

Secular evolution of the solar system over 10 million years

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Summary. We have computed the differential system giving the secular evolution of the 8 main planets of the solar system up to the order 2 with respect to the masses and degree 5 in eccentricity and inclination including lunar and relativistic contributions (Laskar, 1985, 1986). This secular system is numerically integrated over 30 million years. A modified Fourier analysis is performed to obtain a solution for the secular evolution of the orbits on a quasi-periodical form. Comparison with Bretagnon's ephemeris VSOP82 allows to derive uncertainties for the determination of the main frequencies of the secular system. These uncertainties range from 0.001 arcsec/year to 0.02 arcsec/year. Comparisons are made with the results of long term numerical integrations of Applegate et al. (1986) and Carpino et al., (1986) and with the analytical theory of Bretagnon (1974, 1984). The solutions of the outer solar system appear to be more stable than the solutions of the inner solar system where lots of quasi-resonances in the long period frequencies prevent a good convergence of the solutions.

Key words: Celestial mechanics – planetary theories – solar system – secular perturbations

1. Introduction

This paper is the third one on the construction of a solution for the secular evolution of the solar system which was undertaken at Bureau des Longitudes since 1984, following the earlier works of Bretagnon (1974) and Duriez (1979, 1982). Our first paper (Laskar, 1985) describes the methods used for the construction of the secular system at the second order with respect to the masses and up to degree 5 in the variables eccentricity and inclination. In the second paper (Laskar, 1986), a numerical integration of the whole system over 10,000 years allowed us to derive the secular terms in the form of polynomial series of the times which are then directly comparable to the secular terms of classical Le Verrier type solutions like VSOP82 (Bretagnon, 1982). The use of these secular terms allowed us to extend the length of validity of the classical ephemeris solution over 10,000 years (Bretagnon et al., 1985; Bretagnon and Simon, 1986) and to derive new formulae for the precessional quantities up to t^{10} (Laskar, 1986). These two papers will be extensively quoted in the present work, and we shall refer to them respectively as (P1) and (P2). For all the definitions of the variables, the reader is supposed to refer to these previous papers.

In the present work, the same secular system is integrated over 30 million years (–10 to +20 millions). This integration is made possible by the use of a supercomputer CRAY-1. The results are then Fourier analysed in order to derive a quasi-periodic solution which is then comparable to previous works on analytical theories (Bretagnon, 1974 and Duriez, 1979) and to the recent numerical integrations of the outer planets made over such an extended period with a dedicated computer (Applegate et al., 1986) or a CRAY computer (Carpino et al., 1986).

2. Description of the secular system

The organization of the secular system showed itself to be critical as far as its computation and numerical integration are concerned. From the beginning, we started with a different attitude than in the usual analytical works dealing with Poisson quasi-periodical series: the computations are made in a very extensive way, without neglecting any of the terms which we could possibly encounter (of course, up to a given order with respect to the masses and to a given degree in the variables eccentricity and inclination) (P1). This led to a huge number of terms (153824 in the secular system) but these terms can be handled in a much easier way because the structure of the series is then made very simple and remains always the same.

The secular system is a polynomial system with imaginary coefficients and can be expressed in the form

$$\dot{\alpha} = \sqrt{-1} (\Phi_1 \alpha + \Phi_3(\alpha, \bar{\alpha}) + \Phi_5(\alpha, \bar{\alpha})) \quad (1)$$

where $\alpha = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_8, \bar{\zeta}_1, \bar{\zeta}_2, \dots, \bar{\zeta}_8)$

and where Φ_1 is a real matrix with constant coefficients, Φ_3 gathers all the terms of degree 3 and Φ_5 the terms of degree 5.

All the terms are generated according to (P1) and obey the following rules:

– The series in $\bar{z}$ are even in $\zeta, \bar{\zeta}$ while the series in ζ are odd in $\zeta, \bar{\zeta}$.

– For any given term T , the characteristic of the monomial $c_M(T)$ is equal to $+1$ ($c_M(T) = \delta(\bar{\alpha}) - \delta(\alpha)$ where $\delta(\alpha)$ is the degree in α). This last rule is the result of d'Alembert's relations in the expansions of the disturbing function (See the Appendix).

These two simple rules allow to generate all the possible monomials that will actually appear in the computations. Up to the order 2 with respect to the masses, a monomial can only involve the elements of 1, 2 or 3 planets. Up to the degree 5, the number of monomials even or odd in inclination (or in ζ) is given

Table 1. Number of monomials involving 1, 2, or 3 planets in the secular system

No. of planets involved	No. of combinations in each equations	Monomials	
		Even in inclination	Odd in inclination
1	8	10	10
2	28	102	102
3	21	318	318

in Table 1. We can construct a standard list of monomials for each case of Table 1. This represents only $2(10 + 102 + 318) = 860$ typical monomials. All the other monomials are obtained by changing the name of the planets involved in the different list of monomials. We have 8 planets, 28 couples of planets and 56 groups of three planets. Each of these combinations will generate a different list of monomials, using the corresponding typical list. For any of the 16 equations in (1), the left member contains $8 \times 10 + 28 \times 102 + 21 \times 318 = 9614$ monomials (in the monomial involving 3 planets, one of the planets is the planet corresponding to the given equation) which are stored in three different arrays $TT1(10 \times 8)$, $TT2(102 \times 28)$, $TT3(318 \times 21)$. In each array, the first index is the monomial index and the second one the combination of planets index. The total system (1) contains then $16 \times 9614 = 153824$ monomials which are then all generated by the 860 typical monomials and the different combinations of planets. The whole system is stored in three real arrays $TT1(10, 8, 16)$, $TT2(102, 28, 16)$ and $TT3(318, 21, 16)$. Apart from the typical monomials, there is no need to keep any information about the exponents in a given term: everything is contained in the location of the coefficient in the arrays.

3. Numerical integration. Vectorisation

The secular system (1) is close to its linear part

$$\dot{\alpha} = \sqrt{-1} \Phi_1 \alpha \quad (2)$$

which is called the Laplace-Lagrange system. This system is easily integrated and gives some idea of the aspect of the solution. The solutions of (2) are linear combinations of periodic terms of frequencies ranging from about 50,000 years to several million years. The secular system (1) is very smooth, and we can use a very large stepsize for its numerical integration of about 500 years instead of about 0.5 days for the direct numerical integration of Newton's law (and even less if we also include the Moon). The suppression of the short period terms allows to use a very large stepsize in the integration, but we must not forget that the new system (1) is only an approximation of second order of the secular system. The other drawback is that our secular system (1) is very large (153824 terms).

With such a huge second hand member, the best results will be obtained with high order numerical integration routines. We used a well experienced predictor-corrector Adams routine D02CBF from the NAG library with varying order and stepsize and with an output every 500 yr. The use of this routine on a CRAY-1 computer imposed to make some experiments for the precision of the integration over a long time. A typical try is made by integrating a well known circular function of frequency 25 s/yr and 50 s/yr (period of about 52,000 and 26,000 yr) over

10 million yr. The evaluation of the error is then simply made by difference with the known value of the function. The relative accumulated error goes from 5×10^{-8} to 2×10^{-7} after 10 million years which means from 2×10^{-7} to 8×10^{-7} after 20 million years. This precision is good enough for our computation and will ensure that there are no problems coming from the numerical integration itself. We must mention that on a CRAY computer, computation are made with only 14 digits which is rather low compared to the 16 usual digits on other computers.

The main part of the computing time in the numerical integration is used in the evaluation of the 153824 terms of the second member of (1). Typically, the values of z_i, ζ_i are given and a standard term of degree 5 is of the form

$$C \times X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5} \quad (3)$$

where C is a real coefficient, and X_{i_k} any of the complex variables $z_i, \zeta_i, \bar{z}_i, \bar{\zeta}_i$.

In a CRAY-1, computations are vectorised if the most inner loop is long, the optimum being obtained by a multiple of 64 which is the length of the accumulators. The difficulty is that the vectorisation is not possible with indirect addressing (the elements to be multiplied need to be physically in the accumulators before the multiplication). The only way we found to deal with this problem was to choose first the monomial type (corresponding to the first index in $TT1$, $TT2$ and $TT3$). The inner loop is then made over all the different combinations of planets involved in all the different equations. This mean an inner loop of 28 for the 2-planets monomials of $TT2$ and of $21 \times 8 = 168$ for the 3-planets monomials of $TT3$. The vectorisation obtained is then very efficient and reached the possible limit of the CRAY. The program was tested also on the NAS9080 computer of the CIRCE. The NAS9080 is a conventional computer with about the same speed as the scalar computation on the CRAY. We gained a factor of 3 with the CRAY due to the very rapid direct memory of the CRAY, and the vectorisation of the program added a factor of 5, which means a total factor of 15.

I should mention again that this vectorisation was only possible because of the very simple organization of the series. As no term was neglected, all the different series were alike and once a monomial was selected, we could loop on all the possible combinations of the planets involved. We obtain the somewhat surprising fact that keeping more terms leads to faster computations!

The total evaluation of the second member of (1) takes 0.06 second. The numerical integration of the secular system over 10 million years took about 30 minutes which means about 20,000 evaluations of the second member (about one every 500 yr). The integration was made first 10 million years backwards, and then 20 million years forwards for a total computer time of 1.5 hour. Of course, there is nothing which prevents us from integrating

the system over a much longer time (1.5 hours is very reasonable), except that our system is an approximation of the real system, and we need first to analyse our results before thinking of extending the integration.

The output consists of all the different variables z_i, ζ_i every 500 years from $-10,000,000$ years to $20,000,000$ years.

4. Fourier analysis of the solution

If we integrate the secular system (1) in an analytical manner as in Duriez (1979) or Laskar (1984), we obtain a quasi-periodic solution where the arguments are linear integer combinations of the basic frequencies of the system. As was outlined in Laskar (1984), in the Birkhoff normalization of the system (1), serious problems of convergence appear due to the presence of numerous small divisors for the 8 planet systems. This leads us to prefer a numerical integration of the system which avoids all these problems, and provides an accurate solution, although not very utilizable.

With the Fourier analysis of the solution, we try to obtain a more handy representation of the solution, similar to the result of an analytical solution.

Different methods have been already used for a similar purpose by Applegate et al., (1986) and Carpino et al. (1986). The main problem is to get a very good resolution in frequency, which is not given by a usual Fourier transform. Our concern was also not to anticipate at all on the form of the solution, even if we could have an idea of the solution by the knowledge given by analytical integration. This excludes recognition of combination of frequencies and least-squares adjustments during the frequency analysis like in Carpino et al. (1986). All the solutions for the different variables and planets are analysed separately, and only the analysis of the results will show the existence of common frequencies in the solutions. The reason to do this is that in fact if we can forecast the possible results when the normalization process converges correctly, very little is known in case of bad convergence.

The only assumption is that we search a solution on a quasi-periodic form, i.e. for each planet and variable, a solution in the form

$$\alpha = \sum_{i=1}^N A_i \exp \sqrt{-1}(v_i t + \phi_i) \quad (4)$$

where A_i, v_i, ϕ_i are real numbers, and we will try to represent the solution with the minimum possible number of arguments. The principle of the process is simple. We make first a FFT (Fast Fourier Transform) of the solution. Then we search for the term of maximum amplitude of the FFT. Around this term, we search for the value of v which gives the maximum of the power spectrum

$$\phi(v) = \int \alpha(t) \exp \sqrt{-1}(-vt) dt \quad (5)$$

this fine tuning of the frequency is made with a precision of 0.00002 arcsec/year. Once the frequency is found, the power spectrum gives the amplitude of the first term which is then subtracted from the solution. The process is then started again to search for the following term. As the two terms $\exp \sqrt{-1}(v_1 t)$ and $\exp \sqrt{-1}(v_2 t)$ are not orthogonal, on the second step, the amplitude of each term will not be given by the corresponding power spectrum like in regular Fourier analysis. We have then to orthog-

onalize our basis of function by the Gauss algorithm, and then compute the amplitude of each term by computing the inner products

$$\int \alpha(t) \overline{f_i(t)} dt \quad (6)$$

where the f_i are the functions of the orthogonal basis made with the $e_i = \exp \sqrt{-1}(v_i t)$ (the $f_i(t)$ are linear combinations of the $e_i(t)$). We have a decomposition of the solution on the orthogonal basis $(f_i(t))_{i=1,N}$ which gives then the decomposition on the non-orthogonal basis $(\exp \sqrt{-1}(v_i t))_{i=1,N}$.

The advantage of this method is that there is no suppositions on the analytical form of the solution and all the different solutions for the different planets are treated in a completely independent way. The information on the analytical form of the solution will be given by the existence of similar features in all the solutions, and by the recognition of all the basic frequencies from one solution to the other. No least square adjustments are made which could group together frequencies that are close and then prevent further analysis. Of course, the method has its drawbacks, one of which is that the time used for the orthogonalization of the basis of functions grows quadratically with the number of terms. For more than 40 terms, most of the time was used for this orthogonalization.

The solutions for the variables z_i, ζ_i of the different planets are given in Tables 6–13. The presentation is different for the inner planets and for the outer planets. As it will be detailed further, the solutions for the inner planets converge poorly and the terms are not easily identified. They are given in Tables 6–9 with 80 terms ordered by decreasing amplitude. The solutions for the outer planets (Tables 10–13) contain 51 terms ordered by decreasing amplitude. The amplitude of the terms decreases rapidly and several terms are easily identified. In this case the combination of fundamental frequencies corresponding to the term is given in column 1. The residuals between the numerical integration and the series of the solutions are given in Table 5. The fundamental frequencies are easily identified and we can check their internal precision by comparison of the values obtained in the solutions for all the 8 planets. The values of these frequencies are gathered in Table 3.

5. Accuracy of the solution NGT

In every work on analytical theory with perturbation methods one of the most difficult aspects is the analysis of the accuracy of the final solution. Truncations are often made, usually without any real evaluation of the remainder. One basic reason for this is that the problem is too difficult and no definite results have been obtained on the convergence of the series in use. In our solution NGT, we have computed all the terms up to the second order with respect to the masses and degree 5 in eccentricity and inclinations. With these limitations, we obtain very accurate and controlled results (P1), but some estimation needs to be done on higher orders and higher degrees contributions. There exists no way to make comparisons on the whole solution, but the classical Le Verrier type semi-analytical theories give a very efficient comparison at the origin of time of the solution. The solution VSOP82 of Bretagnon (1982) is the most developed solution of this kind so far. It is a semi-analytical solution where the secular terms are expressed in polynomial form, and thus valid only over a few thousand years. The contribution of all

Table 2. Comparison of the fundamental frequencies obtained with different solutions. Unit is arcsec/year

BVW	-Brouwer and Van Woerkom, 1950 . Analytical Theory. Order 1, degree 1 with some terms of order 2 in the Jupiter-Saturn couple							
O1D1	-Solution of order 1 and degree 1							
B74	-Bretagnon, 1974. Analytical theory. Order 2 and degree 3							
B84	-Bretagnon, 1984. Improvement of B74 Relativity, effect of the Moon, least square fit of $E_5, E_6, E_7, E_8, I_6, I_7, I_8$							
NGT	-Present work. Analytical computation of the secular system and then numerical integration Order 2, degree 5, relativity, effect of the Moon.							
ADGSW	-Applegate <i>et al.</i> , 1986. Outer planets numerical integration.							
CMN	-Carpino <i>et al.</i> , 1987. Outer planets numerical integration.							
Dur	-Duriez, 1979. Analytical theory of the 4 outer planets. Order 2, degree 5 in eccentricity, 3 in inclinations.							

	BVW	O1D1	B74	B84	NGT	ADGSW	CMN	Dur
E_1	5.4633	5.4615	5.1991	5.6136	5.5689			
E_2	7.3447	7.3459	7.3461	7.4559	7.4555			
E_3	17.3283	17.3307	17.2205	17.2852	17.3769			
E_4	18.0023	18.0042	17.8573	17.9025	17.9217			
E_5	4.2959	3.7109	4.2072	4.3080	4.2489	4.2421	4.2447	4.3089
E_6	27.7741	22.2868	26.2168	28.1483	27.9606*	28.2318	28.2386	28.3280
E_7	2.7193	2.7014	3.0652	3.1534	3.0695	3.0904	3.0870	3.0982
E_8	0.6333	0.6333	0.6679	0.6735	0.6669	0.6732	0.6727	0.6743
I_1	-5.2015	-5.2007	-5.6109	-5.6109	-5.6043			
I_2	-6.5708	-6.5701	-6.7710	-6.7710	-7.0530			
I_3	-18.7436	-18.7455	-18.8293	-18.8293	-18.8499			
I_4	-17.6333	-17.6358	-17.8188	-17.8188	-17.7614			
I_5	-0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
I_6	-25.7335	-25.7411	-26.2671	-26.2090	-26.3300	-26.3324	-26.3392	-26.2500
I_7	-2.9026	-2.9038	-2.9998	-2.9998	-2.9854	-2.9948	-2.9926	-2.9928
I_8	-0.6775	-0.6777	-0.6914	-0.6914	-0.6917	-0.6923	-0.6914	-0.6905

(*) The corrected value obtained by least square adjustment with VSOP82 is

$$E_6^* = 28.2344$$

degrees is included numerically in the solution. VSOP82 is of order 3 with respect to the masses for the inner planets and the solutions of the outer planets have been extended up to order 6 by an iterative method.

Table 4 of (P2) gives the relative difference between the coefficient of t in the secular terms of VSOP82 and the value of the derivative at the origin of NGT given by the direct substitution of the initial conditions of VSOP82 (which are the ones adopted for NGT) in the second member of the secular system (1). It gives a direct estimate of the precision of the secular differential system (1) which then gives the solution NGT by numerical integration.

The differences in Table 4 of (P2) are used to give an estimate of the precision obtained in the main frequencies of the system given in Table 2. Let us reduce the different solutions of Table 6–13 to their very leading terms

$$\alpha = \sum_i A_i \exp \sqrt{-1} (v_i t + \phi_i) \quad (7)$$

we have then

$$\frac{d\alpha}{dt} = \sum_i \sqrt{-1} v_i A_i \exp \sqrt{-1} (v_i t + \phi_i) \quad (8)$$

at the origin ($t = 0$) we have

$$\frac{d\alpha}{dt} = \sum_i \sqrt{-1} v_i A_i \exp \sqrt{-1} \phi_i \quad (9)$$

If we assume that the error on the value of the derivative comes only from the uncertainty on the frequencies, we have

$$\delta \frac{d\alpha}{dt} = \sum_i \sqrt{-1} \delta v_i A_i \exp \sqrt{-1} \phi_i \quad (10)$$

We thus obtain a system of linear equations which can be solved by least squares and give an estimate of the errors on the main frequencies. This process could even be used to make some corrections on these frequencies, but the least squares method is applied to very few independent observations (30 observations for 15 parameters), and we found that the only completely reliable correction was the very large correction on E_6 . For the others, the results are used only as an indication of the accuracy of the frequencies (Table 3).

As we can see, the determination of the frequencies related to the inner planets are very good, with a precision better than 0.01 arcsec/year. Frequencies in inclination for all the different planets are also very well determined. The main problem is the

Table 3. A least square fit of the fundamental frequencies of NGT on the terms in t of VSOP82 allows to derive either uncertainty on the values of the frequencies ($\delta\nu$) or for the larger one E_6 a correction, which improves the solutions given by Tables 6–13 ($\Delta\nu$). Unit is arcsec/year

	ν	$\delta\nu$	$\Delta\nu$
E_1	5.5689	0.002	
E_2	7.4555	0.007	
E_3	17.3769	0.004	
E_4	17.9217	0.001	
E_5	4.2489	0.007	
E_6	27.9606		+0.2738
E_7	3.0695	0.020	
E_8	0.6669	0.001	
I_1	-5.6043	0.001	
I_2	-7.0530	0.003	
I_3	-18.8499	0.002	
I_4	-17.7614	0.001	
I_5	0.0000		
I_6	-26.3300	0.004	
I_7	-2.9854	0.020	
I_8	-0.6917	0.001	

determination of the frequency E_6 related to Saturn, the correction of which is of 0.2738 arcsec/year which is nearly 1% of its value. Further analysis in the different contributions of VSOP82 order by order shows that this large difference comes from the third order of the outer planets perturbations. The fourth order itself may still give about 10^{-3} of the final result, probably essentially due to the contribution of the $2N_5-5N_6$ and N_7-2N_8 arguments up to high order with respect to the masses. One should note that if this discrepancy on E_6 implies an out of phase on the solution of Saturn after a few hundred thousand years, its effect is much smaller on the solutions of the inner planets and especially on the solutions of the Earth where the amplitude of the term E_6 is small. Even more, in the solution of the inner planets, the correction given in Table 3 can be directly applied to the frequency of the term E_6 in the solution to improve immediately its precision. As we will see it is not so easy for the outer planets.

6. Comparisons of the frequencies with other solutions

The main frequencies are easily identified from the solutions. They usually show very little difference from one planet to another, which gives some idea on the precision of the determinations of these frequencies. The frequencies are compared with the results of other solutions in Table 2. The solution (BVW) is the famous solution of Brouwer and Van Woerkom (1950) which is basically a solution of order one and degree one (compare to the solution (O1D1) of column 2) with the addition of the terms of second order up to degree 5 in eccentricity coming from the great inequality $2\lambda_5-5\lambda_6$ of the pair Jupiter-Saturn. The solution (B74) of Bretagnon (1974) is based on a theory of second order, but limited to degree 3 in eccentricity and inclination. It has been widely used as a reference for the inner planets and especially for the Earth. The improved version (B84) includes also the contribution of relativity and the effect of the perturbations of the Moon for the inner planets (Bretagnon, 1984). This solution is

currently used for paleoclimate computations. In column 5 is our solution (NGT) (Numerical General Theory). The two solutions (ADGSW) and (CMN) are obtained by direct numerical integration of Newton's equations, followed by an averaging method to remove the short period terms. A Fourier analysis is then used to obtain a representation of the solution similar to the analytical solutions. The results of these two numerical integrations are very consistent although the initial conditions and the methods used in the analysis of the results differ. The solution (ADGSW) was made over 200 million years and (CMN) over 9.3 million yr (currently extended to 100 million years). In our study, we found that the influence of the inner planets does not seem to be decisive in the computation of the main long periods of the outer planet system. I would then be inclined to thrust the numerical integrations results up to the level of consistency between the two solutions (ADGSW) and (CMN), that is about 0.01 arcsec/year for the different frequencies. The advantage of the solutions obtained by direct numerical integration is that they include the contributions of the mutual perturbations at all degree and order. The determination of the exact effect of the inner planets and the obtention of initial conditions in the numerical integration are still points which need further analysis to confirm our analysis. The main result which we can use is probably a very good estimate of the frequency E_6 which is difficult to obtain with analytical theories, due to the contribution of high order terms.

The last solution of Table 2 is the solution of Duriez (1979) which was computed for the four outer planets with a completely analytical method up to degree 5 and order 2. Our computations follow essentially Duriez's method with some modifications exposed in (P1) which explain the discrepancies between our solution NGT and Duriez's solution.

As was already stated, up to about 0.01 arcsec/year, the values of the frequencies obtained by the numerical integrations ADGSW and CMN can probably be trusted. At this level of accuracy, there is very good consistency between the two numerical integrations. Moreover, the values of these frequencies enter into the range of uncertainty of the frequencies of our solution NGT deduced from the previous analysis (Table 3) and we have in particular a very good agreement with the values of E_6 given by the numerical integrations and our corrected value obtained by least square adjustment.

7. Comparisons of the amplitudes in the complete solution

In this section we make a comparison on an example between our complete solution and the solutions derived from numerical integration ADGSW and CMN. Comparison is also made with the previous analytical theories B74 and Duriez's. The chosen example consists of the 15 leading terms in the solutions of Jupiter for the two variables z and ζ . This example is chosen because it seems very representative of the general behaviour of the solutions for the outer planets. In ADGSW, amplitudes are given with only 3 digits which explains the * in Table 4.

The first remark is the general very good agreement between the two solutions ADGSW and CMN derived from direct numerical integrations of Newton's laws. The only discrepancies are essentially the absence of the terms Jz_{12} (Jupiter solution, variable z , 12th term) and $J\zeta_6$ (Jupiter solution, variable ζ , 6th term) in ADGSW which were not selected because of their too high frequencies. Here again, this shows a good stability of the solution with respect to small changes in the initial conditions.

Table 4. Comparison of the amplitudes of the principal terms of the solution NGT of Jupiter with the results obtained with numerical integration (CMN and ADGSW) or analytical theories (Dur and B74)

		Jupiter	$z = e \exp \sqrt{-1} \varpi$				
j		ν_j , arcsec/yr	$A_j \times 10^6$	CMN	ADGSW	Dur	B74
1	E_5	4.248956	44130	44166	447**	44544	43416
2	E_6	27.960603	15746	15693	158**	15741	16150
3	E_7	3.069455	1798	1835	186*	1729	2000
4	$2E_6 - E_5$	51.672231	522	571	575	688	298
5	$2E_6 - E_5 + I_6 - I_7$	28.327299	334 (192)	119	117		
6	$E_5 - I_6 + I_7$	27.593819	340 (195)	107	105	12	1
7	$E_6 + E_7 - E_5$	26.781181	180	200	200	419	113
8	$E_6 + E_5 - E_7$	29.140066	176	195	195	412	111
9	$2E_5 - E_7$	5.429258	62	68	69	56	66
10	E_8	0.666784	56	59	59	56	58
11	$2E_6 - E_7$	52.851789	44	50	50	45	23
12	$3E_6 - 2E_5$	75.383894	23	29		14	
13	$3E_6 - 2E_5 + I_6 - I_7$	52.039044	22				
14	$E_6 + I_7 - I_6$	51.305308	20		7		0
15	$2E_5 - E_6 + I_7 - I_6$	3.879769	15				

		Jupiter	$\zeta = \sin i / 2 \exp \sqrt{-1} \Omega$				
j		ν_j , arcsec/yr	$A_j \times 10^6$	CMN	ADGSW	Dur	B74
1	I_6	-26.330003	3148.10	3152.2	316*	3160	3153
2	I_8	-0.691734	580.46	576.4	575	583	587
3	I_7	-2.985415	483.04	482.4	479	483	479
4	$E_5 - E_6 + I_7$	-26.696926	42.75 (24.6)	23.5	23	18	3
5	$E_6 - E_5 + I_6$	-2.618357	29.22 (16.8)	17.5	17		1
6	$2E_6 - I_6$	82.251213	13.76	14.3		10	10
7	$E_5 - E_7 + I_7$	-1.806031	11.26	10.5	11	11	13
8	I_1	-5.604456	11.17				10
9	$-E_5 + E_7 + I_7$	-4.165287	9.63	9.9	10	9	11
10	$E_5 + E_6 - I_6$	58.539542	8.46	8.5	9	9	10
11	$E_5 - E_6 + I_6$	-50.041611	7.56	7.3	7	6	8
12	$2E_5 - I_6$	34.827934	6.26	6.2	6	5	5
13	$E_7 - E_8 + I_7$	-0.577959	5.70	5.3	5	5	3
14	$-E_5 + E_7 + I_6$	-27.509369	4.39	4.6	5	4	5
15	I_2	-7.053069	4.08				4

Let us first analyse the comparisons for the solutions in inclination (variable ζ) which is more straightforward. Initial conditions are not the same in the different solutions, and only the solutions B74 and NGT take into account the effect of the 4 inner planets. We will not therefore consider small discrepancies as significative in the amplitudes of the different terms.

The amplitudes of the main frequencies I_6, I_7, I_8 are very consistent in all the different solutions, but very large differences appear in the amplitudes of some of the other terms.

The solution B74 is lacking the terms of degree 5. They do not appear explicitly in Table 4 but they contribute to the terms of degree 3 ($2E_5 - E_5 - E_6 + I_7 = E_5 - E_6 + I_7$, for example). This may explain some of the large differences found in the amplitudes of the terms of B74. More extended discussion on the changes given by the 5 degree terms can be found in Duriez (1979). The solution of Duriez in the variable ζ is also limited to the degree 3 which may also explain the absence of the term $J\zeta_5 = E_6 - E_5 + I_6$ for the same reason.

The solutions ADGSW, CMN and Dur do not take into account the effects of the inner planets. They are missing the terms I_1 and I_2 . It is quite surprising in fact that the term related to the precessing frequency of Mercury (I_1) gives the 8th

term of the solution for the inclination of Jupiter. The absence of this term will limit the validity of any theory.

Comparison between NGT and CMN shows a very good agreement except for $J\zeta_4 = E_5 - E_6 + I_7$ and for $J\zeta_5 = E_6 - E_5 + I_6$ where the amplitude shows a nearly 100% change. To understand this change, we must go back to the analytical construction of the solution and the effect of the small divisors.

The frequency of $J\zeta_4$ (-26.69 arcsec/year) is very close to I_6 . In the analytical integration we will obtain in the contribution of $J\zeta_4$ the divisor $I_6 - (E_5 - E_6 + I_7) = 0.37$ arcsec/year which is quite small and will increase the amplitude of the term $J\zeta_4$. We will say that there exists a quasi-resonance between the term $E_5 - E_6 + I_7$ and I_6 . As we showed in the previous section, the main difference between our solution NGT and the results of the numerical integrations is a difference in E_6 of 0.27 arcsec/year. This difference does not induce any major difference in the amplitudes of the terms of the solutions, except in the case of a quasi-resonance as in the present case for $J\zeta_4$. The change in E_6 will make a large change in the small divisor $I_6 - (E_5 - E_6 + I_7)$ which becomes equal to 0.64 arcsec/year with the numerical integration frequencies. Exactly the same thing occurs with the term $J\zeta_5$. If we make a proportional adjustment of the amplitude

of the terms $J\zeta_4$ and $J\zeta_5$ according to the change in the small divisor we obtain respectively the new values 24.6×10^{-8} and 16.8×10^{-8} for $J\zeta_4$ and $J\zeta_5$ which is now very close from the results of CMN and ADGSW.

Exactly the same thing occurs for the solutions in the variable z but computations are more involved. The term $Jz_5 = 2E_6 - E_5 + I_6 - I_7$ and $Jz_6 = E_5 - I_6 + I_7$ are in quasi-resonance with E_6 . The proportional change of the divisor due to the change in the value of E_6 does not lead to such a spectacular improvement as in the previous case. The amplitudes of Jz_5 and Jz_6 become respectively 190×10^{-6} and 195×10^{-6} which is already much closer from the results of the numerical integrations. Our empirical corrections are probably a little bit crude in this case involving 5th degree terms. Nevertheless it still explains the large discrepancies observed in the amplitudes of quasi-resonant terms when the small divisor of the resonance involves E_6 . One can see that Jz_7 and Jz_8 are also in quasi-resonance with E_6 but the small divisor is then equal to $E_5 - E_7$ which does not involve E_6 . The amplitude of these terms in NGT and in the numerical integrations are then very close (only 10% difference). We observe about the same small difference on the amplitude of the other terms.

Jz_5 does not appear in Duriez's solution which does not contain the terms of degree 5 involving the inclinations. The large difference in the amplitude of Jz_6 is probably also due to the contribution of missing terms of degree 5.

A very interesting term is $Jz_{13} = 3E_6 - 2E_5 + I_6 - I_7$. This term is of degree 7 and was identified with no possible doubt. This reflects the typical problems which arise in the analytical resolution of the secular system (1). The method used for the integration of such a polynomial system is the method of normalization of Birkhoff (1927) (An exposition of this method adapted to the present case can be found in Brumberg, 1980, or in Laskar, 1984). In this process, we first diagonalize the system (1) by the linear change of variables $\alpha = S\beta$ which leads to the new system

$$\beta = \sqrt{-1}(C\beta + \Psi(\beta, \bar{\beta})) \quad (11)$$

where the matrix C is diagonal. We now search for a change of variable

$$\beta = u + \Gamma(u, \bar{u}) \quad (12)$$

which will transform the system (11) into the "normal form"

$$u = \sqrt{-1}(Cu + F(u, \bar{u})) \quad (13)$$

where $F(u, \bar{u})$ gathers only resonant terms. The system (13) is easily integrable and leads to a quasi-periodic solution

$$u_i = u_i(0) \exp((c_i + \delta c_i)t) \quad (14)$$

In (12) $\Gamma(u, \bar{u})$ is computed degree by degree but is an infinite formal series. Usually, when integrating a system of degree 5, one evaluate $\Gamma(u, \bar{u})$ only up to the same degree 5. Nothing is really known about the remainder, and bad convergence of $\Gamma(u, \bar{u})$ can occur in presence of small divisors in the combinations of the long periods. In this case, the neglected part is much larger than expected and destroys the precision of the solution. Our secular system (1) is of degree 5 and Jz_{13} is a term of degree 7 which appears through the computation of $\Gamma(u, \bar{u})$ even if it is not present in the initial system. In a computation of $\Gamma(u, \bar{u})$ up to degree 5 it would have been neglected. This would be a strong

limitation of the accuracy of the final solution. This shows the difficulty of using Birkhoff's normalization process for real systems when we are far from the ideal initial conditions of very small eccentricities and inclinations. Of course, the amplitude of the term Jz_{13} in our present solution NGT cannot be trusted because we are missing the direct contribution of this term of degree 7 which should be included in the secular system (1). This term is large because it is in quasi-resonance with $2E_6 - E_5$ with the small divisor $E_6 - E_5 + I_6 - I_7$ (the same one again) so its actual size should be also decreased by the same proportional correction of the previous section.

This appearance of higher degree terms in the solution, due to the presence of small divisors in quasi-resonant terms is the main difficulty of either analytical solutions of the secular system or of the analysis of the numerical integration of the secular system. It also reflects the poor convergence of the series involved in Birkhoff normalization process.

8. Analysis of the outer planet solutions

The solutions for the outer planets Jupiter, Saturn, Uranus and Neptune are gathered in Table 10–13. All the different solutions contain 51 terms which correspond to a different precision for each solution. The precision of the analytical representation with respect to the direct output of the numerical integration of the secular system (1) is given in Table 5. All the terms are ordered by decreasing amplitude to show the convergence of the solutions. When the identification of the combination of frequencies giving an argument was possible with no doubt, this combination is given in column 1.

In all these solutions, and especially for the variable ζ , the convergence is very good and the amplitudes of the terms decrease rapidly. We can even say that for these solutions, the 10 leading terms will already give a fair approximation of the solution provided they are correctly determined. The previous analysis showed that in these terms we have already some problems but we prefer to give all 51 terms for possible comparisons in further analysis.

As we already stated, all the frequencies of the terms which are given in these tables are the frequencies directly determined from the Fourier analysis of the different solutions, which explains the slight differences from one planet to another. When the identification of a term was not possible, nothing is given in column 1. Eventually, the same term was identified in the solution for another planet. For example, the first non identified term

Table 5. Maximum eccentricity and inclination over 30 million years. $\Delta e_{\max}$ and $\Delta i_{\max}$ are the maximum values of difference between the numerical integration output of the secular system (1) and its analytical representation given in Tables 6–13

	$e_{\max}$	$\Delta e_{\max} \times 10^3$	$i_{\max}$	$\Delta i_{\max} \times 10^3$
Mercury	0.302	12.17	12.520	1063.0
Venus	0.071	2.27	4.997	171.4
Earth	0.063	1.60	4.362	165.5
Mars	0.125	3.40	8.285	187.2
Jupiter	0.061	0.04	2.062	1.0
Saturn	0.085	0.05	2.595	1.8
Uranus	0.070	0.03	2.761	1.4
Neptune	0.015	0.01	2.387	0.5

in the solution in eccentricity for Jupiter is Jz_{20} which can also be easily identified as Sz_{36} (Saturn solution, variable z , 36th term) or Uz_8 (Uranus solution, variable z , 8th term). Such a term comes probably from higher degree combinations of frequencies. The quasi-resonance with E_5 (0.03 arcsec/year) gives a very small “small divisor” which will increase considerably the amplitude of this term. The same thing applies to most of the other non identified terms. In this case the identification becomes much more difficult because numerous different combinations will lead to the same frequencies and phases. We prefer in this case to give the term just as a representation of the solution with no special physical meaning. When we increase the number of terms, the identification becomes more and more difficult except for the isolated terms which are always easily identified (like Jz_{44} for example).

If the normalization process of Birkhoff converges rapidly, we obtain a solution of the system (1) on a quasi-periodic form valid for all time. The appearance of terms of high degree like Jz_{13} or even more Jz_{20} indicates that the convergence of the Birkhoff process is not sufficient to allow the computation of a completely analytical solution of high precision. It is still possible to obtain an expression which will represent the solution with reasonable accuracy over a limited time. For the outer planets, all the different solutions behave quite well. We have some problems of convergence in the Birkhoff process, but they are limited to terms of small amplitude. Small changes in the initials conditions will not change drastically the solutions, at least over a time span of 10 to 100 million years.

9. Solutions for the inner planets

The solutions for the inner planets are very different from the solutions of the outer planets. They are gathered in Tables 6 to 9. The convergence of the amplitudes of the terms of the solutions is not so good as for the outer planets. We choose to give 80 terms for these solutions. The precision of the analytical representation versus the direct output of the numerical integration of the secular system is given in Table 5. As we can see, the precision is not so good as for the outer planets, because of the poor convergence of the solutions. Even more, it is worthless to increase the number of terms. The improvements obtained after 80 terms are very small compared with the computation time required for the Fourier analysis and, which is more important, the terms have lost all physical signification owing to the limitation of the resolution in the Fourier analysis. The inner planet solutions have a limited accuracy from about 1% to as much as 10% in the inclination of Mercury, compared with less than $\frac{1}{1000}$ for the outer planets with only 50 terms. This internal limitation gives a rather low accuracy on the solution, but it would already be very interesting if we could be sure of this accuracy over 100 millions of years.

9.1. Identification of the frequencies

In the case of the inner planets solutions, the identification of the terms found by Fourier analysis is much more difficult than for the outer planets. Except for the very leading terms and for the main frequencies of the system, the identification is in fact generally impossible. The problems of quasi-resonances which were present in the outer planets solutions on terms of small

amplitude occurs here in some of the leading terms making all identification impossible.

A very significative example is the solution of Mars for the variable z . First of all the two leading terms are the two fundamental frequencies E_3 and E_4 which are very close ($E_4 - E_3 = 0.5448$ arcsec/year) this leads to a beat of about 2 millions years which has already been observed in previous solutions. But the resonances in the solutions of Mars are much more important. In the inclinations, the leading terms are also I_3 and I_4 and we have a strong resonance $2(E_4 - E_3) - (I_4 - I_3) = 0.00126$ arcsec/year (period of about 1 billion year) with a phase of about 183.5 degrees. Owing to the uncertainty on the determination of the frequencies, this value is not known with a great accuracy. Over 100 million years, we can consider that we have probably a strict resonance. This implies that some of the leading terms cannot be separated in the Fourier analysis of a numerical solution over only 30 million years. For example $Maz_4 = 2E_3 - E_4 = E_4 + I_3 - I_4$. The two combinations have the same frequency and the same phase (modulo π). Same thing for $Maz_6 = 2E_4 - E_3 = E_3 + I_4 - I_3$, for $Maz_{15} = 3E_4 - 2E_3 = E_4 + I_4 - I_3$ and for $Maz_{45} = 3E_3 - 2E_4 = E_3 + I_3 - I_4$. Moreover, all these terms are in quasi-resonance with E_3 and E_4 with differences equal to $E_4 - E_3$ and $2(E_4 - E_3)$. Some other interesting terms are Maz_5 , $Maz_7 = Maz_5 + (E_4 - E_3)$, $Maz_{11} = Maz_5 + 2(E_4 - E_3)$. The term Maz_7 has a frequency which is the mean of E_3 and E_4 , which makes all the frequencies of the terms already quoted differ by multiple of $(E_4 - E_3)/2$ (It was Dr. James Williams from JPL who draw my attention on this fact when I showed him the results of my Fourier analysis of the solution). All these quasi-resonances in the terms of the solution are not described here in a very complete manner. They will be more deeply analysed in a next paper (from preliminary analysis, it appears that the frequency $(E_4 - E_3)/2$ is also the frequency of the libration related to the resonance $2(E_4 - E_3) - (I_4 - I_3) \approx 0$).

Apart from the terms already quoted, we have a lot of terms with frequencies ranging from 16.5 to 19.5 arcsec/year. For most of them the identification is not possible. All these terms probably come from higher degree combinations which are in resonances with the other terms already quoted. The spectrum of the solution is completely different from what it is in the outer planets solutions. Instead of well isolated lines, we have cluster of lines in the 16.5 to 19.5 arcsec/year range which makes the spectrum look much more continuous.

The behaviour of the solution in inclination (variable ζ) is very similar to the behaviour of the solution in eccentricity.

The solutions for the Earth are somewhat different. We still find all the same terms of frequencies ranging from 16.5 to 19.5 arcsec/year, but we have also a cluster of terms in the 4 to 8 arcsec/year range, all these terms in quasi-resonance with one of E_1 , E_2 , or E_5 . The solutions of the Earth are more stable because all the leading terms are single frequencies. Their amplitude will not change very much under little changes of the initial conditions. The same resonance as in Mars solution exists, but with a much smaller amplitude. We obtain nearly the same discussion for the inclinations. The solutions for Venus are very similar to the solutions for the Earth.

The solutions for Mercury are different from the previous ones. We are in presence of lots of quasi-resonances in the 4 to 8 arcsec/year range, and effects over a very extended span of time which makes our integration over 30 million years a little insufficient for easy Fourier analysis. Even the very leading terms

are not identified. The term Mez_2 cannot be identified although there is no doubt about its determination. The only terms that are really identified in this case are the fundamental frequencies of the system. These solutions can still be used as convenient representation of our solution over 30 million years with the internal precision given in Table 5.

9.2. Comparisons with previous solutions

Apart from the Brouwer Van Woerkom solution which is reduced to the Laplace-Lagrange solution for the inner planets, the only solution including higher degree and order so far are the solutions B74 and the improved version B84 of Bretagnon (1974, 1984). This solution is completely analytical, of degree 3 and order 2. As was already stated in (Laskar, 1984), the computation of the secular system is quite similar to the one made in the present work, and the essential difference comes from the integration of this system. Bretagnon uses an analytical method of resolution which is very similar to Birkhoff normalization process for low degrees. Although this process may work in the computation of a solution of moderate accuracy for the outer planets, all the problems encountered in the identification of the terms in the inner planets solutions shows that the normalization process does not converge well in the case of the inner planets, because of the existence of lots of quasi-resonances, even in the leading terms of the solutions. The analytical solution of Bretagnon is very different from the present one in its analytical representation, and even some very large terms will not appear in B84 when they are due to quasi-resonances. We preferred in this case to give the different plots of some of the most characteristic example (Fig. 1a–h). The two solutions B84 and NGT will coincide at the origin, and be very similar over a few 100,000 years, but they will diverge rapidly afterwards.

The solutions of Venus and the Earth in NGT are close to B84 over a longer time than the solutions of Mars or Mercury (about 1.5 million years). In these solutions, the effects of resonances appear only on terms of small amplitude.

The comparison of the two solutions NGT and B84 with the ephemeris VSOP82 at the origin (over about 1000 years) showed that the accuracy of NGT is from 10 to 100 times better than B84 (P2). We can thus consider that NGT is always much closer to the right solution than B84.

10. Practical use of the solutions

In practice, one wishes to know how to use the solutions and with which uncertainty over a given time. This question is difficult to answer since for the inner planets we have not identified the terms that may come from higher degree combinations of frequencies. We still can give an approximate answer.

First of all, one should use the corrected value of E_6 given in Table 3. This means that the adopted value of E_6 will be 28.2344 arcsec/year with an empirical uncertainty of 0.01 arcsec/year. For all the other frequencies the adopted value is the computed value with the uncertainty given in Table 3. In the different solutions of Tables 6–13 one should merely change the value of E_6 for the corrected value given above, that is Ez_{10} and Mz_8 for the inner planets. As there are no small divisors involved in the amplitudes of these terms, no change is needed in the amplitudes. For the outer planets, E_6 is respectively Jz_2 , Sz_1 , Uz_4 and Nz_6 .

The uncertainty on the solutions is estimated by the uncertainty on the main frequencies given in Table 3. Beside the truncation error given in Table 5, the average error of all the different solutions is estimated to be about 3% of the maximum amplitude after 1 million years and 30% after 10 million years.

Tables 6–13 of this paper can be obtained directly through EARN/BITNET network on computer readable form on request to the author at LASKAR@FRIAP51.

11. Conclusion

The analytical and numerical solution NGT presented here gives a solution for the orbital evolution of the solar system over 10 million years. A new set of fundamental frequencies is given with for the first time a computation of an estimate of their precision. This solution includes Lunar perturbations and relativistic effects. It should be preferred to Bretagnon's solution in long term studies on the solar system spanning more than 1 million years. For the outer planets, after the correction given in Sect. 10, we obtain a very good agreement with the recent long term numerical integrations. For the inner planets, there exist no such numerical integrations and the analytical solution is still the only one which can give any results.

In the outer planets solutions the convergence is rapid and it is possible to identify easily most of the terms. On the contrary, the convergence of the inner planets solutions is slow and even some of the leading terms of the solutions are not possibly identified. This suggests that the solutions for the outer planets are more stable than the solutions of the inner planets. In the inner planets solutions, we have lots of quasi-resonances in the long term frequencies, and even the strict resonance $2E_3 - 2E_4 = I_3 - I_4$. The analysis of the effects of all these resonances in the secular evolution of the solar system is still under study, but it seems that the solutions for Mercury and Mars are the most susceptible of instabilities.

One of the main conclusions of this study is that in the inner solar system we have long term phenomena of very long period. Integration of the solutions over 1 million years does not reveal much more than the Brouwer Van Woerkom or Bretagnon's solution. In Fig. 2a–d we can see some features of several millions years period, and our 30 million years integration is even a little short for the analysis of Mercury's solution. We must aim now to integrations of about 100 millions years in order to improve our analysis. In this context the inner planets (and thus the Earth) seem out of reach of direct numerical integrations of the whole solar system and we still need to trust analytical works for the moment. Hopefully long term numerical integrations of the whole solar system will be made soon in order to have a better estimate of the accuracy of analytical works like NGT. Even a 100,000 years numerical integration of the whole solar system would be informative considering that the longest reliable numerical integration of the whole solar system including relativity and lunar perturbations is still DE102 (Newhall et al., 1983) spanning (only !) 4400 years.

Note. We just received a paper from (Richardson and Walker, 1987) reporting a 1 million years numerical integration of the 9 planets of the solar system. This is very promising and can provide a basis for some comparisons but the above statement is still valid since this last numerical integration does not include relativistic or lunar perturbations.

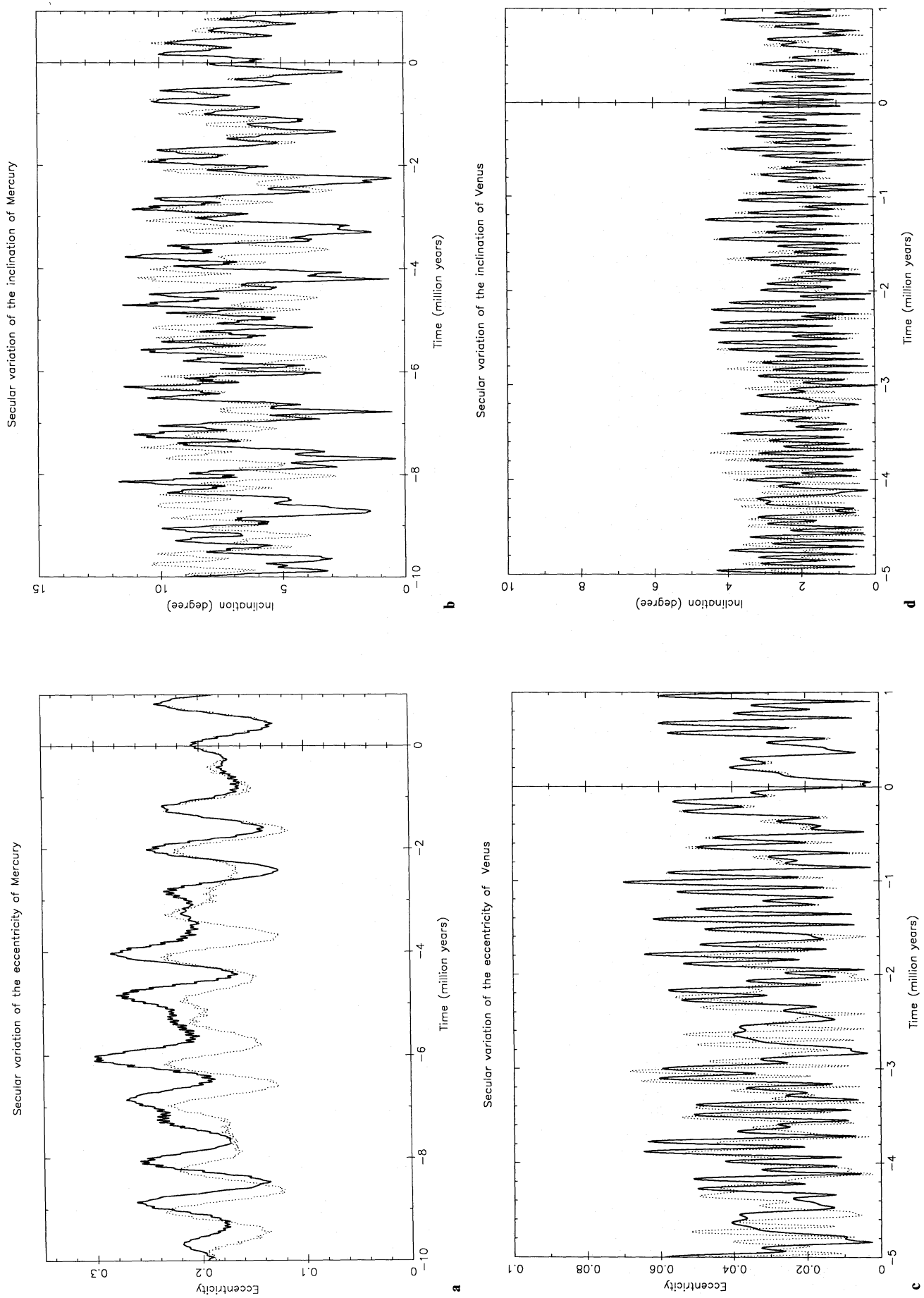


Fig. 1a–h. Comparison of the solution in eccentricity and inclination of NGT (solid line) and the solution B84 of (Bretagnon, 1984) (dotted line) for the inner planets. For the clarity of the figures, the comparison is made over 6 million years in the case of Venus and the Earth, and over 11 million years for Mercury and Mars. The initial conditions of the two solutions are the same but the solutions diverge after about 1 million years. In particular the eccentricity of Mercury in NGT reaches nearly 0.3 while in B84 it stays below 0.24.

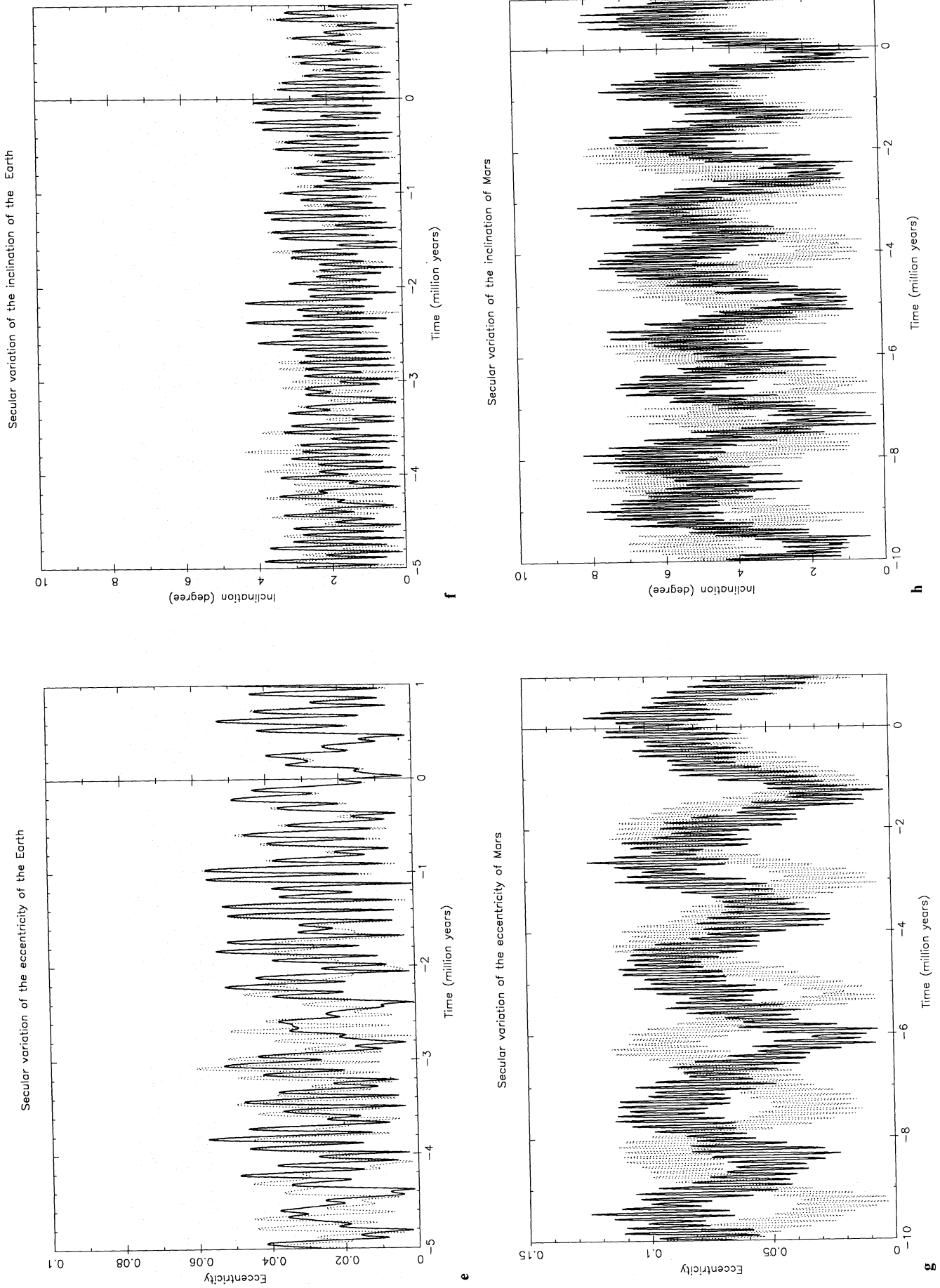


Fig. 1 (continued)

Tables 6–13. Solutions for the secular evolution of the planets over 10 million years. For the outer planets, when the identification was possible, the corresponding combination of fundamental frequencies is given in the first column. All the solutions are on the form

$$x = \sum_i A_i \exp \sqrt{-1}(\nu_i t + \phi_i)$$

The origin of time is J2000 (JD 2451545.0) and the reference frame is the ecliptic and equinox J2000

Mercury $z = e \exp \sqrt{-1} \omega$					Mercury $\zeta = \sin i/2 \exp \sqrt{-1} \Omega$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	5.568529	232736	17306668	75.977574	0	0.000000		1372070	107.594161
2	5.478493	236561	4785465	256.051183	1	-5.604456	231244	4279585	17.416694
3	4.250265	304922	4453487	28.396897	2	-7.052893	183754	2439128	324.319159
4	5.640596	229762	3897512	118.853071	3	-6.962603	186137	1793732	129.085405
5	5.696562	227505	3280716	64.177646	4	-5.515329	234981	1638897	9.708957
6	5.436160	238403	3038297	274.562485	5	-7.149550	181270	1356291	152.816113
7	7.455272	173836	2768956	20.027316	6	-7.003568	185048	1313707	297.114107
8	5.517818	234875	2563571	227.256472	7	-6.861219	188887	1077693	124.097424
9	5.350675	242212	1864760	85.748533	8	-7.189363	180266	967548	123.564855
10	6.100972	212425	1306737	310.439256	9	-5.458035	237448	680966	29.952243
11	6.924977	187148	1279835	135.399150	10	-6.808859	190340	525424	152.295541
12	5.736258	225931	1117739	115.618474	11	-5.563939	232928	519593	188.519274
13	5.605881	231185	942326	130.527171	12	-5.690577	227744	510718	229.662811
14	5.392617	240328	882169	67.857720	13	-7.095850	182641	429806	345.843672
15	5.314534	243859	829776	62.348769	14	-7.410146	174895	423259	105.049925
16	4.344656	298297	771600	194.135547	15	-7.275434	178133	411900	317.627469
17	4.211643	307718	658338	178.675307	16	-5.158036	251258	384297	88.458176
18	6.015780	215433	606339	124.356667	17	-5.067375	255753	352048	265.682662
19	6.055397	214023	574702	142.935441	18	-6.745178	192137	342759	114.828377
20	6.216993	208460	562433	321.664432	19	-5.829321	222324	300275	194.304131
21	4.116295	314846	547654	4.233718	20	-7.494675	172922	293784	66.551373
22	5.244889	247097	540077	232.824403	21	-5.744090	225623	291774	21.933322
23	5.034066	257445	525206	22.818583	22	-3.716962	348671	268260	145.525059
24	4.386246	295469	519956	227.031962	23	-7.236090	179102	261315	3.654596
25	4.154858	311924	515136	16.142113	24	-5.401016	239954	242367	354.433481
26	6.812550	190237	469706	132.118520	25	-6.914567	187430	238144	258.072595
27	4.295918	301681	444334	221.164999	26	-5.198025	249325	234785	60.863858
28	7.020150	184611	427464	298.824842	27	-5.645296	229571	226776	178.907794
29	5.206970	248897	421889	260.286351	28	-7.351896	176281	219854	86.558862
30	4.015809	322724	419200	215.105082	29	-9.039876	143364	197745	206.063267
31	5.829047	222334	415728	281.246337	30	-8.380532	154644	190938	271.271186
32	5.278373	245530	405408	249.716864	31	-5.302425	244416	188532	281.942573
33	5.097168	254258	369021	100.160383	32	-6.680533	193996	171575	130.137720
34	6.138910	211112	357805	130.552760	33	-8.945523	144876	171135	27.130997
35	5.780758	224192	346248	207.839750	34	-6.632849	195391	160052	177.037241
36	6.889813	188103	274949	263.959821	35	-2.984341	434266	159813	137.804219
37	3.978848	325722	273437	187.985074	36	-18.849996	68753	156939	243.823860
38	6.764846	191578	259543	133.530951	37	-6.887918	188155	156379	326.510126
39	7.547300	171717	244264	183.289976	38	-4.282851	302602	145433	248.720458
40	6.666392	194407	238196	310.619609	39	-5.017074	258317	130181	229.038113
41	6.970923	185915	228989	323.861079	40	-6.529473	198484	122675	34.496203
42	7.053830	183729	222505	356.307529	41	-8.292352	156288	122305	115.242738
43	5.967159	217188	221466	163.840848	42	-17.761383	72967	120029	287.275940
44	4.069607	318458	218708	115.447303	43	-5.788024	223910	118674	170.627102
45	6.258583	207075	215001	8.307416	44	-8.806689	147160	118324	9.060807
46	5.142138	252035	211515	52.555759	45	16.740352	77417	115361	120.038647
47	4.440052	291888	201682	186.429674	46	-5.364523	241587	115256	33.467705
48	6.188000	209437	176485	261.970212	47	18.095064	71621	112958	20.128421
49	5.863919	221012	169181	7.605548	48	-5.922541	218825	110536	9.097354
50	17.922413	72311	165869	153.097031	49	-8.901804	145588	106701	13.437936
51	17.376902	74581	158226	317.550710	50	-9.135399	141865	105454	30.628992
52	3.903050	332048	148542	4.357524	51	-5.246080	247041	95154	281.072841
53	6.721752	192806	148204	271.817912	52	-8.473185	152953	95000	87.583483
54	6.480148	199995	147392	264.827069	53	-5.105743	253831	93146	292.959775
55	4.979429	260270	133762	177.353966	54	16.866354	76839	90364	141.161263
56	-16.775908	77253	133397	138.476045	55	-6.473723	200193	82417	337.811279
57	6.583485	196856	133205	99.249709	56	-7.583793	170890	81668	243.402841
58	6.420659	201848	133093	307.886307	57	-7.541460	171850	81442	270.707796
59	3.587484	361255	130983	150.458188	58	-9.176677	141227	81412	16.162848
60	2.895360	447612	127786	139.490055	59	-5.871790	220716	81290	224.477732
61	-18.224364	71113	127548	93.232422	60	-8.233488	157405	78312	60.311209
62	6.383698	203017	118311	106.527952	61	-8.519053	152129	75110	81.719180
63	7.497956	172847	114055	349.799239	62	-7.311302	177259	72565	330.051103
64	7.361212	176057	113238	23.251996	63	-9.080938	142716	71908	199.526028
65	8.811982	147072	112672	93.731360	64	-0.691773	1873446	70222	22.246819
66	6.849511	189210	112617	206.678878	65	-8.999184	144013	69127	186.152518
67	7.592973	170684	110711	209.358883	66	-8.630817	150159	68605	248.161346
68	6.339295	204439	109027	270.051438	67	16.807939	77106	66316	229.139922
69	-18.136105	71459	108413	264.567956	68	-4.228897	306462	63536	280.038504
70	3.467206	373788	107960	308.683587	69	-3.844331	337119	63435	144.840571
71	3.716865	348680	107317	316.581620	70	-6.581825	196905	62899	164.377885
72	3.936964	329187	100522	59.459843	71	-8.752961	148064	62694	181.343832
73	8.912429	145414	93424	251.729555	72	-8.424759	153832	62663	250.045131
74	8.708969	148812	85114	80.061899	73	18.190087	71247	61410	178.078605
75	-19.679372	65855	82524	20.803803	74	-5.964776	217275	61103	359.293926
76	7.100850	182513	82095	348.268455	75	15.423241	84029	59946	82.751415
77	3.504538	369806	79913	325.365426	76	-8.346246	155279	56403	288.235197
78	4.528526	286185	78230	169.514503	77	-8.668942	149499	55903	194.277584
79	3.059338	423621	77662	134.924441	78	-8.851854	146410	55521	173.080690
80	3.547261	365352	77413	146.828999	79	-7.729783	167663	52391	288.301033

Table 7

Venus $z = e \exp \sqrt{-1} \omega$					Venus $\zeta = \sin i/2 \exp \sqrt{-1} \Omega$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	7.455565	173829	2109545	199.744469	0	0.000000		1377076	107.583043
2	4.249054	305009	1971999	30.459172	1	-18.850211	68752	872193	65.339653
3	17.922569	72311	1331270	331.873900	2	-5.604299	231251	695917	17.123730
4	17.377703	74578	1017605	130.215456	3	-17.761481	72966	584757	107.043504
5	5.568568	232734	608960	76.793394	4	-7.052893	183754	336520	143.869559
6	17.105263	75766	312224	213.767970	5	-18.305162	70799	283617	83.590225
7	6.922595	187213	275647	137.634553	6	-6.962954	186127	269037	311.758232
8	16.833846	76987	245483	289.935738	7	-5.513981	235038	232339	8.095041
9	17.655585	73404	222421	217.057439	8	-7.004642	185020	185569	119.405617
10	5.480876	236458	172153	253.270998	9	-7.150390	181248	170673	332.867817
11	17.328567	74789	166902	109.084024	10	-6.861532	188879	169389	304.134487
12	7.363048	176014	166863	208.303597	11	-19.394140	66824	160382	222.547728
13	5.635452	229972	153391	121.623228	12	-7.191316	180217	115488	303.797551
14	17.827006	72698	136559	265.999984	13	-19.123738	67769	105370	329.656760
15	17.164382	75505	131169	13.542894	14	-5.686944	227890	100095	229.889167
16	5.692226	227678	130646	67.162601	15	-2.985786	434056	99485	142.143700
17	17.966543	72134	128689	201.610897	16	-6.811847	190256	93277	329.280733
18	7.548550	171688	125147	9.187381	17	-5.458211	237440	88626	34.409374
19	18.196122	71223	123968	66.909672	18	-5.835239	222098	85368	195.022461
20	18.012723	71949	119308	227.357195	19	-5.563783	232935	77671	184.888895
21	7.017591	184678	115365	300.829637	20	-18.893227	68596	76168	47.857659
22	17.874065	72507	105333	289.376286	21	-5.746746	225518	70560	19.037938
23	5.434774	238464	95198	276.836871	22	-6.752893	191917	66093	300.627809
24	17.713699	73163	94238	1.851587	23	-0.692144	1872442	66025	23.974152
25	19.012385	68166	93023	192.374581	24	-6.134731	211256	57771	128.952298
26	18.462012	70198	90264	4.362565	25	-7.096397	182627	54257	167.116894
27	16.560223	78259	85711	190.648086	26	-18.775020	69027	46716	179.993584
28	17.430641	74351	81042	2.472764	27	-6.638474	195225	46004	355.599926
29	6.102886	212358	78342	305.237898	28	-18.035719	71857	45905	193.793946
30	17.063653	75950	78273	182.659581	29	-7.275727	178126	45403	131.925825
31	16.288426	79565	72923	87.676800	30	-7.407529	174957	44823	284.103392
32	7.059084	183593	72529	348.608686	31	-5.936877	218296	42684	12.670673
33	5.519576	234800	69186	228.182668	32	-18.964700	68337	42673	131.668646
34	6.801280	190552	67458	129.878889	33	-5.790329	223821	40972	181.204786
35	7.309778	177296	64727	179.314266	34	-6.681334	193973	40790	301.173236
36	17.481001	74137	63648	356.469417	35	-6.544473	198029	38828	242.757834
37	7.408975	174922	54788	29.812783	36	-18.582010	69744	37420	174.151610
38	5.346378	242407	54784	76.521346	37	-7.236266	179097	36707	179.952750
39	3.069143	422267	54130	121.380636	38	-6.912634	187482	36378	103.230886
40	17.287200	74968	52899	105.924984	39	-6.478664	200041	35857	170.065826
41	17.597843	73645	52495	262.564849	40	-17.675223	73322	32471	74.617553
42	17.211343	75299	50300	249.614850	41	-17.844623	72626	32163	354.288258
43	6.891981	188044	49942	262.362211	42	-5.643401	229648	31842	165.617873
44	18.259431	70977	47893	224.352658	43	-5.398750	240055	30536	345.782466
45	18.059782	71761	46345	115.222535	44	-7.495241	172909	28361	243.820201
46	7.500612	172785	45531	169.179321	45	-5.147372	251778	27933	102.146885
47	16.631618	77923	44919	315.521478	46	-5.183045	250046	27018	59.154383
48	5.733308	226047	44076	103.692573	47	-19.461266	66593	26475	250.270037
49	7.599602	170535	42169	22.164067	48	-6.071452	213457	24369	197.456247
50	16.895573	76706	38343	91.658301	49	-6.383979	203008	23986	230.547099
51	6.018612	215332	37572	133.474165	50	-5.296655	244682	23352	278.744124
52	16.968988	76374	36659	276.266715	51	-5.059992	256126	23050	266.308361
53	5.385246	240657	35114	84.976542	52	-7.345236	176440	22868	251.220397
54	18.424563	70340	34346	287.421304	53	-17.561369	73798	22235	286.447078
55	6.967388	186009	33023	334.713034	54	-19.055479	68011	21257	88.628046
56	6.228477	208076	30275	331.760372	55	-18.377601	70520	20342	125.305302
57	18.153762	71390	29816	56.445357	56	-5.982471	216632	19577	353.451685
58	6.840976	189446	29471	178.630717	57	-18.714582	69250	18818	94.147966
59	7.243570	178917	26649	188.929786	58	-5.367492	241453	18439	44.776350
60	6.062976	213756	25205	127.629009	59	-18.990881	68243	17636	25.919308
61	6.739710	192293	23612	171.042344	60	-5.108711	253684	17443	292.049369
62	16.516074	78469	23114	166.438857	61	-6.597469	196438	16598	13.883371
63	16.792040	77179	22972	240.540088	62	-6.182004	209640	16450	278.376243
64	5.240338	247312	22922	241.205567	63	-6.255087	207191	16387	51.534981
65	7.106319	182372	22341	19.914703	64	-5.888079	220105	16217	276.757905
66	16.238554	79810	21426	66.769059	65	-17.882815	72471	16039	119.864286
67	6.142113	211002	19874	132.012418	66	-6.435550	201381	15783	16.861066
68	16.727540	77477	19768	247.719570	67	-17.974199	72103	14571	139.063748
69	6.363561	203659	19420	175.894491	68	-17.795376	72827	14183	323.375121
70	6.193450	209253	19353	298.402627	69	-19.512583	66418	13833	311.060571
71	17.565608	73780	18634	214.290164	70	-19.602071	66115	13629	78.139358
72	6.698002	193490	18608	306.270127	71	-18.536513	69916	13617	358.241475
73	18.915298	68515	18490	131.389842	72	-18.482716	70119	12902	230.760522
74	5.313616	243901	18387	77.529704	73	-6.720756	192835	12434	281.685670
75	17.751403	73008	17643	325.837439	74	-19.939481	64996	11743	211.148660
76	16.019372	80902	17336	187.155626	75	-18.674027	69401	11131	251.304557
77	8.000418	161991	16395	50.469005	76	-8.383012	154598	11078	108.599361
78	18.388890	70477	16213	209.591780	77	-9.040110	143361	10406	26.210282
79	18.811611	68893	15264	199.594929	78	-19.425222	66717	9951	212.301028
80	6.455062	200772	13883	255.856705	79	-16.671996	77735	9828	147.989771

Table 8

Earth $z = e \exp \sqrt{-1\omega}$					Earth $\zeta = \sin i/2 \exp \sqrt{-1\Omega}$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	4.248976	305014	1897001	30.613021	0	0.000000		1377224	107.584264
2	7.455487	173831	1631751	199.811097	1	-18.850133	68752	798880	245.275610
3	17.922491	72311	1298909	151.997510	2	-5.604358	231248	515448	17.092978
4	17.377742	74578	813631	310.028552	3	-17.761344	72967	402485	286.939567
5	5.568471	232738	387047	77.058878	4	-7.052737	183758	284848	143.691432
6	17.104716	75768	255684	36.737898	5	-18.304928	70800	268489	263.476756
7	17.654277	73409	224715	38.339829	6	-6.962954	186127	224745	311.907333
8	6.922888	187205	199607	143.317597	7	-5.513825	235045	170037	8.033427
9	16.833084	76991	183885	110.602530	8	-7.004642	185020	154555	119.415527
10	27.960642	46350	152836	126.426025	9	-7.150155	181254	146128	332.581197
11	17.329172	74787	133092	287.612479	10	-6.861649	188875	139612	303.906805
12	17.826517	72700	131928	87.910250	11	-19.393984	66824	136764	41.333570
13	7.363107	176012	127119	207.854098	12	-26.330042	49221	133576	127.730350
14	17.966230	72135	122639	22.356505	13	-7.190984	180225	97416	302.731053
15	18.196357	71223	122535	248.338542	14	-19.123718	67769	96140	149.577002
16	18.012859	71948	115231	46.053080	15	-2.985806	434053	89461	142.209681
17	18.464923	70187	108585	180.405296	16	-6.812257	190245	76207	329.815268
18	5.480739	236464	107889	253.472094	17	-5.687881	227852	73737	230.338124
19	17.874670	72504	106811	106.742036	18	-18.892954	68597	70771	228.756309
20	17.717449	73148	98232	177.096332	19	-0.692105	1872547	64551	23.954607
21	5.637581	229885	98055	119.800741	20	-5.835219	222099	64516	195.802291
22	7.548843	171681	97695	7.458757	21	-5.457957	237451	64375	33.992450
23	17.165456	75500	89902	195.541472	22	-5.564213	232917	56644	184.825149
24	7.018099	184665	87993	298.666882	23	-6.753010	191914	53528	298.867151
25	5.692988	227648	81629	67.462170	24	-5.746648	225522	51936	20.336317
26	16.559286	78264	75883	9.010922	25	-6.135942	211214	46909	130.580295
27	16.288328	79566	70627	268.405839	26	-7.096475	182625	46667	166.151146
28	17.063556	75951	68303	356.553594	27	-18.774864	69028	42744	358.972630
29	19.012717	68164	66867	12.487296	28	-7.281801	177977	40348	138.865689
30	17.430133	74353	61517	185.466186	29	-18.964075	68339	39059	310.863584
31	5.435125	238448	59594	275.737596	30	-7.412412	174841	36585	287.477655
32	17.483833	74125	56695	172.482398	31	-6.635837	195303	34988	350.632805
33	3.069397	422232	56648	120.881333	32	-6.538301	198216	32432	233.017994
34	7.058713	183602	55182	350.744686	33	-5.936740	218301	32254	14.569983
35	6.103257	212345	54616	305.549188	34	-7.238141	179051	30384	173.708098
36	17.596241	73652	53094	81.997976	35	-5.790817	223802	30137	181.547382
37	7.310423	177281	48907	179.362501	36	-6.681939	193955	29993	308.448574
38	17.286664	74971	46950	296.080374	37	-6.912126	187496	29992	103.381283
39	6.802277	190524	46769	129.866397	38	-18.033844	71864	29401	6.977723
40	5.519029	234823	45431	228.848317	39	-18.583982	69737	28770	358.874443
41	17.209117	75308	45400	76.113474	40	-6.473293	200207	27350	160.153382
42	7.410107	174896	43837	28.989588	41	-5.643596	229640	26198	169.309320
43	18.259197	70977	43468	49.167963	42	-7.495163	172911	24844	246.487916
44	16.631208	77925	42265	148.108759	43	-19.459567	66599	24202	69.190923
45	18.060055	71760	40535	295.691354	44	-17.844760	72626	22922	175.205224
46	7.499635	172808	38588	168.119243	45	-5.398809	240052	22452	345.876033
47	5.346671	242393	37435	74.220879	46	-17.674812	73324	21794	206.396612
48	6.954321	186358	32876	355.821607	47	-17.217945	75270	21471	309.759500
49	7.596418	170606	32834	28.043581	48	-19.055713	68011	20453	268.450750
50	16.967953	76379	32032	97.389637	49	-5.146044	251843	20394	102.034860
51	18.425188	70338	31746	122.666909	50	-18.373500	70536	20245	302.461969
52	16.899624	76688	29840	263.430839	51	-7.357208	176153	20217	262.934811
53	18.157208	71376	28764	224.957155	52	-6.070183	213502	19707	193.247889
54	5.733152	226053	26833	101.965176	53	-5.180545	250166	19368	56.499662
55	6.020565	215262	26364	131.733810	54	-18.715364	69247	18108	270.674294
56	17.136072	75629	26117	221.174830	55	-6.384878	202979	17786	231.976174
57	6.885828	188212	24127	266.250164	56	-5.061535	256048	17283	264.796330
58	5.387012	240578	22519	86.878893	57	-5.297221	244656	16688	279.170633
59	17.564494	73785	22239	19.039806	58	-18.990861	68243	16025	205.846704
60	16.237734	79814	21569	245.500936	59	-5.981455	216669	15913	353.532787
61	6.226661	208137	21471	333.827955	60	-17.561779	73796	14839	101.517121
62	7.245465	178870	20835	182.635148	61	-6.593778	196548	14571	8.101066
63	18.390179	70472	19803	50.826279	62	-6.250079	207357	13578	43.329104
64	6.838125	189525	19466	182.824898	63	-5.367589	241449	13312	45.071407
65	16.018864	80904	19398	7.799465	64	-6.178664	209754	12813	278.392835
66	7.999481	162010	18455	50.394971	65	-6.718510	192899	12763	281.202373
67	16.788837	77194	17479	44.029078	66	-5.108282	253705	12759	287.697184
68	16.517129	78464	17170	337.136548	67	-5.890110	220029	12554	277.503745
69	6.746936	192087	16734	160.346542	68	-19.510728	66424	12318	133.104569
70	7.724412	167779	16631	118.062945	69	-6.433851	201434	12075	4.209053
71	7.106475	182368	16427	20.508333	70	-19.601250	66118	11894	258.333055
72	16.715598	77532	16187	65.005623	71	-18.487344	70102	11530	61.461009
73	16.078060	80606	15558	119.927732	72	-9.039993	143362	11369	25.979732
74	15.746529	82303	15435	64.916613	73	-8.376450	154719	11322	115.961917
75	6.064792	213692	15185	131.594214	74	-17.883166	72470	10221	301.940311
76	16.663910	77772	14970	4.654480	75	-18.541806	69896	10161	194.386015
77	18.117824	71531	14637	7.332000	76	-18.220810	71127	9471	159.388286
78	6.141449	211025	14537	131.594937	77	-18.673480	69403	9370	65.685463
79	6.700053	193431	13971	294.448095	78	-16.672250	77733	9324	328.200686
80	18.914477	68518	13931	309.135289	79	-19.939560	64996	9231	31.221436

Table 9

Mars $z = e \exp \sqrt{-1\omega}$					Mars $\zeta = \sin i/2 \exp \sqrt{-1\Omega}$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	17.921710	72314	4697234	333.843257	0	0.000000		1375241	107.587176
2	17.376863	74581	3781713	312.399445	1	-17.761481	72966	3552594	286.709047
3	4.248917	305018	2030249	30.692973	2	-18.849918	68753	1368762	67.263903
4	16.832850	76992	1770697	110.866158	3	-18.305729	70797	564704	87.742848
5	17.105068	75767	1378617	31.990991	4	-17.216245	75277	531504	127.631151
6	18.465294	70185	1223112	358.562896	5	-26.330023	49221	456346	127.752831
7	17.651581	73421	1115320	229.652752	6	-18.036481	71854	310276	16.197008
8	27.960622	46350	719760	126.367604	7	-17.844349	72628	205923	178.938034
9	17.876330	72498	556671	281.750425	8	-17.674960	73324	189894	204.547907
10	17.328997	74787	531349	281.033519	9	-18.887700	68616	181212	33.626883
11	18.193477	71234	518616	76.944893	10	-19.123699	67769	178837	331.174656
12	17.825970	72702	513723	263.744544	11	-17.563264	73790	164653	100.068286
13	17.428239	74362	470424	177.930574	12	-17.968691	72125	136947	302.447070
14	17.164343	75505	469885	197.150286	13	-18.095806	71618	122260	228.606443
15	19.010041	68174	423316	15.777767	14	-18.573260	69777	118441	337.947404
16	17.961464	72154	421925	217.216597	15	-5.604202	231255	108366	16.659659
17	18.013035	71947	336456	220.457870	16	-18.960208	68353	105934	105.492896
18	17.730329	73095	327013	324.929318	17	-17.796509	72823	96002	87.303624
19	16.782431	77223	306287	86.973556	18	-18.342280	70656	95620	48.129534
20	7.455604	173828	299282	199.180035	19	-17.129901	75657	81016	50.272630
21	17.063692	75950	275844	355.595858	20	-18.256521	70988	78417	101.803239
22	17.606671	73608	269464	221.941713	21	-18.724992	69212	76425	94.535803
23	18.424270	70342	227814	300.781136	22	-17.441774	74304	75847	204.956012
24	17.563889	73787	220819	161.390264	23	-17.635527	73488	67986	110.312501
25	16.885202	76753	212786	311.763686	24	-17.724001	73121	66781	249.647516
26	18.541161	69898	181810	282.927722	25	-7.053108	183748	65883	144.957652
27	18.385980	70488	160157	242.451248	26	-18.787755	68981	65631	169.130319
28	16.565282	78235	159290	357.939740	27	-18.427395	70330	65631	97.663195
29	17.208199	75312	158285	68.682414	28	-2.985474	434101	63897	141.702150
30	15.746412	82304	139160	245.843831	29	-19.302265	66830	62310	43.967466
31	16.985442	76300	125397	38.675900	30	-0.691773	1873446	60870	23.480866
32	17.783579	72876	124976	39.234906	31	-17.018137	76154	58980	300.617177
33	17.690398	73260	123373	90.437101	32	-17.331126	74778	56022	111.691650
34	17.464927	74205	113343	208.378398	33	-17.287669	74966	55062	346.270534
35	18.161973	71357	111451	0.073815	34	-17.877229	72494	53329	298.562137
36	17.284340	74981	109312	285.635454	35	-6.963110	186123	51833	311.798757
37	18.506825	70028	108426	236.666960	36	-18.519189	69981	48060	123.542964
38	16.017848	80909	105198	169.887663	37	-19.067636	67968	41425	104.880826
39	16.712532	77546	99129	138.285690	38	-18.999982	68210	41306	4.520941
40	18.917993	68506	98760	284.671523	39	-18.636753	69540	39365	40.158623
41	19.280139	67219	98341	120.350965	40	-18.923637	68485	36877	201.957970
42	18.263377	70961	98279	246.878426	41	-7.002513	185076	36568	118.034667
43	16.637074	77898	87536	128.450011	42	-19.159938	67641	36510	253.703715
44	18.062868	71749	83293	119.265014	43	-5.513708	235050	36305	5.509858
45	16.283660	79588	77284	98.385121	44	-7.148319	181301	34416	327.622257
46	3.069455	422224	73617	121.489889	45	-16.678695	77703	33628	158.979979
47	16.940308	76503	70600	189.236302	46	-16.128577	80354	32092	347.613212
48	18.963978	68340	68474	303.206524	47	-6.860594	188904	30560	298.460307
49	5.569213	232707	67548	73.362253	48	-18.164083	71349	30456	0.717092
50	18.345034	70645	62344	22.716930	49	-7.189558	180261	29700	301.393902
51	18.123937	71507	60902	206.506442	50	-17.496313	74072	28425	232.791412
52	17.248207	75138	60146	82.287605	51	-17.597296	73647	25971	252.039784
53	19.091581	67883	57569	296.909394	52	-17.177175	75448	24573	123.204276
54	16.526680	78418	55207	15.804432	53	-18.002449	71990	24340	68.246378
55	18.090474	71639	54055	40.079048	54	-19.254076	67310	22800	13.912605
56	18.823505	68850	46669	331.345715	55	-17.396023	74499	22661	118.419132
57	16.080365	80595	45960	314.174779	56	-16.880827	76773	22328	32.571361
58	18.598522	69682	45722	203.127625	57	-18.387093	70484	21552	113.201183
59	18.651987	69483	43640	198.935781	58	-19.936063	65007	21430	21.822338
60	6.920876	187259	42491	145.413962	59	-19.200415	67498	18145	86.243538
61	15.979989	81101	38943	146.709572	60	-18.604245	69661	17923	229.166680
62	15.699236	82551	38563	244.513106	61	-6.813234	190218	17608	330.778441
63	19.548900	66295	37523	41.301998	62	-19.445730	66647	17010	58.577324
64	17.523939	73955	35054	18.279314	63	-18.690160	69341	16986	271.916892
65	9.353176	138562	31489	171.760501	64	-19.329307	67048	16865	108.111411
66	19.395253	66820	31173	130.452146	65	-18.213896	71154	16697	347.899043
67	15.811811	81964	30120	29.023479	66	-16.975277	76346	16199	178.599204
68	19.052295	68023	29822	298.036456	67	-5.459070	237403	14669	33.253945
69	25.400584	51022	29349	91.643562	68	-5.687354	227873	14613	233.275888
70	16.355181	79240	28070	262.573295	69	-5.835180	222101	13441	196.138860
71	16.743868	77401	28014	9.385539	70	-16.785986	77207	13431	330.178515
72	16.486652	78609	27592	333.132684	71	-16.285262	79581	13016	94.997994
73	19.140203	67710	26522	223.094058	72	-17.091552	75826	12760	307.505521
74	7.363107	176012	25485	218.181551	73	-25.786146	50259	12471	150.463581
75	18.867281	68690	22924	72.492493	74	-17.051993	76002	11705	5.416377
76	16.396303	79042	22667	106.946865	75	-6.750999	191971	11614	285.604903
77	26.491610	48921	21032	311.811703	76	-5.745457	225569	11419	16.542474
78	15.887141	81575	20814	214.791000	77	-19.296955	67160	11310	180.733642
79	19.482340	66521	19668	299.162163	78	-26.874323	48224	11226	105.176256
80	-53.444661	24249	19409	62.092549	79	-18.472501	70158	11117	51.080130

Table 10

Jupiter					$z = e \exp \sqrt{-1-\varpi}$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	E_5	4.248956	305016	4413049	30.642727	1	E_5	4.248956	30.642727
2	E_6	27.960603	46350	1574620	306.496518	2	E_6	27.960603	306.496518
3	E_7	3.069455	422224	179782	121.235372	3	E_7	3.069455	121.235372
4	$2E_5 - E_5$	51.672231	25081	52205	42.373249	4	$2E_5 - E_5$	51.672231	42.373249
5	$2E_6 - E_6 + I_7$	28.327299	45750	33422	28.724860	5	$2E_6 - E_6 + I_7$	28.327299	28.724860
6	$E_5 - I_6 + I_7$	27.593819	46967	30969	44.270192	6	$E_5 - I_6 + I_7$	27.593819	44.270192
7	$E_6 + E_7 - E_5$	26.781181	48392	18023	216.907827	7	$E_6 + E_7 - E_5$	26.781181	216.907827
8	$E_6 + E_5 - E_7$	29.140066	44474	17584	215.939497	8	$E_6 + E_5 - E_7$	29.140066	215.939497
9	$2E_5 - E_7$	5.429258	238706	6241	119.051016	9	$2E_5 - E_7$	5.429258	119.051016
10	E_8	0.666784	1943656	5615	72.631169	10	E_8	0.666784	72.631169
11	$2E_6 - E_7$	52.851789	24521	4389	311.547274	11	$2E_6 - E_7$	52.851789	311.547274
12	$3E_6 - 2E_5$	75.383894	17192	2272	138.027779	12	$3E_6 - 2E_5$	75.383894	138.027779
13	$3E_6 - 2E_5 + I_6 - I_7$	52.039044	24904	2186	124.381311	13	$3E_6 - 2E_5 + I_6 - I_7$	52.039044	124.381311
14	$E_6 + I_7 - I_6$	51.305308	25260	2038	140.339465	14	$E_6 + I_7 - I_6$	51.305308	140.339465
15	$2E_5 - E_6 + I_7 - I_6$	3.879769	334040	1539	132.131517	15	$2E_5 - E_6 + I_7 - I_6$	3.879769	132.131517
16	E_2	7.455526	173830	1376	19.296417	16	E_2	7.455526	19.296417
17	$2E_6 - 2E_5 + E_7$	50.492731	25667	1253	312.746735	17	$2E_6 - 2E_5 + E_7$	50.492731	312.746735
18	$2E_5 - E_6$	-19.462731	66588	1176	295.053187	18	$2E_5 - E_6$	-19.462731	295.053187
19	$-E_6 + 2I_6$	-80.620542	16075	1174	128.898237	19	$-E_6 + 2I_6$	-80.620542	128.898237
20	4.277617	302972	1142	162.989808	718	20	4.277617	302972	1142
21	E_1	5.568900	232720	933	255.295124	21	E_1	5.568900	255.295124
22	$E_6 - I_7 + I_6$	4.614695	280841	919	295.466356	22	$E_6 - I_7 + I_6$	4.614695	295.466356
23	$E_7 - I_6 + I_7$	28.030346	46235	827	214.670714	23	$E_7 - I_6 + I_7$	28.030346	214.670714
24	$E_7 - I_6 + I_7$	26.413723	49065	791	315.664214	24	$E_7 - I_6 + I_7$	26.413723	315.664214
25	$-E_5 + 2I_6$	4.213479	307584	718	117.432194	25	$-E_5 + 2I_6$	4.213479	117.432194
26	$-E_5 + 2I_6$	-56.908993	22773	684	44.590633	26	$-E_5 + 2I_6$	-56.908993	44.590633
27	$2E_7 - E_5$	27.992467	46298	650	107.611473	27	$2E_7 - E_5$	27.992467	107.611473
28	$E_5 - I_6 + I_6$	1.894642	684034	618	204.842857	28	$E_5 - I_6 + I_6$	1.894642	204.842857
29	$E_5 - I_6 + I_6$	28.694376	45165	539	137.737014	29	$E_5 - I_6 + I_6$	28.694376	137.737014
30	$E_5 - I_6 + I_6$	30.579136	42381	538	11.224065	30	$E_5 - I_6 + I_6$	30.579136	11.224065
31	$E_5 - I_6 + I_6$	28.297369	45799	443	242.269066	31	$E_5 - I_6 + I_6$	28.297369	242.269066
32	$E_5 - I_6 + I_6$	27.137906	47756	426	311.966758	32	$E_5 - I_6 + I_6$	27.137906	311.966758
33	$E_5 - I_6 + I_6$	27.911317	46432	419	234.157359	33	$E_5 - I_6 + I_6$	27.911317	234.157359
34	$E_8 - I_7 + I_8$	2.960739	437728	419	311.519633	34	$E_8 - I_7 + I_8$	2.960739	311.519633
35	$E_8 - I_7 + I_8$	4.315047	300344	411	335.123879	35	$E_8 - I_7 + I_8$	4.315047	335.123879
36	$E_8 - I_7 + I_8$	28.775409	45038	370	311.089675	36	$E_8 - I_7 + I_8$	28.775409	311.089675
37	$3E_6 - E_5 - E_7$	76.563577	16927	367	47.396327	37	$3E_6 - E_5 - E_7$	76.563577	47.396327
38	$3E_6 - E_5 - E_7$	27.209603	47630	337	219.766539	38	$3E_6 - E_5 - E_7$	27.209603	219.766539
39	$E_5 - I_6 + I_8$	27.565588	47015	323	75.198837	39	$E_5 - I_6 + I_8$	27.565588	75.198837
40	$E_5 - I_6 + I_8$	29.887246	43362	317	284.643243	40	$E_5 - I_6 + I_8$	29.887246	284.643243
41	$E_6 - E_5 + E_8$	24.377221	53164	310	172.366706	41	$E_6 - E_5 + E_8$	24.377221	172.366706
42	$E_6 - E_5 + E_8$	25.342451	51139	297	62.449244	42	$E_6 - E_5 + E_8$	25.342451	62.449244
43	$E_5 - E_6 + E_7$	-20.641684	62785	296	24.764343	43	$E_5 - E_6 + E_7$	-20.641684	24.764343
44	$E_5 - E_6 + E_7$	5.498415	235704	295	54.923877	44	$E_5 - E_6 + E_7$	5.498415	54.923877
45	$E_5 + E_6 - E_8$	27.827589	46572	295	95.750063	45	$E_5 + E_6 - E_8$	27.827589	95.750063
46	$E_5 + E_6 - E_8$	31.542481	41087	293	262.153837	46	$E_5 + E_6 - E_8$	31.542481	262.153837
47	$E_5 + E_6 - E_8$	27.872578	46497	250	166.407355	47	$E_5 + E_6 - E_8$	27.872578	166.407355
48	$E_5 + E_6 - E_8$	29.508987	43918	216	116.931996	48	$E_5 + E_6 - E_8$	29.508987	116.931996
49	$E_5 - I_5 + I_6$	6.919880	187286	212	323.736874	49	$E_5 - I_5 + I_6$	6.919880	323.736874
50	$E_5 - I_5 + I_6$	-22.081762	58690	186	232.026059	50	$E_5 - I_5 + I_6$	-22.081762	232.026059
51	$E_5 - I_5 + I_6$					51	$E_5 - I_5 + I_6$		

Jupiter					$\zeta = \sin i / 2 \exp \sqrt{-1-\Omega}$
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	
0	I_5	0.000000	1377458	107.586989	
1	I_6	-26.330003	314810	307.664495	
2	I_8	-0.691734	1873552	58046	
3	I_7	-2.985415	434110	141.649322	
4	$E_5 - E_6 + I_7$	-26.696926	48544	225.677741	
5	$E_6 - E_5 + I_6$	-2.618357	494966	43.523999	
6	$2E_6 - I_6$	82.251213	15756	305.286347	
7	$E_5 - E_7 + I_7$	-1.806031	717595	51.005233	
8	I_1	-5.604456	231244	197.552922	
9	$-E_5 + E_7 + I_7$	-4.165287	311142	52.870069	
10	$E_5 + E_6 - I_6$	58.539542	22138	29.383305	
11	$E_5 - E_6 + I_6$	-50.041611	25898	31.937019	
12	$2E_5 - I_6$	34.827934	37211	113.629387	
13	$E_7 - E_8 + I_7$	-0.577959	2242372	4.675708	
14	$-E_5 + E_7 + I_6$	-27.509369	47111	38.102574	
15	I_2	-7.053069	183749	323.606883	
16	$E_6 - E_5 + 2I_6 - I_7$	-25.963152	49916	208.509891	
17	$E_5 - E_7 + I_6$	-5.514137	235032	380	
18	$E_5 - E_7 + I_6$	-25.150553	51529	36.895272	
19	$-E_7 + E_8 + I_8$	-6.962896	186129	339	
20	$-E_7 + E_8 + I_8$	-3.094034	418870	335.539215	
21	$-E_7 + E_8 + I_8$	-0.728090	1779999	71.550711	
22	$-E_6 + E_7 + I_7$	-0.662605	1955917	169.624417	
23	$-E_6 + E_7 + I_7$	-27.876856	46490	137.073736	
24	$-E_5 + E_7 + I_8$	-1.872505	692121	295.228563	
25	$-E_5 + E_7 + I_8$	-7.004525	185023	299.623380	
26	$-E_5 + E_7 + I_8$	-7.150604	181243	151.554673	
27	$-E_5 + E_7 + I_8$	-6.861102	188890	122.281866	
28	$-E_5 + E_7 + I_8$	-5.687510	227867	49.277566	
29	I_4	-17.761305	72967	107.084402	
30	$E_5 - E_7 + I_8$	-5.457762	237459	207.867735	
31	$E_5 - E_7 + I_8$	0.488001	2655730	294.483988	
32	$E_5 - E_7 + I_8$	-24.402913	53108	287.226836	
33	$E_5 - E_7 + I_8$	-7.191648	180209	121.636600	
34	$E_5 - E_7 + I_8$	-5.563783	232935	8.593109	
35	$E_5 - E_7 + I_8$	-6.810968	190281	147.940439	
36	$E_5 - E_7 + I_8$	-5.835747	222079	15.767438	
37	$E_5 - E_7 + I_8$	-0.763684	1697036	300.107920	
38	$E_5 - E_7 + I_8$	-5.746531	225527	193.622306	
39	$E_5 - E_7 + I_8$	-6.134653	211258	306.526709	
40	$E_5 - E_7 + I_8$	11.483288	112859	279.542882	
41	$E_5 - E_7 + I_8$	-6.751135	191967	109.670359	
42	$E_5 - E_7 + I_8$	-26.360628	49164	359.764650	
43	$-E_6 + E_7 + I_8$	82.617979	15686	27.371844	
44	$-E_6 + E_7 + I_8$	-25.589034	50646	202.348020	
45	$-E_6 + E_7 + I_8$	-5.790133	223829	355.902962	
46	$-E_6 + E_7 + I_8$	-7.095518	182650	348.994945	
47	$-E_6 + E_7 + I_8$	-5.935353	218352	196.193598	
48	$-E_6 + E_7 + I_8$	81.883731	15827	44.117898	
49	$-E_6 + E_7 + I_8$	33.648356	38515	204.268348	
50	$-E_6 + E_7 + I_8$	0.597245	2169964	95.857029	

Table 11

Saturn						Saturn					
j	I_j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	I_j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
0	I_5	0.000000		1377389	107.586942	1	E_6	27.960583	46350	4816143	126.521420
1	I_6	-26.330003	49221	784256	127.663855	2	E_5	4.248956	305016	3314591	30.643731
2	I_8	-0.691773	1873446	55960	23.485141	3	$-E_5 + 2E_6$	51.672231	25081	175566	222.374299
3	I_7	-2.985415	434110	39215	141.650604	4	E_7	3.069436	42227	151099	121.282265
4	$E_5 - E_6 + I_7$	-26.690945	48544	10810	45.700021	5	$-E_5 + 2E_6 + I_6 - I_7$	28.327241	45751	102847	208.892519
5	$2E_6 - I_6$	82.251232	15756	3423	125.194597	6	$E_5 - I_6 + I_7$	27.593897	46966	94345	224.198962
6	$E_5 + E_6 - I_6$	58.539562	22138	1976	209.401924	7	$-E_5 + E_6 + E_7$	26.781161	48392	54749	36.970003
7	$2E_5 - I_6$	34.827895	37211	1585	293.679594	8	$E_5 + E_6 - E_7$	29.140066	44474	54225	35.940887
8	I_1	-5.604280	231251	1434	197.426240	9	$2E_6 - E_7$	52.851809	24521	14828	131.541690
9	I_4	-17.761403	72967	1301	107.076900	10	$2E_5 - E_6$	-19.462692	66588	14286	114.803420
10	$E_5 - E_7 + I_7$	-1.808895	717649	1041	50.846994	11	$-2E_5 + 3E_6$	75.383855	17192	7782	318.070141
11	$-E_5 + E_6 + 2I_6 - I_7$	-27.509544	47110	1008	218.427543	12	$-2E_5 + 3E_6 + I_6 - I_7$	52.039123	24904	7359	304.272910
12	$E_5 + E_6 + 2I_6 - I_7$	-25.963426	49916	971	29.173356	13	$E_6 - I_6 + I_7$	51.305347	25260	6932	320.265998
13	$E_5 - E_7 + I_6$	-25.150475	51529	962	216.883783	14	E_8	0.666980	1943087	5752	73.720010
14	$E_5 - E_6 + I_6$	-50.041572	25898	881	211.923378	15	$2E_5 - E_7$	5.429531	238694	4259	118.110307
15	$-E_6 + E_7 + I_7$	-27.876817	46490	759	316.953005	16	$-2E_5 + 2E_6 + E_7$	50.492711	25667	4217	132.778144
16	$-E_6 + E_7 + I_7$	-4.165365	311137	671	52.676491	17	$-E_6 + 2I_6$	-80.620523	16075	2512	309.105256
17	I_2	-7.052951	183752	584	323.546560	18	$E_7 - I_6 + I_7$	28.001764	46282	2398	49.027071
18	$E_7 - E_8 + I_7$	-0.578448	2240479	550	6.385727	19		3.880726	333958	1790	129.040166
19		-5.514020	235037	484	188.543628	20		-20.642270	62783	1780	205.642911
20		-6.962993	186126	475	132.280053	21	$E_5 - E_6 + E_7$	28.693556	45166	1639	317.085063
21	$E_5 - E_6 + E_8$	-7.004798	185016	324	299.996020	22		28.299459	45795	1610	65.743050
22		-24.402893	53108	310	107.423164	23		30.579078	42381	1431	190.486873
23		-6.861493	188880	298	122.568108	24	$E_5 + I_5 - I_6$	4.614909	280828	1382	294.077352
24		-7.150468	181246	295	152.024546	25	$E_6 + I_6 - I_7$	76.563519	16927	1266	227.310353
25		-0.662409	1956494	256	171.524060	26	$-E_6 + 3E_6 - E_7$	27.138902	47754	1260	129.059524
26		-3.092600	419064	251	334.022534	27		27.565881	47014	1169	252.628745
27	$-E_5 + E_6 + I_6$	-2.618318	494974	245	42.911323	28		28.773905	45040	1154	130.525480
28		-1.870883	692720	207	292.172302	29		7.455721	173826	1140	18.443851
29		-5.687530	227866	205	50.046162	30		27.212454	47625	987	33.074865
30		-7.191414	180214	197	122.933522	31		29.887070	43363	952	103.122716
31		-0.728754	1778377	196	72.423404	32		24.377572	53163	948	350.916625
32		82.618077	15686	191	207.295171	33	$E_5 - I_6 + I_8$	31.543184	41086	905	81.990385
33		-5.457684	237463	181	208.416559	34	$-E_5 + E_6 + E_8$	4.278144	302935	901	166.303533
34		-25.589249	50646	176	22.922454	35	$E_5 + E_6 - E_8$	25.342568	51139	883	241.824701
35		-26.360706	49164	174	185.538557	36		28.087796	46141	824	154.140899
36		-6.811886	190255	169	150.440130	37		27.872520	46497	774	305.525300
37		-5.564271	232914	165	8.188206	38		26.035376	49778	721	325.647293
38		35.195149	36823	161	14.998549	39		29.508440	43919	709	298.341525
39		-5.835551	222086	156	14.243132	40		5.569994	232675	690	251.267030
40	$E_5 - E_7 + I_8$	-0.485267	2670695	144	294.353203	41		1.630725	794738	612	145.482020
41		-5.747430	225492	139	195.773718	42		27.828664	46570	595	264.372163
42		-17.215699	75289	137	306.078039	43		54.290294	23871	590	286.405974
43		20.726095	62529	130	58.529327	44		55.253877	23455	589	179.169116
44		-6.752522	191928	121	115.680569	45		1.892259	684895	578	208.851134
45		-6.134809	211253	117	305.648331	46		52.485355	24692	574	227.179903
46		-19.394530	66822	117	223.892832	47	$-E_5 + 2E_7$	28.048862	46205	557	92.856956
47		-27.063827	47886	116	325.206178	48		28.359253	45699	545	176.924019
48		-18.035954	71856	115	192.624671	49		4.212991	307619	527	119.117085
49		-26.291557	49293	111	231.512944	50					
50		-1.439062	900586	102	133.531197	51					

Table 12

Uranus					$z = e \exp \sqrt{-1-\epsilon}$				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	E_6	4.248897	305020	210.722209	1	E_6	4.248897	373253	210.722209
2	E_7	3.069455	422224	121.237583	2	E_7	3.069455	2907876	121.237583
3	E_8	0.666843	1943486	73.753451	3	E_8	0.666843	163776	73.753451
4	E_6	27.960622	46350	1533958	4	E_6	27.960622	46350	1533958
5	$-E_5 + 2E_7$	1.890130	685667	41024	5	$-E_5 + 2E_7$	1.890130	685667	41024
6	$2E_6 - E_7$	5.428535	238738	300.056629	6	$2E_6 - E_7$	5.428535	40273	300.056629
7	$E_8 - I_7 + I_8$	2.960817	437717	7865	7	$E_8 - I_7 + I_8$	2.960817	437717	7865
8	$-E_5 + E_6 + E_7$	4.277168	303004	6489	8	$-E_5 + E_6 + E_7$	4.277168	303004	6489
9	$E_6 + I_6 - I_7$	26.781044	48392	5519	9	$E_6 + I_6 - I_7$	26.781044	48392	5519
10	$2E_5 - E_6$	4.616999	280701	5474	10	$2E_5 - E_6$	4.616999	5474	110.512673
11	$E_7 + I_7 - I_8$	-19.462731	66588	4050	11	$E_7 + I_7 - I_8$	-19.462731	66588	4050
12	$-E_5 + 2E_6$	0.774602	1673116	3939	12	$-E_5 + 2E_6$	0.774602	3939	58.633348
13	$E_5 - E_6 + E_7$	51.672231	25081	3928	13	$E_5 - E_6 + E_7$	51.672231	3928	42.390494
14	$E_5 - E_6 + E_7$	-20.642231	62783	3662	14	$E_5 - E_6 + E_7$	-20.642231	3662	205.526495
15	$E_5 - I_6 + I_7$	4.212912	307625	3585	15	$E_5 - I_6 + I_7$	4.212912	307625	73.064912
16	$E_5 - E_7 + E_8$	27.593487	46967	3562	16	$E_5 - E_7 + E_8$	27.593487	46967	44.745969
17	$-E_6 + 2E_6 + I_6 - I_7$	1.845024	702429	3313	17	$-E_6 + 2E_6 + I_6 - I_7$	1.845024	702429	168.395425
18	$E_5 + I_7 - I_8$	28.327866	45750	3209	18	$E_5 + I_7 - I_8$	28.327866	45750	28.006264
19	$E_5 + I_7 - I_8$	1.958810	661626	2162	19	$E_5 + I_7 - I_8$	1.958810	661626	144.124467
20	$E_5 - I_6 + I_7$	4.314227	300401	1725	20	$E_5 - I_6 + I_7$	4.314227	300401	295.658522
21	$E_5 - E_7 + E_8$	2.699381	480110	1557	21	$E_5 - E_7 + E_8$	2.699381	480110	221.800348
22	$E_5 - E_7 + E_8$	3.437783	376987	1551	22	$E_5 - E_7 + E_8$	3.437783	376987	200.692733
23	$E_5 - E_7 + E_8$	2.323592	557757	1224	23	$E_5 - E_7 + E_8$	2.323592	557757	48.883257
24	$E_5 - E_7 + E_8$	5.363253	241644	1082	24	$E_5 - E_7 + E_8$	5.363253	241644	4.861483
25	$E_5 - E_7 + E_8$	3.886507	333461	1054	25	$E_5 - E_7 + E_8$	3.886507	333461	304.344078
26	$E_5 - E_7 + E_8$	3.112452	416392	942	26	$E_5 - E_7 + E_8$	3.112452	416392	268.022723
27	$E_5 - E_7 + E_8$	4.172514	310604	875	27	$E_5 - E_7 + E_8$	4.172514	310604	310.785057
28	$E_5 - E_7 + E_8$	-10.219727	126813	870	28	$E_5 - E_7 + E_8$	-10.219727	126813	252.658451
29	$E_5 - E_7 + E_8$	3.033158	427277	858	29	$E_5 - E_7 + E_8$	3.033158	427277	99.019743
30	$E_5 - E_7 + E_8$	5.477145	236619	855	30	$E_5 - E_7 + E_8$	5.477145	236619	338.709912
31	$E_5 - E_7 + E_8$	29.140027	44474	848	31	$E_5 - E_7 + E_8$	29.140027	44474	35.558494
32	$E_5 - E_7 + E_8$	0.699058	1853922	776	32	$E_5 - E_7 + E_8$	0.699058	1853922	16.763429
33	$E_5 - E_7 + E_8$	6.652506	194813	703	33	$E_5 - E_7 + E_8$	6.652506	194813	72.747720
34	$E_5 - E_7 + E_8$	3.179120	407659	672	34	$E_5 - E_7 + E_8$	3.179120	407659	110.735458
35	$E_5 - E_7 + E_8$	-19.095741	67868	629	35	$E_5 - E_7 + E_8$	-19.095741	67868	196.777728
36	$E_5 - E_7 + E_8$	-21.822044	59389	589	36	$E_5 - E_7 + E_8$	-21.822044	59389	116.648788
37	$E_5 - E_7 + E_8$	4.120806	314501	573	37	$E_5 - E_7 + E_8$	4.120806	314501	207.820731
38	$E_5 - E_7 + E_8$	6.540684	198144	536	38	$E_5 - E_7 + E_8$	6.540684	198144	94.574002
39	$E_5 - E_7 + E_8$	7.235445	297561	532	39	$E_5 - E_7 + E_8$	7.235445	297561	72.309381
40	$E_5 - E_7 + E_8$	-6.747170	192080	451	40	$E_5 - E_7 + E_8$	-6.747170	192080	43.830450
41	$E_5 - E_7 + E_8$	1.262740	1026339	440	41	$E_5 - E_7 + E_8$	1.262740	1026339	69.321302
42	$E_5 - E_7 + E_8$	-9.040188	143359	432	42	$E_5 - E_7 + E_8$	-9.040188	143359	342.587933
43	$E_5 - E_7 + E_8$	-7.926125	163509	410	43	$E_5 - E_7 + E_8$	-7.926125	163509	314.910714
44	$E_5 - E_7 + E_8$	26.414113	49064	405	44	$E_5 - E_7 + E_8$	26.414113	49064	315.382260
45	$E_5 - E_7 + E_8$	0.084020	15424813	381	45	$E_5 - E_7 + E_8$	0.084020	15424813	334.460283
46	$E_5 - E_7 + E_8$	6.054929	214040	379	46	$E_5 - E_7 + E_8$	6.054929	214040	87.803732
47	$E_5 - E_7 + E_8$	4.406665	294099	360	47	$E_5 - E_7 + E_8$	4.406665	294099	181.715965
48	$E_5 - E_7 + E_8$	51.305406	25260	340	48	$E_5 - E_7 + E_8$	51.305406	25260	140.343890
49	$E_5 - E_7 + E_8$	4.086149	317169	332	49	$E_5 - E_7 + E_8$	4.086149	317169	78.449596
50	$E_5 - E_7 + E_8$	5.566244	232832	328	50	$E_5 - E_7 + E_8$	5.566244	232832	78.320450

Uranus					$\zeta = \sin i / 2 \exp \sqrt{-1-\Omega}$				
j	I_j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg				
0	I_6	0.000000		1377351	107.586768				
1	I_7	-2.985415	434110	890207	321.644465				
2	I_8	-0.691890	1873129	56153	203.635670				
3	$-E_5 + E_6 + I_6$	-2.618298	494978	52868	223.437236				
4	I_6	-26.330003	49221	34707	307.665387				
5	$-E_5 + E_7 + I_7$	-4.164858	311175	18505	232.160742				
6	$E_5 - E_7 + I_7$	-1.806012	717603	18121	231.173759				
7	$-E_7 + E_8 + I_8$	-3.093877	418891	6046	155.894357				
8	$E_6 - E_7 + I_6$	-1.438906	900684	2717	313.014795				
9	$2E_5 - I_7$	11.483269	112859	1906	99.609416				
10	$-E_5 + E_7 + I_8$	-1.871176	692612	1705	113.995681				
11	$E_5 - E_6 + I_7$	-26.697063	48544	1244	225.923311				
12	$E_5 + E_7 - I_7$	10.303670	125780	1206	10.263795				
13	I_1	-5.603440	231286	1064	16.436367				
14		-3.798502	341187	961	134.618856				
15	$-E_5 + E_6 + I_7$	20.726213	62529	757	237.407714				
16	$-E_5 + E_6 + I_8$	-4.274570	303188	735	68.504594				
17	$E_5 + E_6 - I_7$	35.194935	36823	673	195.483495				
18		-2.251329	575659	515	305.607299				
19	$E_5 - E_7 + I_6$	-25.150417	51529	453	36.862996				
20	$E_5 + E_6 - I_6$	58.539523	22138	448	209.386336				
21		-2.953356	438822	439	275.354227				
22	$E_5 - E_7 + I_8$	0.488079	2655305	376	291.134436				
23		-5.515544	234972	375	10.101094				
24	$E_5 - E_6 + I_6$	-50.041670	25898	365	212.149398				
25	$E_8 + E_7 - I_8$	8.010301	161791	331	309.132956				
26	$-E_6 + E_7 + I_7$	-27.876524	46490	329	136.447005				
27	$-E_5 + E_7 + I_6$	-27.509642	47110	321	218.463484				
28	$2E_7 - I_7$	9.124345	142037	318	280.884381				
29		58.906504	22000	280	291.439513				
30	I_2	-7.053694	183733	280	144.611378				
31	$E_7 - E_8 + I_7$	-0.577881	2242675	249	188.898590				
32	$-E_7 + E_8 + I_7$	-5.395039	240220	247	91.576139				
33		21.905830	59162	240	326.959477				
34		-3.021634	428906	232	180.890088				
35		-6.963716	186107	227	313.699541				
36		-3.060315	423485	207	2.087320				
37		34.827504	37211	198	114.278699				
38		6.830722	189731	180	220.572108				
39		34.015454	38100	178	105.748335				
40		-5.454969	237581	174	17.368184				
41		-5.346222	242414	173	123.348139				
42		-0.619042	2093558	159	143.617204				
43		-7.004759	185017	157	119.667185				
44		-6.863114	188835	145	307.687312				
45		-5.687159	227881	144	232.014438				
46		33.648141	38516	143	24.843593				
47		2.589188	500542	138	7.472644				
48		-7.149315	181276	138	333.025950				
49		-1.913079	677441	137	62.998472				
50		-5.562435	232991	135	183.026284				

Table 13

Neptune					Neptune				
j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg	j	ν_j , arcsec/yr	P_j , years	$A_j \times 10^8$	ϕ_j , deg
1	0.666921	1943258	911797	73.645347	0	I_6	0.000000	1377351	107.586768
2	3.069494	422219	360541	301.195148	1	I_8	-0.091988	1872864	203.750011
3	4.248956	305016	177053	30.642278	2	I_7	-2.985415	434110	106.064
4	$E_7 + I_7 - I_8$	1671515	15856	59.591434	3	$-E_5 + E_6 + I_6$	-2.618220	494992	8082
5	E_6	46350	9951	306.520290	4	$E_7 - E_8 + I_7$	-0.584257	2218203	5823
6	$2E_5 - E_7$	238750	4634	120.502310	5	I_6	-26.329984	49221	3752
7	$-E_5 + E_7 + E_8$	252386	4584	164.291145	6	$-E_5 + E_7 + I_7$	-4.164740	311183	2652
8	$E_6 - E_7 + E_8$	701197	4096	159.686795	7	$E_5 - E_7 + I_8$	0.487708	2657326	1648
9	1.890716	685454	1933	269.607351	8	$-E_5 + E_7 + I_6$	-1.872192	692236	1463
10	1.957248	662154	1461	146.732040	9	$E_6 - E_7 + I_6$	-1.439941	900036	1236
11	6.652213	194822	1035	257.171153	10	$E_6 - E_7 + I_6$	-0.721645	1795897	906
12	0.699175	1853611	876	15.794347	11	$-E_7 + E_8 + I_6$	-3.094170	418852	860
13	0.629070	2060185	833	298.224854	12	$E_6 - E_7 + I_7$	-1.805113	717960	702
14	2.956364	438376	793	139.327008	13	$E_5 - E_6 + I_7$	0.596561	2172450	575
15	5.472781	236808	581	165.510685	14	$E_5 - E_6 + I_7$	-0.657273	1971784	509
16	5.359718	241803	514	188.236197	15	$E_6 + E_7 - I_7$	10.303728	125779	502
17	7.831616	165483	514	346.274862	16	$E_6 + E_7 - I_8$	8.010164	161794	324
18	6.543965	198045	466	269.822808	17	$E_6 + E_7 - I_8$	-0.618358	2095872	304
19	0.733754	1766259	454	250.986213	18	$2E_5 - I_8$	9.189841	141025	299
20	0.592108	2188789	449	164.521344	19	$2E_5 - I_7$	11.483288	112859	281
21	3.041009	426174	437	316.425609	20	$2E_5 - I_7$	-0.763489	1697470	244
22	3.106163	417235	417	81.325363	21	$2E_7 - I_7$	9.125790	142015	216
23	-1.736406	746369	334	27.529225	22	$-E_5 + E_8 + I_8$	-4.275312	303135	173
24	2.328240	556643	303	222.488280	23	$-E_7 + E_8 + I_7$	-5.389316	240475	170
25	4.615808	280774	298	293.451893	24	$-E_7 + E_8 + I_7$	0.006085	212966318	169
26	0.555283	2333943	287	34.554142	25	$-E_5 + I_7 + I_8$	-3.795944	341417	167
27	-7.926457	163503	249	135.155453	26	$2E_7 - I_8$	6.830976	189723	156
28	3.438994	376854	241	17.637374	27	$-E_5 + E_8 + I_7$	-6.566376	197369	152
29	1.817242	713168	233	215.683391	28	$-E_5 + E_8 + I_7$	-0.546006	2373599	150
30	0.811290	1597455	232	29.142645	29	$E_7 - E_8 + I_8$	1.711738	757125	148
31	-6.746917	192087	228	224.344986	30	$E_7 - E_8 + I_8$	-0.803439	1613066	139
32	51.672465	25081	225	41.840595	31	$-E_5 + E_8 + I_7$	-2.252149	575450	109
33	-2.915750	444482	220	298.043813	32	$E_7 - E_8 + I_8$	-1.074719	1205896	98
34	-1.627541	796293	216	13.323372	33	$E_5 - E_8 + I_8$	2.889676	448493	88
35	28.327358	45750	208	28.578944	34	$E_5 + E_8 - I_7$	7.901300	164023	74
36	27.593741	46967	188	44.561842	35	$E_5 + E_8 - I_7$	-0.842286	1538669	69
37	1.144805	1132070	184	318.787308	36	$E_7 + E_8 - I_7$	6.721831	192804	68
38	-10.219766	126813	182	72.442500	37	$2I_7 - I_8$	-5.277807	245556	68
39	0.518224	2500847	175	259.470825	38	$2I_7 - I_8$	-1.897591	682971	65
40	29.139890	44475	165	216.213439	39	$-I_7 + 2I_8$	-0.315956	4101831	64
41	-9.040364	143357	161	162.221845	40	$-I_7 + 2I_8$	1.601752	809114	62
42	3.187468	406592	139	295.836486	41	$2I_5 - I_8$	-2.952985	438877	59
43	-0.404469	3204200	126	151.512241	42	$2I_5 - I_8$	-2.590790	500233	57
44	0.293808	4411044	103	130.074619	43	$2I_5 - I_8$	0.693336	1869224	57
45	1.043128	1242416	88	174.172806	44	$E_7 + E_8 - I_8$	-0.221038	5863258	56
46	-4.452747	291056	85	159.016505	45	$E_7 + E_8 - I_8$	4.427915	292688	55
47	1.993877	649989	79	274.440352	46	$E_6 - E_8 + I_6$	0.964499	1343703	45
48	-0.026629	4868060	74	163.690456	47	$E_6 - E_8 + I_6$	-0.491497	2636840	39
49	0.853505	1518443	72	163.157929	48	$E_5 - E_6 + I_7$	-1.106270	1171504	36
50	-5.632639	230087	62	194.654991	49	$E_5 - E_6 + I_7$	-26.696770	1469678	35
51					50	$E_5 - E_6 + I_7$	-5.607131	231134	34
					51				26.749286

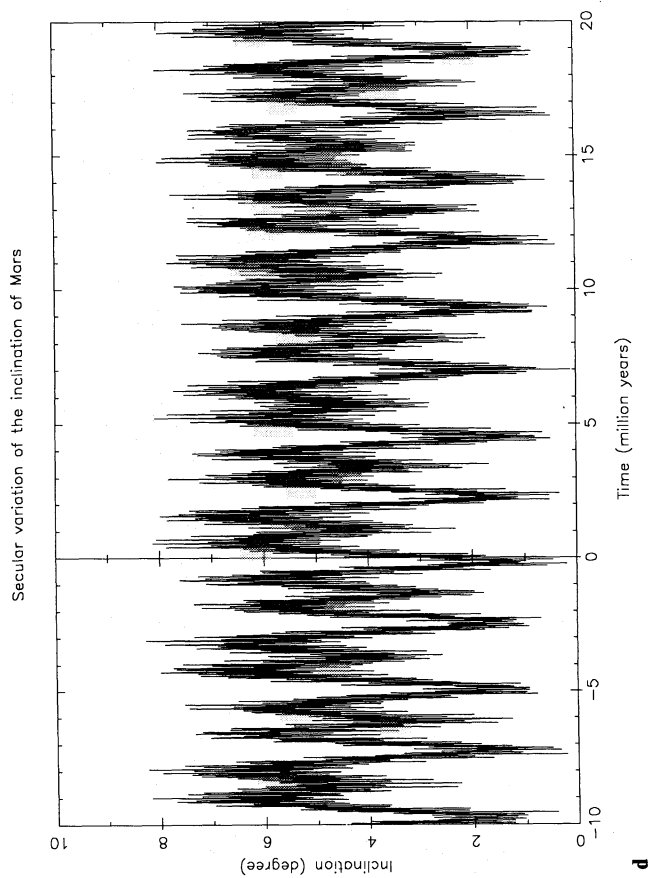
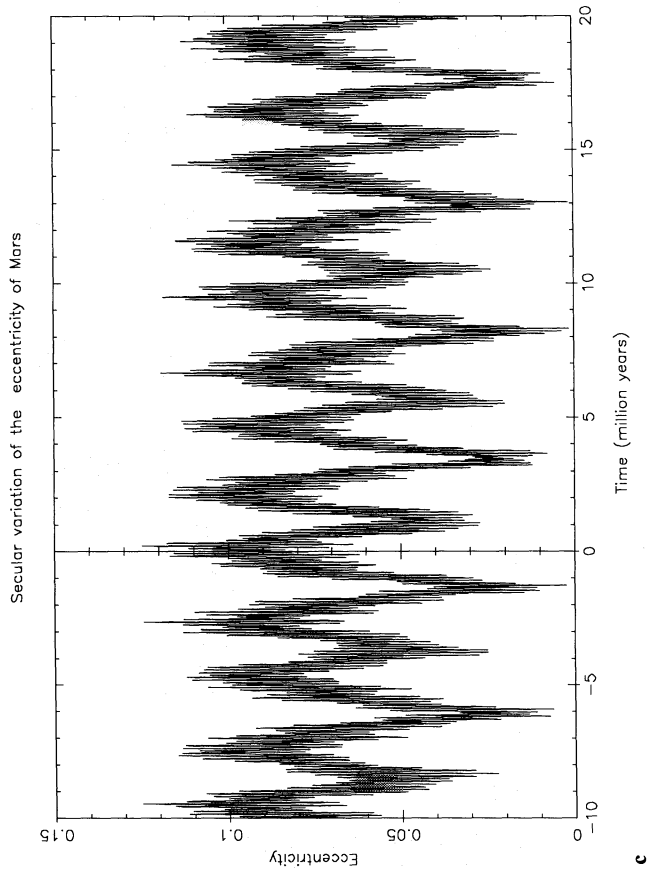
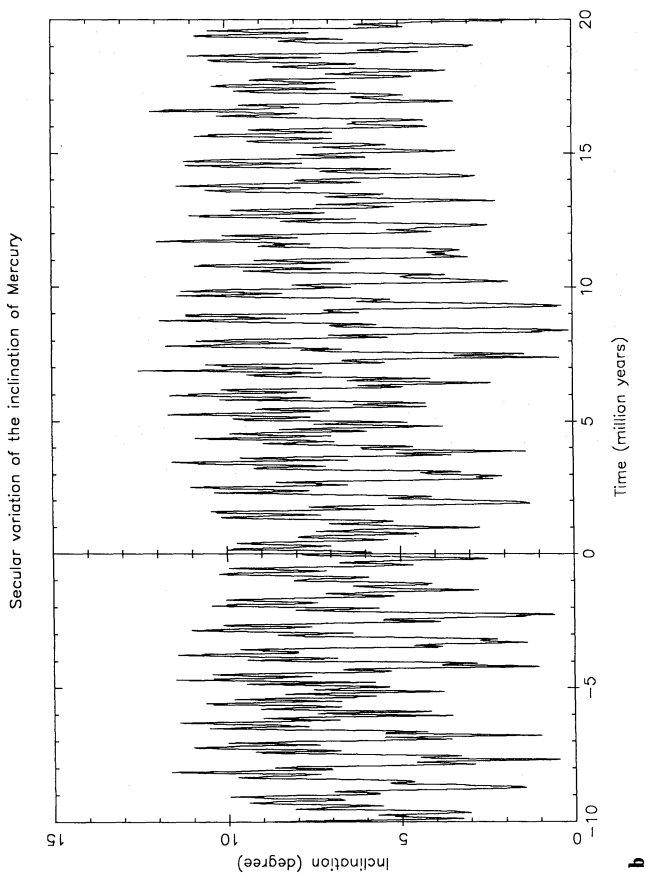
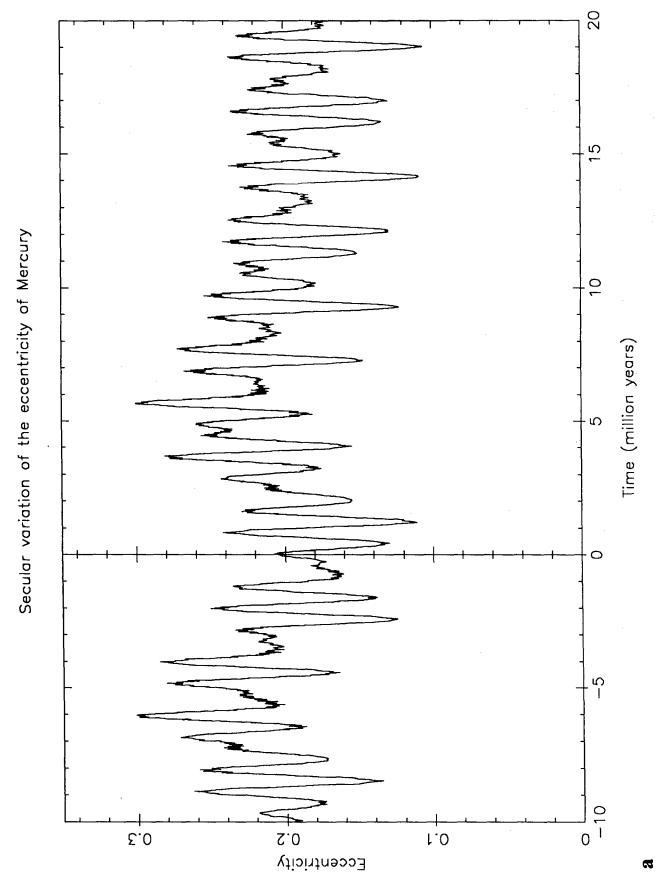


Fig. 2a–d. Solution NGT for the eccentricity and inclination of Mercury and Mars over 30 millions years. In these solutions we have very long periods effects of several millions years. In particular, the 30 millions time span is not even enough for the analysis of Mercury's solutions.

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Appendix: the disturbing function and d'Alembert relations

The d'Alembert relations and characteristic are essential properties of the expansions of the disturbing function which allow to limit the number of terms in the series of the planetary theories. It was known since the early works of Euler and d'Alembert (1754) and according to Brouwer and Clemence (1961) was called d'Alembert relations for the first time by Brown.

I will give here some main ideas on these relations and how they appear in the expressions of the disturbing function following Duriez (1979) and Laskar (1985). A classical exposition can be found in Tisserand (1889) or Brouwer and Clemence (1961).

A1. Notations

For a given couple of planets, we shall denote by P the disturbed planet and P' the disturbing planet. The elements related to P' will be all denoted by a prime (').

$\mathbf{r}$	heliocentric position vector of the planet P
a	semi-major axis
e	eccentricity
i	inclination
Ω	longitude of the ascending node
$\bar{\omega}$	longitude of perihelion
n	mean motion
λ	mean longitude
M	mean anomaly
v	true anomaly
Δ	distance P, P'
N	mean mean motion
p	defined by $n = N(1 + p)$
q	defined by $\lambda = Nt - \sqrt{-1}q$
z	$= e \exp \sqrt{-1} \bar{\omega}$
ζ	$= \sin(i/2) \exp \sqrt{-1} \Omega$

The typical terms involved in our computations on the 8 planet system are of the general form

$$T = C \prod_{i=1}^8 p_i^{g_i} \bar{z}_i^{\bar{n}_i} \bar{r}_i^{\bar{v}_i} z_i^{n_i} r_i^{v_i} \exp \left(\sqrt{-1} \sum_{i=1}^8 k_i \lambda_i \right) \quad (1)$$

where $g_i, n_i, \bar{n}_i, v_i, \bar{v}_i$ are integers. For such a term T we use the following notations

C	coefficient (real number)
$M = \prod_{i=1}^8 p_i^{g_i} \bar{z}_i^{\bar{n}_i} \bar{r}_i^{\bar{v}_i} z_i^{n_i} r_i^{v_i}$	monomial
$I = (k_1, k_2, \dots, k_8)$	inequality
$\delta(T) = \sum_{i=1}^8 \bar{n}_i + \bar{v}_i + n_i + v_i$	total degree
$\delta^+(T) = \sum_{i=1}^8 \bar{n}_i + \bar{v}_i$	
$\delta^-(T) = \sum_{i=1}^8 n_i + v_i$	

A2. Back to d'Alembert relations

Let us consider one of the well known formula of the elliptic motion for the radius vector as a function of the eccentricity and mean anomaly M

$$\frac{a}{r} = 1 + \sum_{k=1}^{\infty} 2J_k(ke) \cos kM \quad (2)$$

where the Bessel functions J_k can be expanded as

$$J_k(x) = \sum_{l=0}^{\infty} (-1)^l \frac{(x/2)^{k+2l}}{(k+l)!l!} \quad (3)$$

In this case, the d'Alembert property can be classically stated as:

The developement in power of the eccentricity which is in factor of $\cos(+kM)$ is at least of degree k and has the same parity as k .

In Abu-El-Ata and Chapront (1975) or Duriez (1979) this property is used in a different form. Introducing

$$X = e \exp \sqrt{-1} M = \bar{z} \exp \sqrt{-1} \lambda \quad (4)$$

the same statement becomes

The ratio a/r can be expanded in a series involving only X and $\bar{X}$

$$\frac{a}{r} = 1 + \frac{1}{2} X + \frac{1}{2} \bar{X} + \frac{1}{2} X^2 + \frac{1}{2} \bar{X}^2 + \frac{9}{16} X^3 + \dots \quad (5)$$

In Laskar (1985), a more general form of this same property was stated, using the expansions of the elliptical expressions in terms of the general form (1).

For any term T of the form (1), we define as usual the characteristic of the inequality of T , $c_I(T)$ and introduce the new notion of characteristic of the monomial of T , $c_M(T)$

$$c_I(T) = \sum_{i=1}^8 k_i \quad (6)$$

$$c_M(T) = \delta^+(T) - \delta^-(T)$$

With these conventions, the d'Alembert rule in (2) becomes

The ratio a/r can be expanded in series of general terms (1) which satisfy the relation

$$c_I(T) = c_M(T) \quad (7)$$

This form of d'Alembert rule is very simple and is conserved through addition, multiplication and complex conjugation as

$$\begin{aligned} c_I(T \cdot T') &= c_I(T) + c_I(T') \\ c_M(T \cdot T') &= c_M(T) + c_M(T') \\ c_I(\bar{T}) &= -c_I(T) \\ c_M(\bar{T}) &= -c_M(T) \end{aligned} \quad (8)$$

The d'Alembert relation (7) holds also for the expansion of

$$\begin{aligned} \theta &= \exp \sqrt{-1} (v - M) = 1 - \bar{X} + X - \frac{1}{8} \bar{X}^2 - \bar{X} X \\ &\quad + \frac{9}{8} X^2 + \dots \end{aligned} \quad (9)$$

From (8), it will be also verified by all powers of r/a and θ .

The disturbing function due to P' is

$$R = Gm(1/\Delta - \mathbf{r} \cdot \mathbf{r}'/r'^3) \quad (10)$$

In Laskar (1985) the expression of the disturbing function are given in Eqs. (6–12). This expansion involves only expressions depending on the masses and semi-major axis and powers of $X, \bar{X}, X', \bar{X}'$, even powers of $Y = \bar{\zeta} \exp \sqrt{-1} \lambda$, $\bar{Y}$, Y' , $\bar{Y}'$ and powers of $\exp \sqrt{-1}(\lambda - \lambda')$

All these quantities verify the d'Alembert relation (7) so will also the disturbing function (10).

A3. The d'Alembert relations for the Lagrange equations

The relation (7) is not kept by derivation. If σ denotes any of the variables $p, q, z, \zeta, \bar{z}, \bar{\zeta}$, we have

$$c_I \left(\frac{\partial T}{\partial \sigma} \right) = c_I(T)$$

while

$$c_M \left(\frac{\partial T}{\partial \sigma} \right) = c_M(T) + \tau(\sigma) \quad (11)$$

with

$$\begin{aligned} \tau(p) &= \tau(q) = 0 \\ \tau(z) &= \tau(\zeta) = +1 \\ \tau(\bar{z}) &= \tau(\bar{\zeta}) = -1 \end{aligned} \quad (12)$$

The relation for the Lagrange equations (Laskar, 1985, Eq. 4) becomes then

$$c_I(T) = c_M(T) + \tau(\sigma) \quad (13)$$

for the different variables σ . For the secular case we have $c_I(T) = 0$ thus the d'Alembert relation reduces to

$$c_M(T) = -\tau(\sigma) \quad (14)$$

which is also the relation which will be verified by the combination of fundamental frequencies in the solution of the secular system.

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