KARL SCHWARZSCHILD LECTURE

THE AESTHETIC BASE OF THE GENERAL THEORY OF RELATIVITY

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May I begin by expressing my gratitude to the officers and councillors of the Astronomische Gesellschaft for their courtesy in asking me to give this lecture in the series established in memory of Karl Schwarzschild.

I.

Karl Schwarzschild is, of course, one of the towering physical scientists of this century. The breadth and range of his contributions are staggering: they cover the entire range of physics, astronomy, and astrophysics of his time.

In physics, they range from electrodynamics and geometrical optics to the then newly developing atomic theory of Bohr and Sommerfeld. In electrodynamics, he derived a variational base for Lorentz's equations of the electron. In geometrical optics, he developed the theory of the aberrations in optical instruments (described by Max Born as "unsurpassed in clarity and rigour by later work") and formulated the principle underlying

the optics of the Schmidt telescope. And in the Bohr-Sommerfeld theory, he worked out, in his last published paper, the theory of the Stark effect and of the Deslander term in the rotational-vibrational spectra of diatomic molecules. (In this last paper he introduced, for the first time, the notions of action and angle variables.)

In Astronomy and Astrophysics, Schwarzschild's contributions are so many and so varied that I shall mention only those of his discoveries to which his name is attached. We have the Schwarzschild exponent in photographic photometry, the Schwarzschild-Milne integral equation in the theory of radiative transfer, the Schwarzschild criterion for the onset of convective instability, the Schwarzschild ellipsoidal distribution of stellar velocities, and, of course, the Schwarzschild solution of Einstein's equations for describing the space-time external to a spherical distribution of mass and of static black-holes.

And all of these in a brief twenty years!

It is possible that the announced title of my lecture has puzzled some of you: it is not addressed to any concrete topic as the earlier lectures in this series have been; and I am afraid that it will scarcely have any astronomical overtone. My lecture, however, will bear on Schwarzschild's attitude and approach to scientific problems, as I can discern them from his published papers; and it will bear very directly on his solution of the equations of general relativity.

II.

I shall consider three examples from Schwarzschild's work which, to my mind, illustrate his approach to scientific problems.

The first relates to his work on star-streaming. The phenomenon was discovered by J.C. Kapteyn: and it was adequately interpreted, almost at once, by Eddington on the basis of his and Kapteyn's hypothesis of two star-streams. Schwarzschild expressed his reaction to this hypothesis

as follows*:

The magnitude of the proper motions of the stars in the two streams and along with them, presumably also, their average distances from the sun would appear to be equal. The stars in the two streams, during their motion through each other, must, therefore, share common fluctuations; and it is problematical how this can be brought about.

I have, therefore, believed that the same observational material on which Eddington has based himself should be reworked on a more unified hypothesis concerning stellar motions.

On these grounds, Schwarzschild formulated his ellipsoidal distribution of the peculiar velocities of stars; and this formulation has been the basis of all subsequent discussions bearing on stellar motions and the dynamics of stellar systems. What is remarkable to me, however, is his argument: a description of nature must be natural; it cannot be *ad hoc*.

As a second example, I shall take a still earlier publication of Schwarzschild. At a meeting of this Gesellschaft in 1900, Schwarzschild addressed himself to the question whether the geometry of the three-dimensional space of astronomy might be non-Euclidean. He stated the problem as follows.

As must be known to you, during this century [meaning the 19th century] *one has developed non-Euclidean geometry (besides Euclidean geometry), the chief examples of which are the so-called spherical and pseudo-spherical spaces. We can wonder how the world would appear in a spherical or a pseudo-spherical geometry with possibly a finite radius of curvature.... One would then find oneself, if one will, in a geometrical fairyland; and one does not know whether the beauty of this fairyland may not in fact be realized in nature.*

We can only marvel at Schwarzschild's scientific imagination and curiosity in addressing himself to such a question some fifteen years before the founding of general relativity. But to Schwarzschild, it was more than simple imagination. He actually estimated limits to the radius of curvature of

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* This and the other translations from German of the originals (including Einstein's "Gedächtnisrede" in the appendix) are the author's.

the three-dimensional space with the astronomical data available at his time and concluded that if the space is hyperbolic its radius of curvature cannot be less than 64 light years and that if the space is spherical its radius of curvature must at least be 1600 light years.

We need not argue about Schwarzschild's particular estimates. It is far more relevant that Schwarzschild allowed his imagination to contemplate a world that may have features of a fairyland!

My third example bears on Schwarzschild's discovery of the solution of Einstein's vacuum-equations appropriate to the exterior of a central spherical distribution of mass - undoubtedly the most important discovery in relativity after its founding.

Schwarzschild's paper in which he derived his solution was communicated by Einstein to the Berliner Akademie in 1916 January 13, just about two months after Einstein himself had published the basic equations of his theory in a short communication - his detailed paper with full derivations was still six months in the future - and had deduced the theoretical rate of the precession of the perihelion of Mercury and of the magnitude of the deflection a light ray will experience as it grazes the limb of the sun. In acknowledging Schwarzschild's paper, Einstein wrote on 1916 January 9,

I have read your paper with greatest interest, I had not expected that one could obtain the exact solution of the problem so simply. The analytical treatment of the problem appears to me splendid.

The circumstances under which Schwarzschild derived his now famous solution were heroic. During the spring and summer of 1915, Schwarzschild was serving in the German army at the Eastern front. While at the Eastern front with a small technical staff Schwarzschild contracted pemphigus, a fatal disease; and he died on 1916 May 11. It was during this period of illness that Schwarzschild wrote his two papers on general relativity - the second one dealt with the equilibrium of a homogeneous mass and showed that no hydrostatic equilibrium is possible if the radius of the object is less than 9/8 of the Schwarzschild radius, $2\ GM/c^2$ - and the fundamental one on the Bohr-Sommerfeld theory to which I have already

referred.

About Schwarzschild's last illness, Eddington wrote in a moving obituary notice:
His end is a sad story of long suffering from a terrible illness contracted in the field, borne with great courage and patience.

Parenthetically, I may add a footnote, Richard Courant told me in the late thirties that he had met Karl Schwarzschild proceeding to the Eastern front while he, as a member of the general staff, was with a party retreating from the same front; and Courant said that he was surprised that someone as distinguished as Karl Schwarzschild would be proceeding towards a front that was considered too dangerous for the general staff!

Let me return to Schwarzschild's original paper and his reasons for seeking an exact solution to the problem which Einstein had solved earlier by an approximate procedure. Schwarzschild began with the statement: *It is always satisfying to obtain an exact solution in a simple form. It is even more important, in the present instance, to have the uniqueness of the solution established and remove whatever doubts there may be concerning Mr. Einstein's treatment of the problem, since, as it will appear below, it is difficult, in the nature of this problem, to establish the validity of an approximate procedure.*

While Einstein in his letter of acknowledgement to Schwarzschild (quoted earlier) argued that there can be no doubts about the validity of his approximate procedure in solving the equations, it is significant that Schwarzschild, undaunted, whished to solve exactly the problem which he realized was a fundamental one in the newly formulated theory. I said "undaunted" in view of the great much-ado that was to be made soon afterwards (and for some decades, to the detriment of the theory) about the "difficulty" of Einstein's theory in general and of finding exact solutions in particular.

I shall return later to the role of exact solutions for the understanding of general relativity. But I must pass on now to the main subject of my Lecture.

III.

The general theory of relativity has often been described as an extremely beautiful theory; and it has even been compared to a work of art (as by Rutherford and by Max Born, for example). In the same vein, statements like the following ones by Dirac are not uncommon:
What makes the theory so acceptable to physicists, in spite of its going against the principle of simplicity, is its great mathematical beauty. (1939)
The Einstein theory of gravitation has a character of excellence of its own. (1978)

These and similar characterizations of the general theory of relativity raise the following questions:

> What is the aesthetic base of the theory? And, more importantly, to what extent is an aesthetic sensibility to its excellence relevant to the formulation and solution of problems which will lead to a deeper understanding of the theory?

To answer this question without descending to dilettantism, it is first necessary to appreciate the present peculiar position of the general theory of relativity with respect to its confirmation by observation and experiment; and the reasons for its inspiring confidence in spite of inadequate empirical support.

During the past twenty years a great deal of commendable effort has been expended to verify the lowest first-order departures from the Newtonian theory that the general theory of relativity predicts. These efforts have been successful and the predictions of the theory relating to the differing rates of time-keeping in locations of differing gravity; to the deflection a light ray experiences when traversing a gravitational field and the consequent time delay; to the precession of a Kepler orbit; and, finally, to the slowing down of the orbital period of a binary star in an eccentric orbit by virtue of the emission of gravitational radiation, have all been confirmed within the limits of observational and experimental errors and uncertainties. But all these effects relate to departures from the predictions of the Newtonian theory by a few parts in a million; and of no

more than three or four parameters in a post-Newtonian expansion of the Einstein field-equations. And, so far, no predictions of general relativity, in the limit of strong gravitational fields, have received any confirmation; and none seems likely in the foreseeable future.

Should one not argue that a confirmation of a theory, which generalizes a theory as well tested in its domain of validity as the Newtonian theory, should refer to predictions which relate to major aspects of the theory, rather than to small first-order departures from the theory which it replaces? Would the status of Dirac's theory of the electron, for example, be what it is today if its only success consisted in accounting for Paschen's 1916 measurements of the fine structure splittings of the lines of ionized helium? The real confirmation of Dirac's theory that inspired confidence was the discovery, in accordance with the theory, of electron-positron pairs in cosmic-ray showers. Similarly, would our faith in Maxwell's equations of the electromagnetic field be as universal as it is without Hertz's experiments on the propagation of electromagnetic waves with precisely the velocity of light and without Poincare's proof of their invariance to Lorentz transformations? In the same way, a real confirmation of the general theory of relativity will be forthcoming only if a prediction characteristic of the theory, and only of that theory, is confirmed. The occurrence of black holes as one of the final equilibrium states of massive stars in the natural course of their evolution is not a confirmation of a prediction of general relativity in any real sense. The notion that light cannot escape from a sufficiently strong gravitational field is an inference not based on any exact prediction of the theory; it depends only on the empirical fact that light is affected by gravity. On the other hand, since the general theory of relativity provides an exact description of the space-time around black holes, only a confirmation of the metric of the space-time around black holes can be considered as "establishing" the theory in any real sense. It is well known that the Kerr solution with two parameters provides the unique solution for stationary black holes that can occur in the astronomical universe. But a confirmation of the metric of the Kerr space-time (or, some aspect of it) cannot even be contemplated in the foreseeable future.

Perhaps, I may digress here to indicate how one may eventually have a confirmation of the space-time around a rotating Kerr black hole. If

one imagines a Kerr black hole with an accretion disc of free electrons in the equatorial plane, then the polarization of the light emerging from it, after traversing the strong gravitational field of the black hole, will manifest so non-uniform a distribution that one should be able to map it. Will Nature be generous enough to provide a clean example which will enable such a mapping? I am afraid that this is the only time my talk will bear on an astronomical observation.

IV

As I have said, we have, as yet, no exact feature of general relativity that has been confirmed by observation; and none appears feasible in the foreseeable future. Why then do we have faith and confidence in the theory? One should respond more explicitly than merely to say, as some have, that our confidence derives from the "beauty of the mathematical description of nature which the theory provides".

To the solid ground of Nature trusts the mind that builds for aye!

So said Wordsworth. There is no solid ground for the general theory of relativity. On what then do we build our trust? We build our trust on the internal consistency of the theory and on its conformity with what we believe are general physical requirements; and, above all, on its freedom from contradiction with parts of physics not contemplated in the formulation of the theory. Let me illustrate by some examples.

The causal character of the laws of physics requires that, given complete initial data on a space-like 3-surface, the future is uniquely determined in the space-time domain bounded by the future-directed in-going null rays emanating from the boundary of the spatial slice. More formally stated: the basic equations of any physical theory must allow an initial-value formulation which determines uniquely the future development in the entire domain of dependence of the initial data on a spatial slice. The field equations of general relativity do allow such a formulation though the proof of this fact is not straightforward: it was provided only in 1944 by Lichnerowicz.

As a second example, consider the notion of energy that is so central to physics. In physics, one is accustomed to define a local energy that is globally conserved. The fact, that an isolated body which is not static or stationary will emit gravitational waves that contribute to the energy, implies that in general relativity we cannot expect to have a local definition of energy. But one should expect that, if space-time is asymptotically flat (in some well-defined sense) one should be able to define, globally, for the entire space (extending to infinity) a total energy that is positive. In 1962, Bondi was able to show that if the space-time is asymptotically flat at null-infinity (i.e., as we go to infinity along null-rays), then one can define a mass-function - the Bondi mass - that is a decreasing function of time; and, further, that the rate of decrease of this mass-function is exactly equal to the rate at which energy is radiated to infinity in the form of gravitational waves. But a proof that the Bondi mass always remains positive finally emerged only in recent years (Schoen & Yau 1981, Witten 1981, and Horowitz & Perry 1982). The proof requires that the energy-momentum tensor, T_{ij}, satisfy some "energy conditions". For a perfect fluid, for which

$$T_{ij} = (\epsilon + p)\, u_i u_j - p g_{ij} \ .$$

the required conditions are equivalent to

$$\epsilon \geq |p| \ .$$

The foregoing two examples, deep in the structure of the general theory of relativity, illustrate its internal consistency - a consistency by no means obvious or self-evident.

An even more remarkable feature of the general theory of relativity is that it does not violate the laws of other branches of physics not contemplated in its formulation, such as thermodynamics or quantum theory, so long as one does not transgress the domain of validity of the theory. (I shall return presently to the meaning I attach to the phrase "the domain of validity of the general theory of relativity".)

My first example derives from a consideration of the behaviour of electron waves, described by Dirac's equation, in the space-time of a Kerr black hole. It is known that one can extract the rotational energy of the

black hole by processes which result in the slowing down of its rotation. More precisely, if we have waves, with a time (t) and an azimuthal-angle (ϕ) dependence given by

$$e^{i(\sigma t+m\phi)} \qquad (m = 0, \pm 1, \pm 2, \ldots),$$

with a frequency $\sigma(>0)$ less than the critical value

$$\sigma_s = -am/2Mr_+ \qquad (m = -1, -2, \ldots),$$

where a and M are the Kerr parameters and r_+ is the radius of the event horizon, then one has super-radiance (by which one means that the reflection coefficient for the incident waves exceeds unity). The super-radiance is a necessary consequence of a theorem due to Hawking that every interaction of a black hole with an external source must always result in an increase of the surface area of the event horizon provided only that the energy-momentum tensor of the external source is compatible with the positive-definite character of the energy. However, when one considers the reflection of Dirac waves by the Kerr black hole, one finds, by a well-defined mathematical algorism of the theory, that they do not exhibit the phenomenon of super-radiance. Apparently, then, Hawking's theorem is violated. But one soon realizes that the energy momentum tensor of the Dirac waves, provided by the quantum theory, does not satisfy the positive-energy requirement. Had the standard algorism predicted super-radiance, we should have had a contradiction between the premises of the general theory of relativity and the premises of the quantum theory. But no such contradiction occurs!

Let me consider a second example. Hawking showed in 1975 that, when one considers the curvature of space-time as providing a classical potential for electron (or photon) scattering according to the rules of the quantum theory, one must observe, from the event horizon of a black hole, an emission of electrons (or photons) with a Fermi (or a Planck) distribution at a temperature determined by the constant surface-gravity of the event horizon. Associated with this temperature and the rate at which energy is lost by the emission of the particles, one can define, formally, an entropy. When one pursues this line of reasoning, one finds that the notion of entropy one derives is entirely consistent with all the known laws of thermodynamics and of statistical mechanics.

Thermodynamics and statistical mechanics were not contemplated in the formulation of the general theory of relativity; and yet the consequence that follow from the theory do not violate the laws of thermodynamics and of statistical mechanics.

The foregoing illustrations disclose not only the internal consistency of the general theory but also its consistency with the entire domain of physics outside the realm originally contemplated. These are probably sufficient grounds for one's confidence and faith in Einstein's general theory of relativity.

V.

There is another feature of the theory that is related to its aesthetic base.

Every valid physical theory is circumscribed by limitations inherent to it. Thus, the Newtonian theory of gravity is limited by the requirement that bodies should be moving with velocities small compared to that of light. The classical laws of mechanics and of electrodynamics are similarly limited by the requirement that the relevant actions are large compared to Planck's quantum of action, h. Likewise, we may expect the general theory of relativity to be limited by the requirement that the intervals of time and of distance are large compared to the Planck scales, $(\hbar G/c^5)^{\frac{1}{2}}$ ($\backsim 5.4 \times 10^{-44}$ sec.) and $(\hbar G/c^3)^{\frac{1}{2}}$ ($\backsim 1.6 \times 10^{-33}$ cm), respectively ($\hbar = h/2\pi$).

Any physical theory, which replaces an earlier theory by overcoming its limitations, will envisage circumstances that are peculiar to the theory and whose exact description will provide a basis for its confirmation. In the Newtonian theory of gravitation, the solution of the two-body problem provides an example: its exact solution provides a quantitative explanation for Kepler's laws. Similarly, Bohr's theory of one-electron systems provides an exact derivation of Balmer's formula and an exact basis for determining the ratio of the masses of the electron and the nucleus from the departures of the Pickering series of ionized helium from the Balmer series of hydrogen.

We now ask: what is the essentially new feature of the general theory of relativity? And what are the circumstances which will reveal those features unambiguously?

The essential features of the general theory of relativity are the precise notions regarding space and time which it incorporates. These notions generalize, in magisterial fashion, those that underlie the special theory of relativity. We ask: are there physical circumstances in which these new notions of the theory are manifested in their pristine purity? The space-time around black holes provides the requisite arena. The general theory of relativity solves the problem of these space-times with magnificent completeness. The space-time around black holes is uniquely specified: it is simple and it involves only two parameters: the mass of the black hole and the angular momentum of the black hole; and the behaviour in the space-time of all known test particles is exactly predicted. None of the physical theories that have been explored hitherto provides a problem so characteristic of itself and a solution so complete. This feature of the general theory of relativity appears to me as one of its most aesthetically satisfying aspects; and this leads me to examine, more generally, the aesthetic base of the theory.

To examine a physical theory and to state the source of its aesthetic appeal is beset with difficulties. Like all discussions relating to beauty, it is subject to the tastes and the temperaments of the individuals; and it is difficult, if not impossible, to achieve objectivity. Nevertheless, it seems to me that the question is relevant: as a practitioner of the general theory of relativity for the past twenty and more years, I can ask myself: what aspects of the theory appeal to my aesthetic sensibility and how do the aesthetic ingredients of the theory influence and direct the formulation and solution of problems that lead to a deeper understanding of the physical and the mathematical content of the theory?

I have already referred to the theory of the black holes. It is a remarkable fact, to which I have also made reference, that the general theory of relativity provides, for isolated stationary black holes, a unique solution with just two parameters. As I have said on another occasion,

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their description, they are the simplest objects as well.

But that is not all! Contrary to every prior expectation, the standard equations of mathematical physics, relating to the propagation and scattering of electromagnetic, gravitational, and the Dirac-electron waves, as well as the geodesic equations of particles and of polarized photons, all of them, can be separated and solved exactly. The manner of separation of these equations has led to a re-examination of the century-old problem of the circumstances when partial differential equations in two variables can be separated and solved; and a rich mathematical theory has arisen. As an example, I may refer to the separation of Dirac's spinor-equation of the electron in Kerr geometry. As a corollary, it led to the separation of Dirac's spinor-equation in spheroidal coordinates in Minkowski geometry of special relativity - a separation that had been considered impossible before.

VI.

I now turn to the most difficult question to which I wish to address myself, namely, how sensitiveness to the mathematically aesthetic aspects of the theory enables the formulation and solution of problems of physical significance. In answering this question, I should be precise if I am not to descend to dilettantism. That, I am afraid, will require a somewhat more technical language than I have used so far.

There are two major areas in general relativity in which progress

has been made in recent years: the mathematical theory of black holes and the mathematical theory of colliding waves. Black holes, as resulting from the gravitational collapse of massive stars in the late stages of stellar evolution, are well known. But the relevance of the theory bearing on the collision or scattering of waves by waves in general relativity requires explanation.

In the general theory of relativity one can construct plane-fronted gravitational waves confined between two parallel planes with a finite energy per unit area; and, therefore, we can, in the limit, construct impulsive gravitational waves with a δ-function energy-profile. Parenthetically, I may note that one cannot construct such impulsive waves in electrodynamics. For, a δ-function profile of the energy will imply a square-root of a δ-function profile for the field variables; and the square-root of a δ-function is simply not permissible for physical description.

In 1971, Khan & Penrose considered the collision of two impulsive gravitational waves with parallel polarizations. And they showed that the result of the collision is the development of a space-time singularity not unlike the singularity in the interior of black holes with which we are acquainted. This phenomenon is not manifested in any linearized version of the theory: the occurrence of the singularity, by a focussing of the colliding waves, in no way depends on the amplitude of the waves. Clearly, in this context, nothing short of an exact solution of the problem will suffice to disclose the new phenomenon. In any event, the occurrence of a singularity in this example, suggested to Penrose that a new realm in the physics of general relativity remained for exploration. However, there was no substantial progress in this area before one realized that the mathematical theory of black holes is structurally very closely related to the mathematical theory of colliding waves. This fact is, in itself, surprising: one should not have thought that two theories dealing with such disparate physical circumstances will be as closely related as they are. Indeed, by developing the mathematical theory of colliding waves with a view to constructing a mathematical structure architecturally similar to the mathematical theory of black holes, one finds that a variety of new physical implications of the theory emerge - implications one simply could not have foreseen.

VII.

A description of how the development to which I have referred was accomplished is not possible without some familiarity with the language of general relativity, any more than an analysis of a musical composition is possible without some familiarity with musical notation.

We are concerned with space-times that describe stationary axisymmetric black holes and space-times that describe the collision and scattering of plane-fronted waves. In the former case, the metric coefficients are independent of the time, t, and of the azimuthal angle, ϕ, about the axis of rotation; they depend only on the two remaining spatial coordinates, a radial coordinate r and the polar angle Θ. In the latter case, the metric coefficients are independent of two space-like coordinates, x^1 and x^2, both ranging from $-\infty$ to $+\infty$; they depend only on the time, t, and the remaining spatial coordinate, x^3, normal to the (x^1, x^2)-planes.

It can be shown that the metric appropriate to a description of stationary axisymmetric black-holes can be written in the form:

$$ds^2 = \gamma(\Delta\delta)\ [\chi(dt)^2 - \frac{1}{\chi}\ (d\phi - \omega dt)^2]$$

$$- e^{\mu_2+\mu_3}\ \gamma\Delta\ \left[\frac{(d\eta)^2}{\Delta} + \frac{(d\mu)^2}{\delta}\right], \qquad (1)$$

where

$$\Delta = \eta^2 - 1,\ \delta = 1 - \mu^2 = \sin^2\Theta\ \ (\mu = \cos\Theta)\ , \qquad (2)$$

η is a radial coordinate (measured in a suitable unit) and χ, ω, and $\mu_2 + \mu_3$ are metric functions to be determined. It may be noted, that ω is directly related to the angular momentum of the black hole; it is zero for the Schwarzschild black hole which is static.

In writing the metric in the form (1), we have already arranged for the occurrence, at $\eta = 1$, of a null-surface that will eventually be identified with the event horizon of the black hole.

The central problem of the theory is to solve for χ and ω: once

one has solved for them, the remaining metric function, $\mu_2 + \mu_3$, follows by a simple quadrature.

Associated with the metric (1), we have a "conjugate metric" obtained by the transformation

$$t \to +i\phi \quad \text{and} \quad \phi \to -it \ . \tag{3}$$

By this "conjugation", χ and ω are replaced by

$$\tilde{\chi} = \frac{\chi}{\chi^2-\omega^2} \quad \text{and} \quad \tilde{\omega} = \frac{\omega}{\chi^2-\omega^2} \ . \tag{4}$$

For the reduction of the physical problems, it is essential that we consider, in place of χ and ω, the pair of functions, Ψ and Φ, where

$$\Psi = \frac{\gamma(\Delta\delta)}{\chi} \ , \tag{5}$$

and Φ is a potential for ω defined by

$$\Phi_{,\eta} = \frac{\delta}{\chi^2}\,\omega_{,\mu} \quad \text{and} \quad \Phi_{,\mu} = -\frac{\Delta}{\chi^2}\,\omega_{,\eta} \ . \tag{6}$$

One can similarly define $\tilde{\Psi}$ and $\tilde{\Phi}$ in terms $\tilde{\chi}$ and $\tilde{\omega}$.

In the mathematical theory of black holes, one combines the functions Ψ and Φ and $\tilde{\Psi}$ and $\tilde{\Phi}$ into the pairs of complex functions,

$$Z^\dagger = \Psi + i\Phi \quad \text{and} \quad \tilde{Z}^\dagger = \tilde{\Psi} + i\tilde{\Phi} \ , \tag{7}$$

and defines

$$E^\dagger = \frac{Z^\dagger - 1}{Z^\dagger + 1} \quad \text{and} \quad \tilde{E}^\dagger = \frac{\tilde{Z}^\dagger - 1}{\tilde{Z}^\dagger - 1} \ . \tag{8}$$

Both these functions satisfy the Ernst equation,

$$\begin{aligned}(1 - |E|^2)\{[(1-\eta^2)E_{,\eta}]_{,\eta} - [(1-\mu^2)E_{,\mu}]_{,\eta}\} \\ = -2E^*[(1-\eta^2)(E_{,\eta})^2 - (1-\mu^2)(E_{,\mu})^2] \ .\end{aligned} \tag{9}$$

Turning next to space-times appropriate to the description of colliding waves, we envisage the collision of two plane-fronted impulsive

gravitational waves accompanied, in general, with gravitational and other shock-waves with the same fronts, approaching each other from $+\infty$ and $-\infty$. Prior to the instant of collision, the space-time between the approaching wave-fronts is flat. We are principally concerned with the space-time that develops after the instant of collision (though satisfying the boundary conditions at the collision fronts is not a negligible part of the problem).

The metric of the space-time after the instant of collision can be written in the form,

$$ds^2 = -\gamma(\Delta\delta)\left[\chi\,(dx^2)^2 + \frac{1}{\chi}\,(dx^1 - q_2 dx^2)^2\right] + e^{\nu+\mu_3}\gamma\Delta\left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta}\right], \qquad (10)$$

where, now,

$$\Delta = 1 - \eta^2, \qquad \delta = 1 - \mu^2, \qquad (11)$$

η measures the time (in a suitable unit) from the instant of collision, μ measures the distance normal to the colliding fronts at the instant of the collision, and χ, q_2, and $\nu+\mu_3$ are metric functions to be determined. It may be noted that q_2 is directly related to the varying plane of polarization of the gravitational waves: it is zero when the plane of polarization is unchanging.

In writing the metric in the form (10), we have taken into account, aposteriori, the fact that, as a result of the collision, a curvature or a coordinate singularity develops when $\eta=1$ and $\mu=\pm1$.

As in the case of stationary axisymmetric space-times, the solution to the Einstein field-equations can be completed once we have solved for the metric functions χ and q_2 or, equivalently, for Ψ and Φ related to χ and q_2 by

$$\Psi = \frac{\gamma(\Delta\delta)}{\chi} \qquad (12)$$

and

$$\Phi_{,\eta} = \frac{\delta}{\chi^2} q_{2,\mu} \quad \text{and} \quad \Phi_{,\mu} = \frac{\Delta}{\chi^2} q_{2,\eta} \ . \tag{13}$$

In the present case, we need not consider the process of "conjugation" since it corresponds to a simple interchange of the roles of x^1 and x^2.

We now combine the functions χ and q_2 and Ψ and Φ into the pair of complex functions,

$$Z = \chi + iq_2 \quad \text{and} \quad Z^\dagger = \Psi + i\Phi \ . \tag{14}$$

and define

$$E = \frac{Z-1}{Z+1} \quad \text{and} \quad E^\dagger = \frac{Z^\dagger-1}{Z^\dagger+1} \ . \tag{15}$$

We find that <u>both</u> E and $E^\dagger$ satisfy the same Ernst-equation (9).

When we turn to the consideration of charged black-holes or the collision of gravitational waves coupled with electromagnetic waves, we must supplement Einstein's equations with Maxwell's equations. For space-times with the two symmetries we are considering, the Maxwell field can be expressed in terms of a single complex-potential H; and the entire set of equations governing the problem can eventually be reduced to a pair of coupled equations for

$$H \quad \text{and} \quad Z^\dagger = \Psi + i\Phi + |H|^2. \tag{16}$$

where Ψ is defined as in equations (5) and (12) and Φ is a potential for ω or q_2, defined similarly as in equations (6) and (13) but including additional terms in H on the right-hand sides.

There are two cases when the pair of equations governing $Z^\dagger$ and H can be reduced to a single Ernst equations. These are:

$$\text{Case (i):} \quad H = Q(Z^\dagger + 1) \ , \tag{17}$$

where Q is some real constant; and

$$\text{Case (ii):} \quad Z^\dagger = 1, \quad \Phi = 0, \quad \text{and} \quad \Psi = 1 - |H|^2 \ . \tag{18}$$

In case (i), with the definition,

$$E^{\dagger} = \frac{Z^{\dagger}-1}{Z^{\dagger}+1}, \tag{19}$$

we find that $E^{\dagger}$ satisfies the equation,

$$(1 - 4Q^2 - |E|^2)\{[(1-\eta^2)\, E,_{\eta}],_{\eta} - [(1 - \mu^2)E,_{\mu}],_{\mu}\}$$
$$= -2E^{*}[(1 - \eta^2)(E,_{\eta})^2 - (1 - \mu^2)(E,_{\mu})^2], \tag{20}$$

for <u>both</u> types of space-times we are presently considering. Moreover, it can be shown that if $E_{vac.}$ is a solution of the Ernst equation (9) for the vacuum, then

$$E_{Ei.Ma.} = E_{vac.} \sqrt{(1-4Q^2)} \tag{21}$$

is a solution of equation (20) appropriate for the Einstein-Maxwell equations. (It should be noted that in the stationary axisymmetric case, we should also consider the process of conjugation when the corresponding "tilded" variables will satisfy the same Ernst equation.)

In case (ii), we find that H satisfies the Ernst equation (9) for the vacuum so that we can write

$$H = E_{vac.} \qquad \text{and} \qquad \Psi = 1 - |E_{vac.}|^2. \tag{22}$$

The completion of the solution for the various problems we shall consider, particularly in the theory of colliding waves, often requires fairly elaborate analysis. We shall not describe any of that analysis since it is not needed for exhibiting the structure and the coherence of the entire theory.

VIII.

The origin of the structural similarity of the mathematical theory of black holes and of colliding waves stems from the circumstance that in both cases the Einstein and the Einstein-Maxwell equations are reducible to the same Ernst equation; and, indeed, as we shall see, even the same

solution. This identity is obtained only by the special choice of coordinates that assures the occurrence of an event horizon at a radial distance $\eta=1$ for black holes and the development of a singularity at time $\eta=1$ for colliding waves. The richness and the diversity of the physical situations that are described, in spite of this identify, results from the different combinations of the metric functions which can be associated with the same solution of the Ernst equation.

We shall consider first the solutions derived from the vacuum equations. The solution of the Ernst equation (9), from which the solutions describing the diverse physical situations follow, is the simplest one, namely,

$$E = p\eta + iq\mu \ , \tag{23}$$

where p and q are the two real constants restricted by the requirement,

$$p^2 + q^2 = 1 \ . \tag{24}$$

In the theory of black holes, the solution $p\eta + iq\mu$, applies to $\tilde{E}^{\dagger}$ i.e., to the Ernst equation for $\tilde{\Psi} + i\tilde{\Phi}$ belonging to the conjugate metric. The solution that follows is that of Kerr. It reduces to the Schwarzschild solution when $p = 1$ and $q = 0$. The resulting space-times of the Schwarzschild and the Kerr black holes are adequately described in text books and generally known. I shall mention only that these solutions belong to a special algebraic type, namely, type D in the Petrov classification. Solutions belonging to this type have many special properties. It is to these properties that we owe the separability of all the standard equations of mathematical physics in Kerr geometry.

Turning next to the theory of colliding waves, the fundamental solution is that of Khan and Penrose which describes the collision of two purely impulsive gravitational waves with parallel polarizations. It follows from the solution, $E = \eta$, of the Ernst equation for $\chi + iq_2$. The solution, $E=p\eta + iq\mu$, leads to the Nutku-Halil solution which describes the more general case when the colliding impulsive waves have non-parallel polarizations. Thus, the Khan-Penrose and the Nutku-Halil solutions play the

same role in the theory of colliding waves, as the Schwarzschild and the Kerr solutions play in the theory of black holes.

The combination, $\Psi + i\Phi$, of the metric functions, also leads to the same Ernst equation; and we are invited to consider the solution $p\eta + iq\mu$ for $E^{\dagger}$. The solution that follows has properties that were entirely unexpected: a horizon develops when $\eta = 1$, in place of a curvature singularity, violating a common belief that space-like curvature singularities are the rule when waves collide. In this instance, we must, therefore, extend the space-time beyond $\eta = 1$ and $|\mu| = 1$. When this extension is made, we find that the extended space-time includes a domain which is a mirror image of the one that was left behind and a further domain which includes hyperbolic arc-like singularities reminiscent of the ring singularity in the interior of the Kerr black-hole. It is remarkable that a space-time resulting from the collision of gravitational waves should bear such a close resemblance to Alice's anticipations with respect to the world *Through the Looking Glass: "It [the passage in the Looking-Glass House] is very like our passage as far as you can see, only it may be quite different on* beyond."

The foregoing remarks, concerning the solution derived from $E^{\dagger} = p\eta + iq\mu$, apply only when $q \neq 0$. When $q = 0$ and $p = 1$, a space-like curvature singularity develops at $\eta = 1$; and the space-time cannot be extended into the future.

Finally, it should be noted that the solution derived from $E^{\dagger} = p\eta + iq\mu$ is of type D and shares all the mathematical features of space-times belonging to this type.

Turning next to the Einstein-Maxwell equations, we are led to the solutions appropriate to charged black-holes when we consider the solution $\tilde{E} = p\eta + iq\mu$ $(p^2 + q^2 = 1-4Q^2)$ for the Ernst equation (20) for $\tilde{E}^{\dagger}$, in accordance with equation (21). We obtain the Reissner-Nordstrom solution when $q = 0$ and the Kerr-Newman solution when $q \neq 0$.

There were conceptual difficulties in obtaining the corresponding "elementary" solution of the Einstein-Maxwell equations for colliding waves.

Penrose had raised the question: would an impulsive gravitational wave with its associated δ-function singularity in the Weyl tensor imply a similar δ-function singularity in the energy-momentum tensor? If that should happen, then the expression for the Maxwell tensor would involve the square root of the δ-function; and "one would be at a loss to know how to interpret such a function". Besides, there was the formidable problem of satisfying the many boundary conditions at the various null boundaries. On these accounts, all efforts to obtain solutions compatible with carefully formulated initial conditions failed. However, when it was realized that the Khan-Penrose and the Nutku-Halil solutions followed from the simplest solution of the Ernst equation for $\chi + iq_2$, it was natural to seek a solution of the Einstein-Maxwell equations which will reduce to the Nutku-Halil solution when the Maxwell field is switched off. The problem is not a straightforward one: since, in the framework of the Einstein-Maxwell equations, we do not have an Ernst equation at the level of the metric functions χ and q_2: we have one only for $E^\dagger$ derived from $\Psi + i\Phi + |H|^2$. The technical problems that are presented can be successfully overcome and a solution can be obtained which satisfies all the necessary boundary conditions and physical requirements. That we can obtain a physically consistent solution by this "inverted procedure" is a manifestation of the firm aesthetic base of the general theory of relativity.

Since we do have an Ernst equation for $\Psi + i\Phi + |H|^2$, we can consider the solution $E = p\eta + iq\mu$ $(p^2 + q^2 = 1-4Q^2)$, for the Ernst equation (20) for $E^\dagger$. When $Q = 0$, this solution will reduce to the solution for the vacuum we have described earlier; and we find that, like the vacuum solution, it develops a horizon and subsequently, timelike singularities.

In our consideration of the Einstein-Maxwell equations in §VII, we have distinguished two cases: case (i) and case (ii). They differ in essential ways: when the electromagnetic field is switched off, the space-time, in case (i), reduces to a non-trivial solution of the Einstein vacuum-equations, while in case (ii), it becomes flat. The solutions we have considered hitherto belong to case (i). As we have seen, in case (ii) the complex electromagnetic potential, H, satisfies the Ernst equation (9) for the vacuum. We naturally ask the nature of the space-time that will follow

from the simplest solution, $p\eta + iq\mu$, of the Ernst equation. The solution one then obtains (discovered by Bell and Szekeres by different methods) is a very remarkable one: gravitation, as manifested by a non-vanishing Weyl scalar, is confined exclusively to the δ-function profile describing the impulsive gravitational waves. In other words, except for the presence of the impulsive waves, the space-time is conformally flat. Thus, as an exact solution of the Einstein-Maxwell equation, we have a conformally flat space-time in which plane-fronted electromagnetic shock-waves, accompanying impulsive gravitational waves, collide and develop a horizon. A further feature of the Bell-Szekeres solution is that the solution for $q = 0$ is entirely equivalent to the solution for $q \neq 0$. Therefore, to obtain a solution, in this framework, which will describe a more general physical situation than the Bell-Szekeres solution, we must go outside the range of the simplest solution of the Ernst equation. For this purpose, we take advantage of a transformation due to Ehlers which enables us to obtain a one-parameter family of solutions from any given solution of the Ernst equation. We therefore consider the Ehlers transform of the solution, $E = p\eta + iq\mu$. We find that the resulting solutions are of type D and have all the features of the solution for the vacuum derived from the solution $E^{\dagger} = p\eta + iq\mu$. It is remarkable that we should obtain a one-parameter family of space-times with this abundant structure by applying the Ehlers transformation to the Bell-Szekeres solution.

In Table 1, we describe more fully the various solutions that have been derived for black holes and for colliding waves. The pictorial pattern of this table is a visible manifestation of the structural unity of the subject.

The inner relationships between the theory of black holes and the theory of colliding waves is equally visible (see Table 2), in the simpler context when $\omega = 0$ and $q_2 = 0$. In this case, the basic equation, on which the solutions for both theories depend, is

$$[(1 - \eta^2)\ (\lg\Psi)_{,\eta}]_{,\eta} - [(1 - \mu^2)(\lg\Psi)_{,\mu}]_{,\mu} = 0\ . \qquad (25)$$

This equation can be solved exactly and the solutions that are relevant in the two theories are listed in Table 2.

Table 1

Killing Vectors	Field Equations	Solution for Ernst Equation for E	Solution for Ernst Equation for $E^{\dagger}$	Solution for Ernst Equation for $\tilde{E}^{\dagger}$
∂_t, ∂_Φ	Einstein-vacuum	does not exist		η
∂_t, ∂_Φ	Einstein-vacuum	does not exist		$p\eta+iq\mu$; $p^2+q^2=1$
∂_t, ∂_Φ	Einstein-Maxwell	does not exist		$\eta\nu(1-4Q^2)$
∂_t, ∂_Φ	Einstein-Maxwell	does not exist		$p\eta+iq\mu$; $p^2+q^2=1-4Q^2$
∂_{x^1}, ∂_{x^2}	Einstein-vacuum	η		interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-vacuum	$p\eta+iq\mu$; $p^2+q^2=1$		interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell	does not exist	$E^{\dagger}(E_{vac}=p\eta+iq\mu)$ $x\nu(1-4Q^2)$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-vacuum		η	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-vacuum		$p\eta+iq\mu$; $p^2+q^2=1$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell	does not exist	$\eta\nu(1-4Q^2)$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell	does not exist	$p\eta+iq\mu$; $p^2+q^2=1-4Q^2$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell ($H=E_{vac}$)	does not exist	η	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell ($H=E_{vac}$)	does not exist	$p\eta+iq\mu$; $p^2+q^2=1$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell ($H=E_{vac}$)	does not exist	Ehlers transform of $p\eta+iq\mu$	interchanges x^1 and x^2
∂_{x^1}, ∂_{x^2}	Einstein-hydrodynamics ($\epsilon=\rho$)	$p\eta+iq\mu$		interchanges x^1 and x^2

Table 1

Solution	Description
Schwarzschild	Black hole; static; spherically symmetric event horizon space-like singularity at centre type D; parameter: mass
Kerr	Black hole; stationary, axisymmetric event & Cauchy horizons; ergosphere time-like ring-singularity in equatorial plane type D; parameters: mass and angular momentum
Reissner-Nordström	Charged black-hole; static; spherically symmetric event and Cauchy horizons time-like singularity at centre type D; parameters: mass and charge
Kerr-Newman	Charged black-hole; stationary, axisymmetric event & Cauchy horizons; ergosphere time-like ring-singularity in equatorial plane type D; parameters: mass, charge and angular momentum
Khan and Penrose	Collision of impulsive gravitational waves parallel polarizations develops space-like curvature singularity
Nutku-Halil	Collision of impulsive gravitational waves non-parallel polarizations develops space-like curvature singularity (weaker than Khan-Penrose)
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational and electromagnetic shock waves non-parallel polarizations develops space-like curvature singularity
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational shock-waves parallel polarizations develops very strong space-like curvature singularity type D
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational shock-waves non-parallel polarizations develops a horizon and subsequent time-like arc-singularities type D
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational and electromagnetic shock-waves parallel polarizations develops a horizon and subsequent three-dimensional time-like singularities type D
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational and electromagnetic shock-waves non-parallel polarizations develops a horizon and subsequent time-like arc-singularities type D
Bell-Szekeres	Collision of impulsive gravitational waves and accompanying electromagnetic shock-waves parallel polarizations space-time conformally flat develops a horizon; permits extension with no subsequent singularities
Bell-Szekeres	Same as above
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational waves and accompanying gravitational and electromagnetic shock-waves develops a horizon and subsequent time-like arc singularities type D
Chandrasekhar and Xanthopoulos	Collision of impulsive gravitational and accompanying gravitational shock-waves and null-dust ($R_{ij}=Ck_ik_j$) non-parallel polarizations develops weakened space-like singularity transforms null-dust into a perfect fluid with $\epsilon=p$

Table 2

Basic Equations $[(1-\eta^2)(\lg\Psi)_{,\eta}]_{,\eta} - [(1-\mu^2)(\lg\Psi)_{,\mu}]_{,\mu} = 0$

Killing Vectors	Field equations	Solution	Remarks
∂_t, ∂_Φ	Einstein vacuum	$\lg\Psi = \frac{\eta-1}{\eta+1}$	Schwarzschild solution: spherically symmetric static black hole
∂_t, ∂_Φ	Einstein vacuum	$\lg\Psi = \frac{\eta-1}{\eta+1} + \sum_n A_n P_n(\mu) P_n(\eta)$	Distorted black-holes (when $\Sigma A_{2n+1} P_n(1)=0$) (Weyl's solutions)
∂_{x^1}, ∂_{x^2}	Einstein vacuum	$\lg\Psi = \frac{1+\eta}{1-\eta}$	Khan-Penrose solution for colliding impulsive waves with parallel polarizations
∂_{x^1}, ∂_{x^2}	Einstein vacuum	$\lg\Psi = \frac{1+\eta}{1-\eta} + \sum_n A_n P_n(\mu) P_n(\eta)$	Collision of impulsive gravitational waves with accompanying gravitational shock waves; parallel polarizations
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell ($H=E_{vac}$)	$\lg\Psi = \frac{1+\eta}{1-\eta}$	Collision of impulsive gravitational waves with accompanying electromagnetic shock-waves in conformally flat space-time; parallel polarizations; Bell-Szekeres solution
∂_{x^1}, ∂_{x^2}	Einstein-Maxwell ($H=E_{vac}$)	$\lg\Psi = \frac{1+\eta}{1-\eta} + \sum_n A_n P_n(\mu) P_n(\eta)$	Collision of impulsive gravitational waves with accompanying gravitational and electromagnetic shock-waves; parallel polarizations

As the foregoing discussion demonstrates, the Einstein-Maxwell equations share many of the distinctive features of the Einstein vacuum-equations. The only source, other than a Maxwell field, which when coupled with gravitation, leads to equations which retain, at least, some of the distinctive features of the vacuum equations, is a perfect fluid with the equation of state, energy density (ϵ) = pressure (p). For such a fluid, the Ricci tensor, in accordance with Einstein's equation, is given by

$$R^{ij} = -4\epsilon\ u^i u^j\ , \qquad (26)$$

where u^i denotes the four-velocity of the fluid.

On the assumption, that in the region of the interaction of the colliding waves, after the instant of collision, we have as source a perfect fluid with $\epsilon = p$, we find that <u>prior</u> to the instant of collision, the impulsive gravitational waves <u>must</u> have been accompanied by null-dust with an energy-momentum tensor of the form,

$$T^{ij} = Ek^i k^j = -\tfrac{1}{2}R^{ij}\ , \qquad (27)$$

where E is some positive scalar function and k^i denotes a null vector. In other words, under the circumstances envisaged, a transformation of null dust (i.e., massless particles discribing null trajectories) into a perfect fluid (whose stream lines follow time-like trajectories) occurs at the instant of collision. That such a transformation is required is, in the first instance, surprising. But as Roger Penrose and Lee Lindblom have pointed out, the transformation in question can take place, equally, in the frame-work of special relativity though this fact does not seem to have been notified before.

In developing the theory of colliding waves in parallel with the theory of black holes, we have, in effect, examined systematically the consequences of adopting for the Ernst equation, in its various contexts, its simplest solution (or, in one case, its Ehlers transform). While this approach may appear as an exceedingly formal one, it has nevertheless disclosed possibilities that one could not have, in any way, foreseen, as for example, the development of horizons and subsequent time-like singularities or the transformation of null dust into a perfect fluid. In this

instance, then, exploring general relativity, sensitive to its aesthetic base, has led to a deepening of our understanding of the physical content of the theory.

In his first announcement of his field equations in November 1915, Einstein concluded with the statement:

Anyone who fully comprehends this theory cannot escape its magic.

At least, to one practitioner, the magic of the theory is in the harmonious coherence of its mathematical structure.

REFERENCES

The nature of the lecture precludes giving a list of references in conventional style. However, the particular papers of Karl Schwarzschild quoted explicitly in the text are:

1. "Bemerkungen zur Elektrodynamik", Phys. Zeitsch. 4, 431 (1903).
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3. "Über das zulässige Krümmungsmaß des Raumes", Vierteljahrsschrift der Astronomischen Gesellschaft, 35, 337 (1900).
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5. "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie", Sitzungsberichte (Berliner Akademie), p. 424 (1916 March 23).
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Referring to Schwarzschild's paper on "Bemerkung zur Elektrodynamik", C. Lanczos (in "The Einstein Decade", Academic Press: New York and London, p. 152 (1974)) comments, *"Schwarzschild's remarkable anticipation of the relativistically invariant proper formulation of the principle of least*

action is not recorded in any history of physics".

For an account of Schwarzschild's contributions to the notions of action and angle variables, see

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APPENDIX

On 29 June 1916, Einstein gave a brief memorial address on Karl Schwarzschild at a meeting of the Berliner Akademie. It presents a proper measure of Schwarzschild; and I have thought it worthwhile to append the following translation of Einstein's address.

On May 11 of this year [1916], Karl Schwarzschild, 42 years old, was by death snatched away. This early demise of so gifted and many-sided a scientist is a grievous loss not only to this body, but also to all his astronomer and physicist friends.

What is specially astonishing about Schwarzchild's theoretical work is his easy command of mathematical methods and the almost casual way in which he could penetrate to the essence of astronomical or physical questions. Rarely has so much mathematical erudition been adapted to reasoning about physical reality. And so it was, that he grappled with many problems from which others shrank away on account of mathematical difficulties. The mainsprings of Schwarzschild's motivations in his restless theoretical quests seem less from a curiosity to learn the deeper inner relationships among the different aspects of Nature than from an artist's delight in discerning delicate mathematical patterns. It is therefore understandable that Schwarzschild's earliest contributions were in celestial mechanics, a branch of science whose foundations are more firmly established than any other. In this area, I may recall his investigations on the periodic solutions of the three-body problem and Poincare's theory of the equilibrium of rotating fluid masses.

Among the most important of Schwarzschild's astronomical contributions are his investigations on stellar statistics, i.e., a part of science which seeks by statistical methods to relate the observations on the luminosity, the velocity, and the spectral type of stars to the structure of a large system of many objects to which the Sun belongs. In this area, astronomers are indebted to him for deepening and widening their understanding of Kapteyn's discovery.

Schwarzschild directed his deep knowledge of theoretical physics to the theory of the Sun. Here, one is grateful for his investigations on the equilibrium of the solar atmosphere and for considerations relating to radiative transfer. To this area also belongs his beautiful investigations on the pressure of light on small spherical particles which provided an exact basis for Arrhenius' theory of comet tails. These investigations in theoretical physics, while they were motivated by astronomical questions, seem to have led Schwarzschild to be interested in questions purely in physics. We are indebted to him for his interesting contributions to the foundations of electrodynamics. Besides, in his last year he became interested in the new theory of gravitation: he succeeded in obtaining, for the first time, an exact calculation in the new theory of gravitation. And in the very last months of his life, much weakened by a fell disease, he yet succeeded in making

some profound contributions to quantum theory.

To Schwarzschild's great theoretical contributions also belong his investigations on geometrical optics in which he refined the theory of aberrations of optical instruments of astronomical importance. These studies will remain an enduring edifice for the perfection of the tools of astronomy.

Schwarzschild's theoretical investigations were carried out simultaneously with his efforts as a practical astronomer. From age 24, he worked at observatories without interruption: 1896-99 as an Assistant in Wien; 1901-09 as Director of the Göttingen Observatory; and after 1909 as Director of the Astrophysical Institute at Potsdam. A long series of investigations testify to his efforts as an observer and as a leader of astronomical observations. Moreover, his lively spirit led him to advance his scientifc field by charting new methods of observation. He discovered, in experimental physics, what has been named after him, how the blackening of a photographic plate can be used for the purposes of photometry by photographic methods. He also had the brilliant idea of using extra-focal images of stars for measuring their brightness: only through this idea did photographic photometry, besides visual methods, become feasible.

Since 1912, this modest man has been a member of this Akademie to whose Sitzungsberichte he has, in this short time communicated many important contributions. Now bitter circumstances have taken him away: but his work will bear fruits and have an enduring influence on Science for which he devoted all his strength.