2}\right) + \gamma\alpha_2' x\right]\right\}, \quad \text{if} \quad \delta U_2 \gg \alpha_2' \tag{B7}$$

$$n_\infty(x) \approx n_0 \frac{|\dot{\Omega}_\infty|\gamma_0}{2\Omega_c v_0} \ll 1 \tag{B8}$$

(4) $x \geq 1$:

$$\exp(\beta U_\infty) = \frac{\exp(-\gamma\alpha_2' x)}{\alpha_1}\left[\frac{1}{\alpha_2'} + \frac{\exp[\gamma(\delta U_0 + \alpha_2')/2] - 1}{\delta U_0 + \alpha_2'}\right.$$

$$\left. + \frac{\exp(\gamma\,\delta U_0)\{\exp[\gamma(\alpha_2' - \delta U_0)] - \exp[\gamma(\alpha_2' - \delta U_0)/2]\}}{\alpha_2' - \delta U_0} + \frac{\exp(\gamma\alpha_2' x) - \exp(\gamma\alpha_2')}{\alpha_2'}\right]$$

$$\approx \frac{\exp[-\gamma(\alpha_2' x + \delta U_0/2)]}{\alpha_1 \delta U_0}\left\{1 + \frac{\delta U_0}{\alpha_2'}\exp\left[\gamma\left(\alpha_2' x - \frac{\delta U_0}{2}\right)\right]\right\} \quad \text{if} \quad \delta U_0 > \alpha_2' \quad \text{and} \quad x < \frac{\delta U_0}{2\alpha_2'},$$

$$\approx \frac{1}{\alpha_1 \alpha_2'} \quad \text{if} \quad x > \frac{\delta U_0}{2\alpha_2'},$$

$$n_\infty(x) \approx n_0 \frac{|\dot{\Omega}_\infty|\gamma_0}{2\Omega_c v_0} \ll n_0 \quad \text{if} \quad \delta U_0 > \alpha_2' \quad \text{and} \quad x < \frac{\delta U_0}{2\alpha_2'},$$

which verifies that the vortex-free region will extend to a length scale of order $r_0\,\delta\omega_0/2\Omega_c$.

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