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FOURIER ANALYSIS OF THE LIGHT CURVES OF THE RR LYRAE STARS IN THE GLOBULAR CLUSTERS M15, NGC 6171, AND NGC 6723[1]

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ABSTRACT

We compare the Fourier decomposition parameters of the *B* light curves of the RR Lyrae variables in the globular clusters NGC 6171, NGC 6723 (Osterhoff I), and M15 (Osterhoff II). Indications are that there may be small systematic differences between the Fourier coefficients of the variables of the two Osterhoff classes. The progression of the Fourier coefficients and their interrelations are very similar to (although less pronounced than) those in ω Cen. The significance and possible implications of these progressions on stellar parameters are briefly discussed. The first overtone variables are well separated from the fundamental ones in all parameter relations. A least-squares frequency analysis of those stars, which show greater scatter in the phase diagrams than the observational errors allow, reveals the complexity of their light variations and points to further double-mode suspects in each cluster. Analysis of the two separate data sets of M15, which were taken 20 yr apart, shows that, except perhaps for a few cases, there is no significant change in the light variations between the two epochs. The known double-mode RR Lyrae stars with reliable period identifications have not changed their modal contents either.

Subject headings: clusters: globular — numerical methods — stars: pulsation — stars: RR Lyrae

I. INTRODUCTION

Fourier decomposition has been proved to be a very useful quantitative tool for exhibiting the systematics of the light and velocity curves of variable stars (Simon and Lee 1981; Simon and Teays 1982; Hodson, Cox, and King 1982; Simon and Teays 1983; Simon and Davis 1983; Petersen 1984; Simon and Moffett 1985). In addition, knowledge of the Fourier coefficients of the observed variables gives a firm base for testing nonlinear hydrodynamical results and links the solution of the "amplitude equations" (Buchler and Goupil 1984; Klapp, Goupil, and Buchler 1985; Buchler and Kovács 1986) to the observations. In the case of Galactic Cepheids both the nonlinear hydrodynamical results and the study of the amplitude equations indicate that the observed systematics are caused by a 2:1 resonance between the fundamental and the second overtone, as originally suggested by Simon and Schmidt (1976). In the case of RR Lyrae stars, however, except for the separation of the first overtone variables from the fundamental ones, we either have no progression—i.e., systematic variation of the Fourier parameters with the period—as in the field RR Lyrae stars (Simon and Teays 1982), or, at best, a less pronounced one, as in the ω Cen variables (Petersen 1984). In order to study this issue and the behavior of the double-mode RR Lyrae stars further, we have Fourier analyzed the variables in NGC 6171, NGC 6723 (Osterhoff I), and M15 (Osterhoff II). For the discussion of the physical properties and the classification of these clusters see Sandage (1982, and references therein).

In § II, of this paper we present the Fourier coefficients of the three clusters. In § III we study in more detail some of the stars showing complicated light variations. We use a least-squares frequency analysis to decipher their behavior. The question of the constancy of the modal content of the double-mode RR Lyrae stars in M15 is examined in § IV. A discussion and our conclusions follow in § V.

II. THE FOURIER DECOMPOSITION

The photographic light curves were fitted in the usual way by a least-squares method with the Fourier sum:

$$A_0 + \sum_{k=1}^{M} A_k \sin\,[k\omega(t - t_0) + \phi_k]\,. \tag{1}$$

The order of the fit was generally limited to $M = 5$, because of the quality of the data. In many cases, however, where the phase coverage was good and the noise level was low so that eighth-order fits were meaningful, they gave very similar results, with typical differences of 1%–10% for the low-order amplitudes and phases. As suggested by Simon and Lee (1981) and by the theoretical framework of Buchler and Goupil (1984), we introduced the epoch (t_0) independent phase differences

$$\Phi_{k1} = \Phi_k - k\Phi_1\,, \tag{2}$$

as well as the amplitude ratio, R_{21}, defined by

$$R_{21} = \frac{A_2}{A_1}\,. \tag{3}$$

We note that an addition of $\pi/2$ to our luminosity phase differences brings them in concordance with the convention of Simon and Lee (1981), Simon and Teays (1982), and Petersen (1984).

a) The Cluster M15

We analyzed two data sets of *B* magnitudes of similar quality for this cluster, taken from the papers of Sandage, Katem, and Sandage (1981; hereafter SKS) and Bingham *et al.* (1984; hereafter BCDP). The Fourier decomposition parameters are presented in Tables 1 and 2, respectively. The double-

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TABLE 1

FOURIER PARAMETERS OF *B* LIGHT CURVES OF M15 RR LYRAE STARS OF SKS

STAR No.	PERIOD (DAYS)	A_0	A_1	A_2	A_3	A_4	A_5	ϕ_1	ϕ_{21}	ϕ_{31}	ϕ_{41}	ϕ_{51}
2	0.6842700	16.167	0.292	0.118	0.100	0.038	0.010	0.45	2.30	5.06	1.38	4.01
3	0.3887407	16.155	0.305	0.055	0.021	0.012	0.006	1.55	3.19	5.55	3.65	1.90
4	0.3135750	16.163	0.330	0.087	0.019	0.029	0.018	2.28	2.82	5.32	1.78	4.66
5	0.3842086	16.138	0.302	0.045	0.021	0.012	0.017	6.02	3.17	5.16	3.66	3.70
6	0.6659710	16.132	0.385	0.181	0.154	0.081	0.056	3.79	2.41	4.78	1.28	4.02
7	0.3675556	16.068	0.322	0.065	0.008	0.018	0.016	0.65	2.29	1.74	3.83	6.03
8	0.6462510	16.220	0.390	0.174	0.139	0.099	0.072	4.14	2.20	4.64	1.04	3.72
9	0.7153076	16.133	0.355	0.169	0.098	0.073	0.040	0.07	2.46	5.24	1.58	4.44
10	0.3864102	16.204	0.286	0.053	0.010	0.010	0.007	6.26	2.93	6.02	3.27	3.78
11	0.3432499	16.158	0.342	0.070	0.018	0.021	0.003	3.60	2.92	5.83	3.81	5.48
12	0.5929151	16.279	0.424	0.162	0.126	0.054	0.048	2.81	2.20	4.43	0.34	2.89
13	0.5749610	16.299	0.448	0.160	0.144	0.081	0.061	5.71	2.25	4.66	0.60	3.31
14	0.3819990	16.260	0.311	0.048	0.029	0.004	0.011	5.62	3.09	5.96	2.01	3.72
15	0.5835668	16.333	0.409	0.172	0.113	0.075	0.054	0.38	2.27	4.64	1.04	3.83
16	0.3992340	16.170	0.264	0.033	0.016	0.015	0.018	2.48	3.08	4.93	1.95	0.41
17	0.4288717	16.194	0.172	0.013	0.027	0.022	0.012	0.98	6.26	3.71	2.25	5.99
18	0.3677382	16.172	0.334	0.077	0.032	0.010	0.014	5.72	3.08	5.46	3.76	0.60
19	0.5722930	16.210	0.560	0.246	0.172	0.113	0.067	4.64	2.34	4.78	0.66	3.13
20	0.6969320	16.178	0.358	0.154	0.119	0.080	0.048	1.43	2.31	4.64	1.14	3.91
21*	0.6488000	16.014	0.340	0.188	0.098	0.053	0.026	3.89	2.22	4.82	0.51	3.04
22	0.7202220	16.148	0.386	0.210	0.149	0.080	0.024	4.77	2.28	4.69	0.97	3.21
23*	0.6327203	16.269	0.370	0.121	0.103	0.061	0.041	3.95	2.11	4.39	0.50	3.32
24	0.3696970	16.157	0.290	0.059	0.052	0.017	0.022	6.18	3.08	5.17	4.01	0.62
25*	0.6653290	16.307	0.349	0.128	0.127	0.082	0.067	1.26	2.46	5.15	1.87	4.51
26	0.4022426	16.267	0.204	0.026	0.024	0.004	0.004	4.59	3.17	4.95	4.05	1.88
29	0.5749961	16.331	0.418	0.148	0.057	0.028	0.026	2.64	2.09	4.89	0.55	3.06
30	0.4059760	16.176	0.175	0.032	0.016	0.014	0.003	5.04	3.70	6.12	1.26	5.98
31	0.4082310	16.213	0.196	0.035	0.028	0.079	0.042	0.57	3.35	3.78	0.02	2.49
32	0.6054000	16.105	0.435	0.172	0.142	0.096	0.061	3.41	2.00	4.50	0.90	3.06
33*	0.5836284	15.802	0.385	0.230	0.158	0.109	0.072	5.34	2.12	4.78	0.22	2.85
35	0.3839970	16.238	0.304	0.036	0.025	0.016	0.011	0.57	3.28	5.54	4.70	0.26
36*	0.6241696	16.141	0.425	0.136	0.103	0.057	0.012	4.37	2.25	4.44	0.24	1.41
38	0.3752740	16.168	0.313	0.060	0.024	0.011	0.006	6.05	3.78	5.47	4.34	0.33
39	0.3895680	16.221	0.238	0.038	0.016	0.022	0.012	5.38	2.81	0.52	1.90	0.09
40	0.3773212	16.200	0.324	0.071	0.033	0.025	0.018	0.74	2.98	4.87	4.11	2.54
41	0.3917430	16.049	0.260	0.020	0.016	0.006	0.019	4.99	2.83	0.09	4.14	4.48
42	0.3601746	16.251	0.351	0.082	0.031	0.020	0.021	4.58	3.19	5.28	2.47	0.62
43*	0.3960090	16.226	0.320	0.027	0.016	0.016	0.104	1.88	3.33	5.56	3.50	0.04
44*	0.5956433	16.118	0.298	0.114	0.101	0.047	0.029	3.32	2.09	4.46	6.24	2.63
46*	0.6914840	16.048	0.422	0.172	0.102	0.052	0.045	5.52	2.50	4.84	0.48	3.48
48	0.3649720	16.154	0.328	0.082	0.041	0.013	0.019	2.47	2.96	5.57	2.11	2.44
49*	0.6552020	15.482	0.285	0.183	0.115	0.085	0.074	3.86	1.85	4.22	0.61	3.15
50*	0.2980612	16.215	0.314	0.083	0.034	0.021	0.014	5.01	2.48	4.09	0.47	5.13
51*	0.3969870	16.136	0.238	0.060	0.024	0.017	0.012	4.14	3.31	0.82	5.15	1.78
52	0.5756510	16.382	0.480	0.164	0.134	0.109	0.062	2.04	2.13	4.53	0.73	3.02
53	0.4141614	16.143	0.258	0.004	0.021	0.017	0.007	2.62	3.23	2.05	5.77	3.17
54*	0.3995741	16.090	0.246	0.056	0.007	0.011	0.014	1.01	3.04	1.01	1.72	1.99
55	0.7485900	16.192	0.247	0.087	0.064	0.029	0.018	5.92	2.49	4.75	1.53	5.18
56*	0.5703530	15.889	0.277	0.163	0.129	0.082	0.017	3.58	1.36	3.62	6.24	2.57
57*	0.3496144	16.079	0.243	0.027	0.026	0.044	0.026	0.28	3.48	3.08	5.35	0.72
58	0.4076690	16.036	0.214	0.037	0.046	0.045	0.019	5.58	1.42	2.99	6.22	2.35
61	0.3996600	16.086	0.274	0.055	0.015	0.025	0.017	2.64	2.89	6.04	0.50	1.04
62*	0.3773180	16.027	0.308	0.066	0.023	0.015	0.008	3.96	2.53	5.96	2.92	4.81
64*	0.3641910	15.958	0.348	0.072	0.030	0.043	0.013	4.72	2.99	0.94	3.81	0.56
65	0.7181990	16.191	0.242	0.095	0.066	0.036	0.001	4.62	2.28	4.94	1.50	3.38
66	0.3793547	16.211	0.271	0.033	0.027	0.010	0.032	0.80	2.57	6.26	2.39	0.36
67	0.3998640	16.203	0.183	0.021	0.038	0.035	0.019	2.21	2.10	6.24	4.52	0.37
74	0.2960076	16.156	0.330	0.107	0.018	0.018	0.019	1.49	2.71	5.11	1.50	4.76
97	0.6963420	16.184	0.294	0.134	0.092	0.039	0.020	0.12	2.20	4.83	1.26	4.13
102*	0.7594941	16.164	0.174	0.052	0.024	0.020	0.004	1.48	2.40	5.84	4.35	4.12

NOTE.—Stars denoted by asterisks were frequency analyzed. Amplitudes are measured in magnitudes and phases in radians.

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TABLE 2

FOURIER PARAMETERS OF *B* LIGHT CURVES OF M15 RR LYRAE STARS OF BCDP

STAR No.	PERIOD (DAYS)	A_0	A_1	A_2	A_3	A_4	A_5	ϕ_1	ϕ_{21}	ϕ_{31}	ϕ_{41}	ϕ_{51}
2*	0.6843490	16.094	0.293	0.077	0.056	0.030	0.016	5.77	2.74	4.26	0.89	4.50
3*	0.3887485	16.225	0.290	0.063	0.034	0.019	0.036	5.18	2.35	5.26	0.45	5.08
4	0.3135718	16.200	0.362	0.072	0.021	0.016	0.029	5.11	2.99	3.98	0.64	4.88
5*	0.3842204	16.136	0.267	0.026	0.063	0.031	0.041	5.71	1.90	5.04	3.52	0.66
6*	0.6659592	16.028	0.391	0.167	0.100	0.074	0.019	6.08	2.17	4.67	1.53	4.34
7*	0.3675770	15.906	0.363	0.055	0.017	0.091	0.055	3.96	1.16	1.36	3.05	5.39
8*	0.6462446	16.216	0.385	0.190	0.155	0.087	0.067	1.61	2.23	5.10	1.36	4.49
9*	0.7152819	16.099	0.326	0.157	0.089	0.064	0.030	5.50	2.20	4.75	1.66	4.46
10*	0.3863931	16.193	0.281	0.033	0.012	0.027	0.021	5.80	2.66	3.65	5.24	2.56
11	0.3432150	16.173	0.366	0.074	0.021	0.047	0.021	2.20	2.65	5.77	1.91	4.75
12	0.5928520	16.257	0.390	0.142	0.092	0.059	0.024	0.98	2.46	4.58	1.06	4.16
13*	0.5749230	16.259	0.418	0.205	0.182	0.133	0.081	3.54	1.96	4.52	1.05	4.02
14*	0.3820000	16.252	0.291	0.061	0.025	0.013	0.013	2.08	2.41	5.30	1.54	3.02
15*	0.5835090	16.338	0.473	0.143	0.072	0.104	0.048	3.95	2.43	4.47	1.20	1.29
16*	0.3992180	16.117	0.258	0.062	0.048	0.022	0.008	3.42	3.55	4.99	5.20	6.07
17	0.4288924	16.201	0.215	0.037	0.017	0.013	0.024	5.43	2.36	4.60	4.65	1.03
18*	0.3677459	16.146	0.322	0.030	0.037	0.003	0.022	3.37	2.46	5.35	2.50	2.23
19*	0.5723030	16.254	0.544	0.278	0.170	0.122	0.111	2.16	2.11	4.46	0.90	3.09
20*	0.6969413	16.114	0.366	0.159	0.072	0.092	0.046	3.76	2.32	4.88	1.54	4.61
22*	0.7201734	16.218	0.329	0.154	0.083	0.020	0.031	2.26	2.38	4.81	3.80	0.28
23*	0.6326848	16.235	0.353	0.124	0.089	0.058	0.020	3.82	1.98	4.18	0.25	5.16
24*	0.3696955	16.122	0.316	0.032	0.029	0.009	0.035	3.24	3.66	4.16	4.19	1.96
25*	0.6653286	16.281	0.398	0.172	0.144	0.099	0.039	2.59	2.05	4.57	0.96	3.86
26	0.4022770	16.257	0.250	0.043	0.021	0.020	0.039	3.67	2.98	0.30	2.14	2.04
28*	0.6706464	16.275	0.328	0.205	0.084	0.130	0.044	2.27	2.67	5.64	1.69	5.11
29	0.5749761	16.329	0.433	0.207	0.123	0.065	0.009	1.73	2.23	4.46	0.52	3.53
30	0.4059796	16.102	0.227	0.027	0.033	0.019	0.036	1.70	3.01	3.18	3.30	4.71
31	0.4081781	16.249	0.214	0.042	0.045	0.022	0.025	5.74	2.85	3.97	0.34	3.18
32*	0.6054150	16.021	0.436	0.182	0.182	0.082	0.090	5.31	1.97	4.57	0.30	2.82
35*	0.3840009	16.256	0.294	0.063	0.008	0.026	0.022	5.51	3.05	0.59	1.70	0.98
36*	0.6241424	16.102	0.381	0.159	0.097	0.079	0.048	3.82	2.05	4.44	1.25	3.49
38*	0.3752713	16.159	0.299	0.046	0.022	0.004	0.032	3.02	3.10	5.41	4.88	0.73
39	0.3895696	16.150	0.239	0.039	0.040	0.034	0.047	1.22	2.85	5.26	0.21	4.82
40*	0.3773360	16.202	0.262	0.091	0.040	0.040	0.032	1.84	2.79	5.45	5.33	0.02
41	0.3917493	15.967	0.264	0.045	0.019	0.018	0.029	3.38	2.57	0.46	4.74	2.31
42	0.3601745	16.236	0.358	0.073	0.037	0.020	0.023	4.35	2.85	5.99	3.52	6.12
43*	0.3960156	16.179	0.288	0.039	0.012	0.014	0.024	4.73	2.82	0.15	0.99	4.44
44*	0.5956280	15.995	0.300	0.114	0.109	0.069	0.093	3.21	2.36	3.48	5.57	1.11
48*	0.3649762	16.080	0.331	0.109	0.029	0.048	0.036	2.64	3.46	2.29	3.53	3.51
49*	0.6552054	15.372	0.226	0.061	0.066	0.020	0.027	3.92	2.65	5.01	2.15	3.59
50	0.2980583	16.217	0.310	0.096	0.026	0.013	0.026	1.81	2.54	5.16	2.12	5.17
51	0.3969565	16.068	0.238	0.017	0.021	0.065	0.033	3.67	2.74	3.49	3.00	4.30
52	0.5756132	16.390	0.424	0.146	0.072	0.067	0.041	6.01	2.30	4.44	0.52	2.92
53	0.4141270	16.139	0.256	0.063	0.023	0.048	0.036	1.79	3.59	0.28	4.90	2.16
54	0.3995683	16.019	0.225	0.028	0.018	0.012	0.040	4.96	2.77	2.44	4.31	1.62
55*	0.7486232	16.094	0.281	0.051	0.117	0.058	0.010	4.51	3.08	5.08	3.39	4.94
57*	0.3492640	15.948	0.272	0.043	0.029	0.038	0.030	0.79	3.03	0.98	4.02	4.93
65*	0.7181960	16.136	0.237	0.138	0.055	0.041	0.041	5.60	2.68	5.54	0.72	5.01
66*	0.3793396	16.166	0.248	0.081	0.016	0.025	0.028	6.02	3.17	0.31	1.57	2.89
67	0.4046130	16.087	0.238	0.047	0.073	0.014	0.043	1.49	2.66	4.99	1.58	2.42
74*	0.2960103	16.131	0.309	0.083	0.050	0.011	0.024	1.63	2.57	5.39	2.06	2.00
96	0.3963520	16.272	0.292	0.074	0.020	0.016	0.019	5.85	2.98	5.00	1.16	1.37
97	0.6963330	16.120	0.329	0.147	0.127	0.064	0.013	3.02	2.31	4.61	1.17	4.21
101*	0.4007374	16.179	0.310	0.050	0.056	0.065	0.069	2.09	3.51	4.35	5.10	4.30
102*	0.7594401	16.072	0.238	0.027	0.029	0.032	0.030	2.01	2.76	4.45	3.42	4.13
103*	0.3682424	16.222	0.338	0.078	0.011	0.009	0.036	3.24	3.01	5.06	1.25	6.09

NOTE.—Stars denoted by asterisks were frequency analyzed. Amplitudes are measured in magnitudes and phases in radians.

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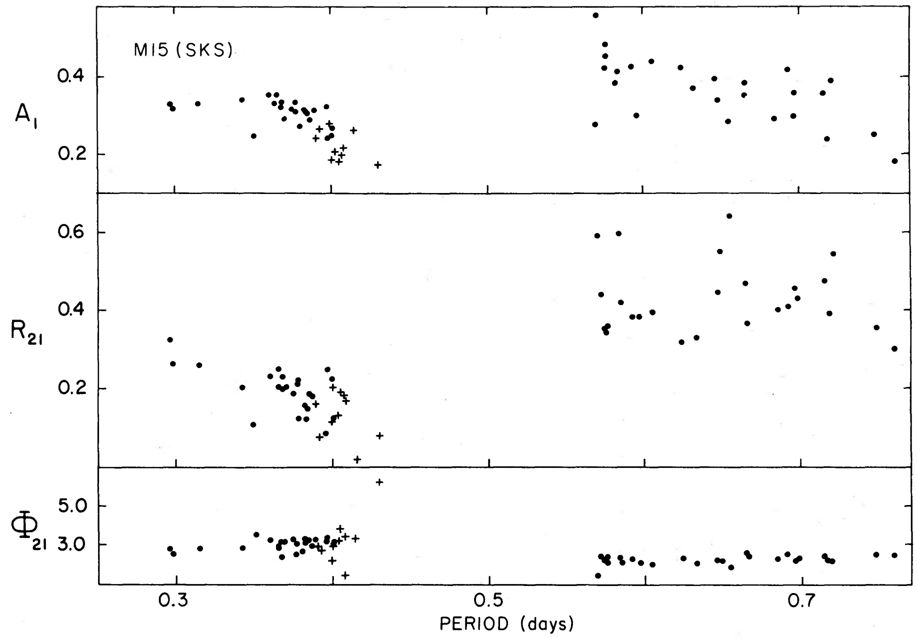

FIG. 1.—Fourier amplitude, amplitude ratio, and phase difference vs. period for the *B* light curves of the RR Lyrae stars in M15 for the data of SKS. Double-mode stars listed by CHC are denoted by crosses.

mode RR Lyrae stars of Cox, Hodson, and Clancy (1983, hereafter CHC) are also included in the tables. They were fitted with a single period corresponding to the first overtone, in order to determine whether they stand out from the single-mode stars in a Fourier decomposition. We note that the variable names are shown according to BCDP, i.e., V67 ≡ III-5, V74 ≡ IV-41, V97 ≡ II-26, and V102 ≡ I-24. We took the Julian Date at the first datum of the *B* observations as an epoch, i.e., $t_0 = 2{,}436{,}136.720$ (except $t_0 = 2{,}436{,}136.738$ for V26 and V33) for the SKS data, and $t_0 = 2{,}442{,}274.506$ (except $t_0 = 2{,}442{,}276.486$ for V96) for the BCDP data. Inspection of the folded light curves (*phase curves*) with the published periods prompted us to a further study of the stars denoted by asterisks. In these cases we either improved the published period or left it unchanged if no improvement could be achieved. In all other cases we accepted the published periods.

The problems concerning the studied stars will be presented in the next section. Here we only remark that, based on the frequency analysis and the smoothness of the phase curves, we adopted our new periods for stars 33, 36, 50, and 54 in the SKS data and for stars 3, 4, 5, 6, 14, 18, 20, 22, 23, 35, 40, 43, 55, 66, 74, 101, and 102 in the BCDP data. The improved periods in the SKS data are closer to those of BCDP. In the case of V15 and V42 of SKS we reanalyzed the data by omitting the suspicious outstanding points. This yielded the following A_1, R_{21}, A_3, A_4, ϕ_{21}, ϕ_{31}, ϕ_{41} values: 0.375, 0.37, 0.082, 0.055, 2.27, 4,74, 1.45 for V15 and 0.336, 0.22, 0.027, 0.009, 3.08, 5.28, 3.45 for V42. Because we had no *a priori* reason to exclude these points from the analysis, Table 1 contains the results including all data. The variables 25, 55, and 96 of BCDP have gapped phase curves; therefore, their Fourier coefficients are less reliable. Variable V44 has a very noisy and different light curve from the other variables of similar periods.

Phase-period, amplitude-period and phase-phase relations according to Tables 1 and 2 are shown in Figures 1 and 2 for the SKS data, and in Figures 3 and 4 for the BCDP data. It is

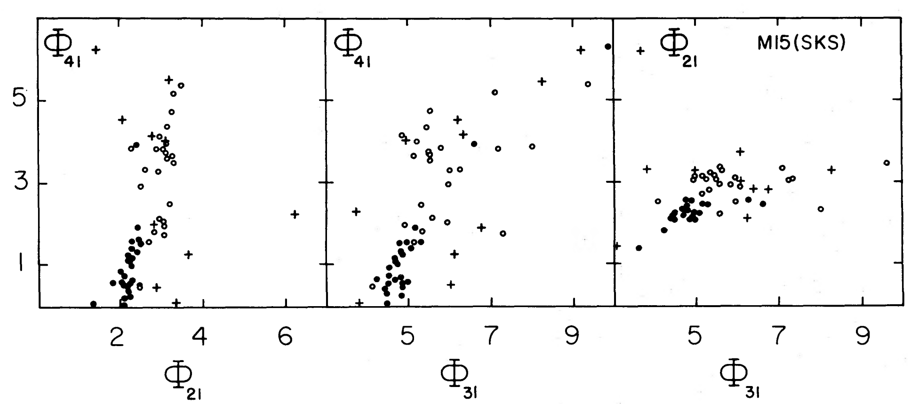

FIG. 2.—Interrelations of the Fourier phase differences for the *B* light curves of the RR Lyrae stars in M15 for the data of SKS. Open circles denote the first overtone, crosses the double-mode (as listed by CHC), and filled circles the fundamental variables. The outstanding point (star V102) has very small Fourier amplitudes (see Table 1).

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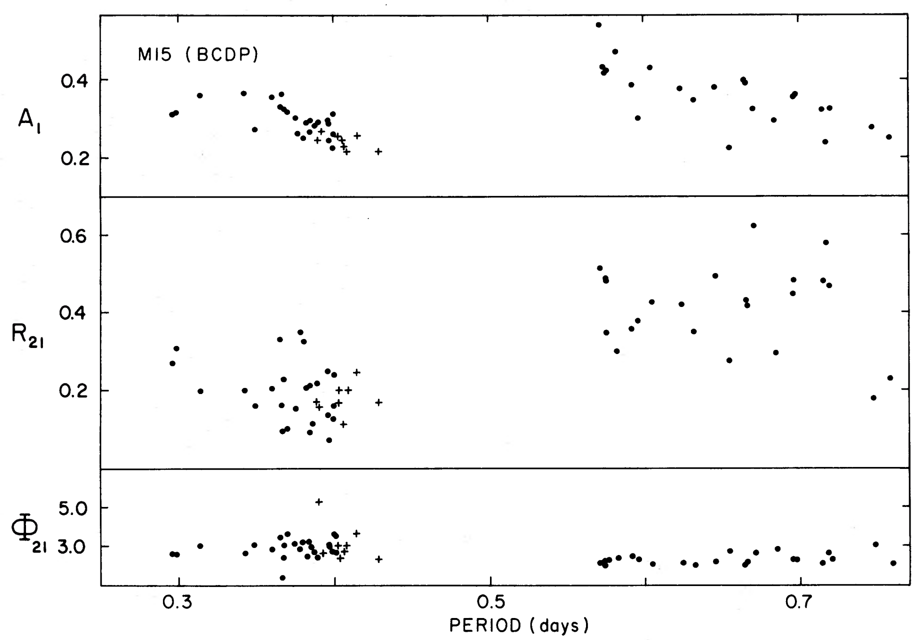

FIG. 3.—Same as in Fig. 1, but for the BCDP data

seen that the SKS data are less noisy (this is also seen by the comparison of the dispersion of the data relative to the Fourier fits, which was ~0.06–0.04 mag for the SKS and 0.10–0.05 for the BCDP data). However, in both cases we can conclude that the period versus ϕ_{21}, R_{21}, and A_1 relations are relatively featureless, except perhaps for the first overtone variables, where a trend similar to that of ω Cen (Petersen 1984) can be suspected. The lack of a well-defined progression for the RRab stars can partly be accounted for by the shorter range of periods for these stars in M15, as compared to the range for ω Cen.

Figures 1–4 show that the double-mode RR Lyrae stars of CHC are not found to stand out clearly from the first overtone variables when they are analyzed as singly periodic.

The phase-phase relations show a similar structure, although a less pronounced one than in the case of the Galactic Cepheids (Simon and Moffett 1985). First overtone and double-mode variables clearly stand out from the fundamental ones in all relations, although phase-phase plots by themselves do not allow such a differentiation.

b) The Cluster NGC 6171

We analyzed the *B* magnitudes published by Dickens (1970). The results are shown in Table 3 and Figures 5 and 6. We used Dicken's periods because our reanalysis of the problematic stars did not improve the fit. Variable 7 was omitted because of the bad phase coverage. The epoch was taken to be t_0 = 2,439,258.976, except for V20, where t_0 = 2,439,259.898.

We see that the features of NGC 6171 are essentially the same as those of M15 and that there are no major differences between the RR Lyrae stars of these two clusters (belonging to different Osterhoff groups) as far as their light curve systematics are considered (see, however, § V).

c) The Cluster NGC 6723

We analyzed the *B* magnitudes published by Menzies (1974). We omitted from our analysis some stars which had a bad phase coverage or for which only few data were available. The results are shown in Table 4 and Figures 7 and 8. In the case of variables 5, 6, 12, 31, 32 we used our periods instead of

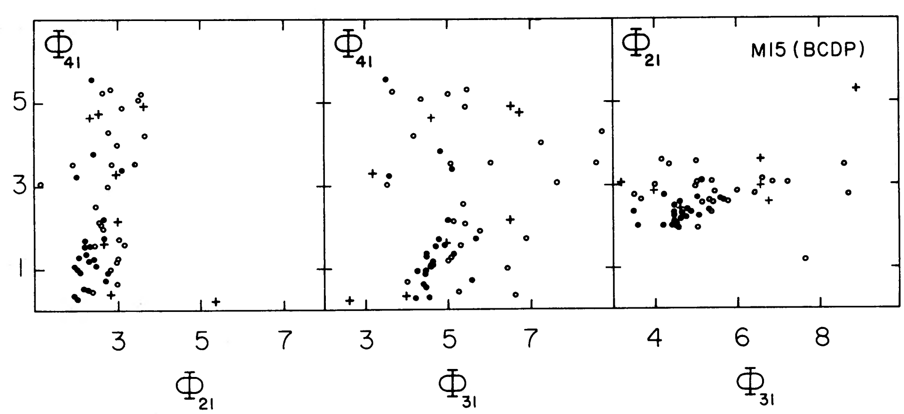

FIG. 4.—Same as in Fig. 2, but for the BCDP data

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TABLE 3

FOURIER PARAMETERS OF *B* LIGHT CURVES OF NGC 6171 RR LYRAE STARS

STAR No.	PERIOD (DAYS)	A_0	A_1	A_2	A_3	A_4	A_5	ϕ_1	ϕ_{21}	ϕ_{31}	ϕ_{41}	ϕ_{51}
1	0.3233436	16.197	0.276	0.037	0.015	0.012	0.023	4.49	2.25	6.12	2.64	6.06
2	0.5710205	16.381	0.394	0.173	0.117	0.062	0.021	4.93	2.39	5.25	1.65	4.78
3	0.5663430	16.138	0.278	0.124	0.056	0.034	0.015	1.73	2.72	5.53	2.30	1.00
4	0.2821320	16.048	0.315	0.038	0.026	0.011	0.015	0.04	3.17	0.68	3.76	5.80
5	0.7024000	16.293	0.244	0.080	0.045	0.004	0.013	5.81	2.75	5.21	3.35	5.20
6	0.2602558	16.069	0.325	0.068	0.012	0.039	0.018	5.53	3.09	0.80	3.44	0.84
8	0.5599240	16.218	0.474	0.248	0.145	0.086	0.042	4.44	2.32	5.24	1.54	4.50
9	0.3206050	16.101	0.311	0.040	0.035	0.018	0.019	1.89	3.44	0.64	5.32	2.09
10*	0.4155450	16.404	0.444	0.237	0.174	0.086	0.047	0.88	2.05	4.70	1.03	3.08
11	0.5928000	16.376	0.327	0.151	0.092	0.046	0.046	1.80	2.41	4.92	2.10	3.96
12	0.4729400	16.382	0.539	0.222	0.114	0.053	0.036	0.45	2.18	4.42	0.21	3.09
13	0.4667980	16.534	0.601	0.272	0.184	0.140	0.114	3.18	2.15	4.71	1.16	3.62
14	0.4816150	16.344	0.598	0.281	0.210	0.145	0.113	4.75	2.38	5.21	1.76	4.61
15	0.2885910	16.162	0.312	0.024	0.019	0.020	0.010	4.22	2.89	6.25	3.82	1.09
16*	0.5228300	16.462	0.449	0.184	0.105	0.067	0.053	4.78	2.58	5.17	1.86	5.19
17	0.5611650	16.279	0.475	0.245	0.154	0.105	0.067	3.44	2.37	5.19	1.61	4.50
18	0.5643780	16.502	0.369	0.225	0.128	0.056	0.037	4.58	2.34	4.98	1.78	4.53
19*	0.2787600	16.322	0.329	0.055	0.029	0.021	0.008	0.64	3.05	5.79	3.38	2.00
20	0.5781500	16.318	0.385	0.173	0.115	0.030	0.014	3.45	2.52	5.14	2.25	0.81
21	0.2583110	17.078	0.280	0.078	0.015	0.020	0.018	1.74	2.91	0.49	2.82	4.77

NOTE.—Stars denoted by asterisks were frequency analyzed. Amplitudes are measured in magnitude and phases in radians.

those of Menzies because they gave rise to smoother phase curves. Because of the bad phase coverage and large noise, the parameters for variables 3 and 12 are less reliable. The inclusion or omission of the outstanding points for the light curves of variables 1 and 18 did not appreciably affect the Fourier parameters shown in Table 4. The epoch is $t_0 =$ 2,438,610.1741, except for V11, V12, V20, V31, and V32, where $t_0 = 2,438,612.0772$.

Except perhaps for a larger ϕ_{21} value for the first overtone variables, we have again parameter relations very similar to those of other groups of RR Lyrae stars (although with larger scatter).

III. LEAST-SQUARES FREQUENCY ANALYSIS

Because the published periods gave rise to a large dispersion of the phase curves, we found it necessary to start by frequency analyzing some of the data. We used the method of the least-squares frequency analysis (see, e.g., Barning 1963), who fitted the data by least squares for each test frequency and calculated

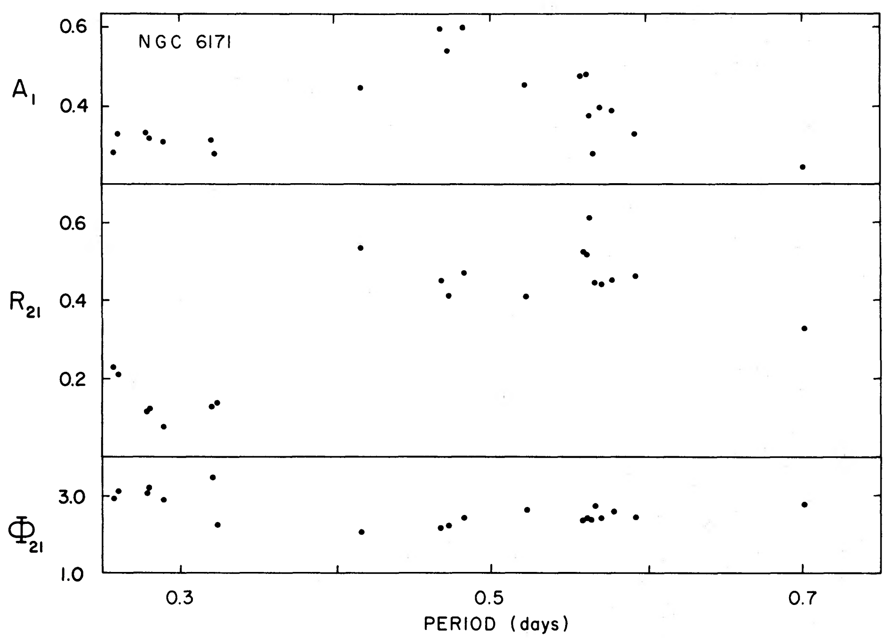

FIG. 5.—Same as in Fig. 1, but for NGC 6171

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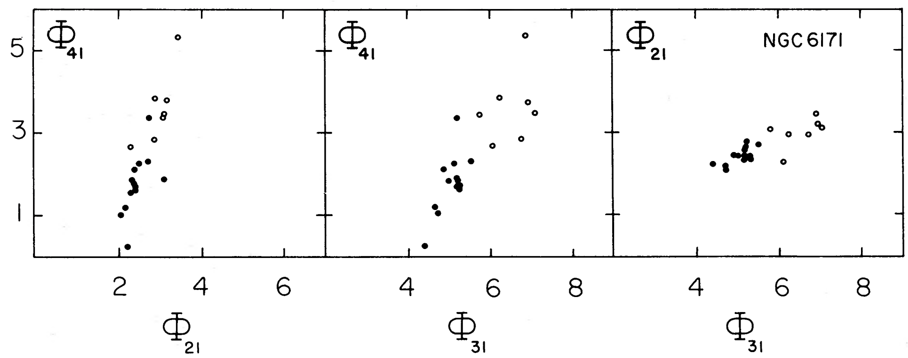

FIG. 6.—Same as in Fig. 2, but for NGC 6171

the fractional decrease of the dispersion of the data; i.e.,

$$R = \frac{\sum_{n=1}^{N} x_n^2 - \sum_{n=1}^{N} (x_n - A_0 - A \cos \omega t_n - B \sin \omega t_n)^2}{\sum_{n=1}^{N} x_n^2}, \tag{4}$$

where t_n and x_n are the time and magnitude values, respectively, and ω is the test frequency. A more convenient expression for calculating the above quantity is

$$R = \frac{AC + BS}{\sum_{n=1}^{N} x_n^2}, \tag{5}$$

where

$$C = \sum_{n=1}^{N} x_n \cos \omega t, \tag{6}$$

$$S = \sum_{n=1}^{N} x_n \sin \omega t_n. \tag{7}$$

Here we summarize our results relating only to those stars which were not listed by CHC as double-mode ones. The latter will be discussed in the next section.

a) *The Cluster M15*

We performed a thorough two-step frequency analysis of the SKS data: first, we did a period search between 0.0 and 5.0 cycle per day (c/d) with a very small frequency step; then we prewhitened with the period of the highest peak and repeated the period search on the residuals. A similar analysis with the BCDP data was very difficult because of the paucity of observations, the longer time base, the worse temporal data distribution, and the higher noise level. Period searches for these data were restricted solely to the very close neighborhood (typically ± 0.01 days) of the published periods, and they were aimed at finding the best single periodic fit to the data (see the previous section for the list of the stars with improved periods).

Our remarks concerning the individual stars analyzed in the SKS data are as follows:

1. *V23, V25, V43, V46, V50, and V51.*—Both the frequency spectrum of the original data and of the prewhitened data are

TABLE 4

FOURIER PARAMETERS OF *B* LIGHT CURVES OF NGC 6723 RR LYRAE STARS

STAR No.	PERIOD (DAYS)	A_0	A_1	A_2	A_3	A_4	A_5	ϕ_1	ϕ_{21}	ϕ_{31}	ϕ_{41}	ϕ_{51}
1*	0.5383594	16.049	0.239	0.046	0.030	0.050	0.051	0.81	2.61	3.78	0.88	2.54
2	0.5035345	15.734	0.689	0.274	0.230	0.140	0.154	3.03	2.10	4.86	1.24	3.64
3*	0.4940990	15.794	0.573	0.274	0.254	0.093	0.189	3.14	2.83	5.02	2.28	4.25
4	0.4510600	15.598	0.614	0.265	0.171	0.104	0.063	6.14	2.23	4.64	0.84	3.59
5*	0.5767548	15.358	0.493	0.206	0.137	0.070	0.023	1.08	2.43	5.87	2.61	0.42
6*	0.4815478	14.966	0.496	0.126	0.142	0.174	0.161	2.44	2.59	0.30	3.28	5.99
7*	0.3076726	15.869	0.288	0.027	0.004	0.010	0.037	6.24	4.15	0.28	4.94	0.11
9*	0.5758030	15.322	0.474	0.255	0.136	0.058	0.047	5.41	2.05	4.30	0.32	4.10
10*	0.2523260	15.747	0.279	0.054	0.024	0.017	0.029	3.07	2.57	4.75	3.07	3.68
11	0.5342835	15.854	0.449	0.230	0.139	0.127	0.094	5.79	2.47	5.12	1.31	3.71
12*	0.4706119	15.001	0.784	0.159	0.328	0.187	0.208	1.46	2.24	5.33	0.62	5.06
15	0.4354410	15.726	0.648	0.348	0.248	0.156	0.091	3.37	2.12	4.56	0.67	3.38
16	0.6962725	15.548	0.382	0.196	0.122	0.049	0.052	0.17	2.63	5.32	1.92	5.27
17	0.5301790	16.071	0.508	0.264	0.138	0.112	0.057	0.95	2.26	4.98	1.15	4.75
18*	0.5264550	15.864	0.385	0.184	0.132	0.082	0.055	2.50	2.27	5.03	1.13	3.52
19	0.5347045	16.000	0.425	0.228	0.155	0.088	0.034	5.08	2.62	5.20	1.67	4.56
21*	0.5948630	15.601	0.433	0.139	0.101	0.057	0.046	5.06	2.42	5.19	2.46	5.64
23*	0.6259000	15.315	0.354	0.096	0.023	0.090	0.025	1.86	2.98	3.80	0.57	3.11
24*	0.3001437	15.820	0.270	0.015	0.010	0.013	0.037	5.88	3.74	0.91	3.15	1.82
27	0.6192480	15.970	0.308	0.148	0.083	0.015	0.004	4.26	2.60	5.21	1.80	4.24
31*	0.6088140	15.638	0.331	0.157	0.072	0.022	0.040	3.64	2.30	5.69	0.36	3.65
32*	0.2875419	15.386	0.222	0.037	0.050	0.026	0.018	0.14	4.80	6.25	2.94	2.50

NOTE.—Stars denoted by asterisks were frequency analyzed. Amplitudes are measured in magnitude and phases in radians.

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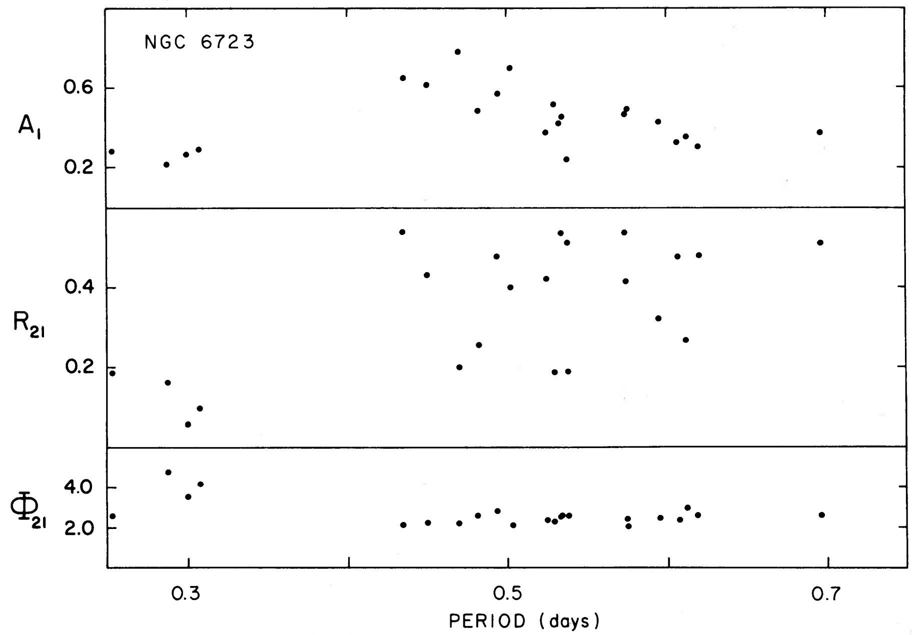

FIG. 7.—Same as in Fig. 1, but for NGC 6723

relatively simple, showing a small amount of noise and in some cases significantly higher harmonic components. The representative case of V46 is shown in Figure 9. We conclude that the scatter in the phase diagrams in these cases is caused by the somewhat greater noise inherent in the data.

2. *V54, V56, V57, V64 and V102.*—Besides the single periodic component, there might be a long-term trend in the data. Figure 10 shows the case of V64. We note, however, that an alternative explanation of the peaks at integer c/d frequencies is afforded by a second periodicity close to 1 or 2 c/d. This assumption, however, is fraught with peril because of the aliases caused by the 1 day periodicity in the data sampling.

3. *V33.*—We have only 46 data on this star with an unfavorable phase coverage. The frequency analysis is ambiguous.

4. *V36.*—There might be some other periodicity of small amplitude.

5. *V44.*—The spectra are unclear, with many peaks of similar height after prewhitening.

6. *V49.*—The main component has a frequency of ~ 1.5 c/d; therefore, the first harmonic component appears at an unfavorable integer c/d frequency, which causes the period identification to be somewhat ambiguous.

7. *V21 and V62.*—These stars might be candidates for double-mode pulsation as their prewhitened spectra indicate (see Figs. 11 and 12). However, the peaks in the prewhitened spectra are again somewhat close to integer c/d frequencies which suggests some caution. In the case of V21, for example, an alternative possibility is that a long-term trend is superposed on the first harmonic of the fundamental oscillation, resulting in a larger peak at ~ 2 c/d. Further discussion of the possible implications of the double periodicity of these stars will be presented in § V.

b) The Cluster NGC 6171

We have some comments on individual stars in this cluster:

1. *V10.*—The spectra are clear; the small dispersion of the phase curve at low brightness is probably caused by observational errors.

2. *V16 and V21.*—Prewhitened spectra might indicate some other component of small amplitude.

3. *V19.*—This is a double-mode suspect (see Fig. 13). Problems with the physical interpretation of this possibility are discussed in § V.

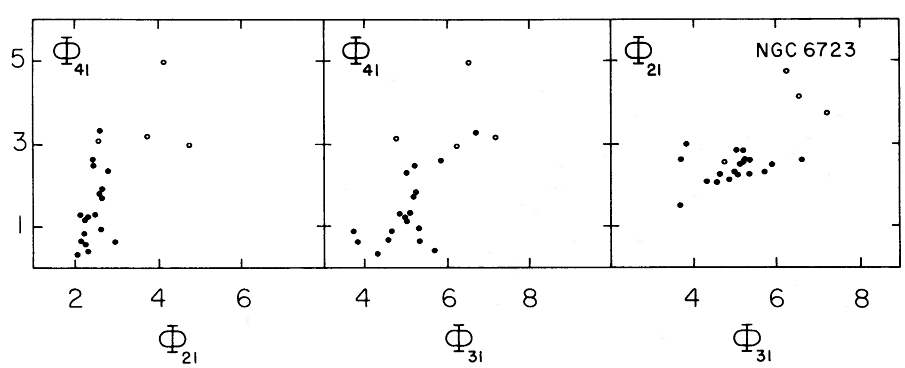

FIG. 8.—Same as in Fig. 2, but for NGC 6723

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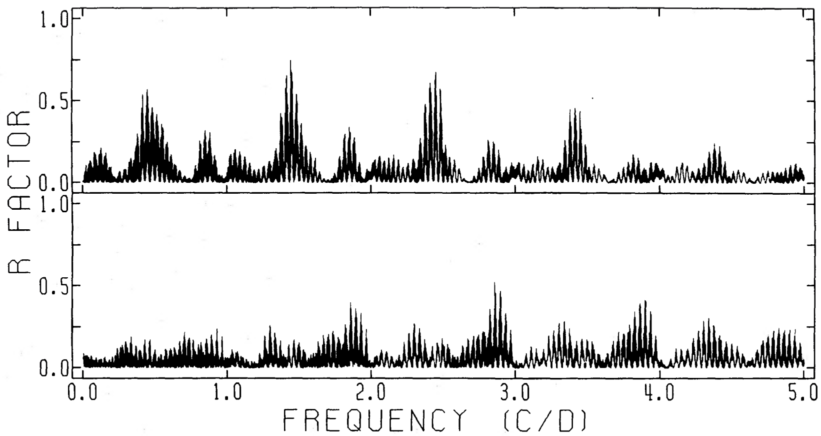

FIG. 9.—Least-squares frequency spectra of V46 (SKS data) calculated with eq. (5). Spectrum of the original data is shown in the upper panel, that for the prewhitened data is shown in the lower one. The frequency used for prewhitening corresponded to the frequency of the largest peak of the spectrum of the original data.

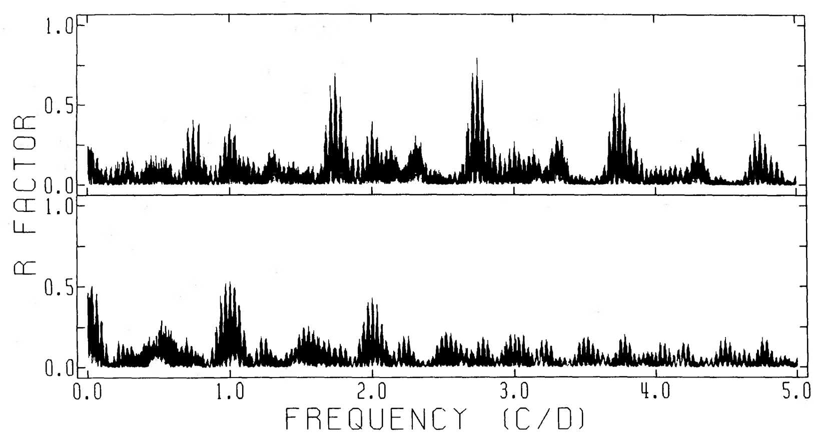

FIG. 10.—Same as in Fig. 9, but for V64 of SKS data

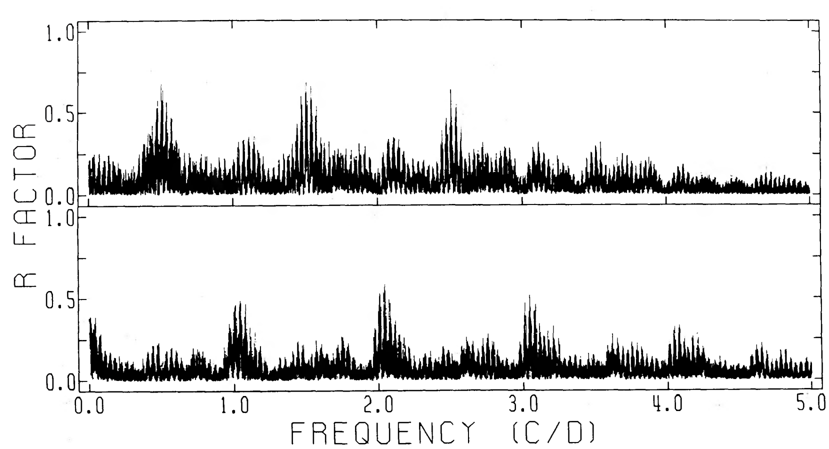

FIG. 11.—Same as in Fig. 9, but for V21 of SKS data

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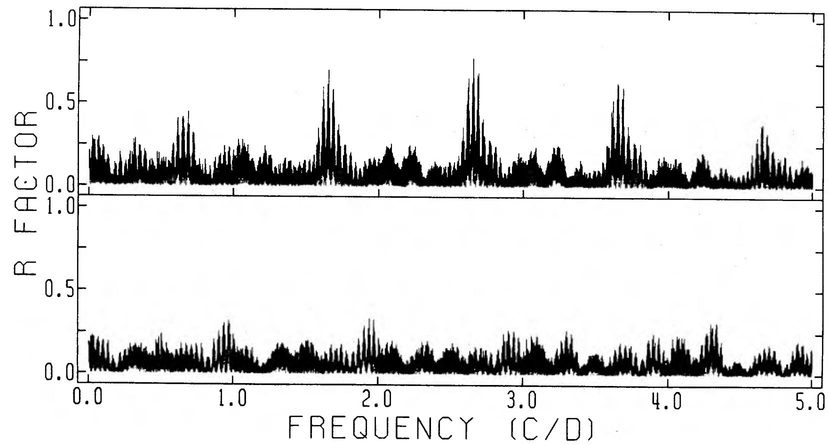

FIG. 12.—Same as in Fig. 9, but for V62 for SKS data

c) *The Cluster NGC 6723*

Because of the unfortunate data distribution, we first performed the thorough frequency analysis for the problematic stars of this cluster only between Julian Dates 2,438,993.9070–2,439,001.9248. However, for the final, more accurate period search we used the whole data set. Our comments on individual stars are as follows:

1. *V10, V12 and V21.*—The frequency spectra are simple with very small power content in the 0–5 c/d band (except for the first harmonic component). Scatter of the phase curve is caused by observational noise.
2. *V7, V9, V17, V24, V31 and V32.*—Beside the occasional first harmonic component, the prewhitened spectra indicate some long-term trend (or oscillation with a frequency of integer c/d, as probably for the case of V32).
3. *V6 and V23.*—The spectra are unclear (V6 is shown in Fig 14).
4. *V5.*—This is a double-mode suspect (see Fig. 15). The consequence of this possibility is discussed in § V.

IV. DOUBLE-MODE STARS IN M15

In order to study the constancy of the modal content of the double-mode RR Lyrae stars listed by CHC we thoroughly Fourier analyzed the stars in both data sets. A similar analysis has already been performed by Hodson and Cox (1982, hereafter HC) for the SKS data and the data of Filippenko and Simon (1981).

To obtain accurate amplitude values we had to identify the component frequencies with high confidence and accuracy. Our *standard search* consisted of the following procedure: We prewhitened the original data with a given P_0, chosen to be the highest peak in the power spectrum of the original data. The largest peak of the Fourier analysis of these prewhitened data then yielded the second period P_1. Because the identification of the main peak in the unprewhitened data was biased by the second component, we also performed an *iterative search* of the two frequencies. First, a standard search was performed with a chosen P_0 (generally the largest peak or one of its aliases) to give P_1. This was followed by a standard search with P_1 (or one of its aliases) which gave P'_0; the next search used P'_0 to yield P'_1, and so forth. If the procedure converged and the satellite structure around the second period of the prewhitened data was symmetric, the result of the iterative search was considered *consistent*. It has been comforting that with the exception of a few cases this method yielded frequencies (and amplitudes) very close to those obtained with the simple prewhitening method.

Two stars (V26 and V41) showed a very small and insignificant fundamental component; we therefore left these stars out of consideration (see Fig. 16 which illustrates the case of V41).

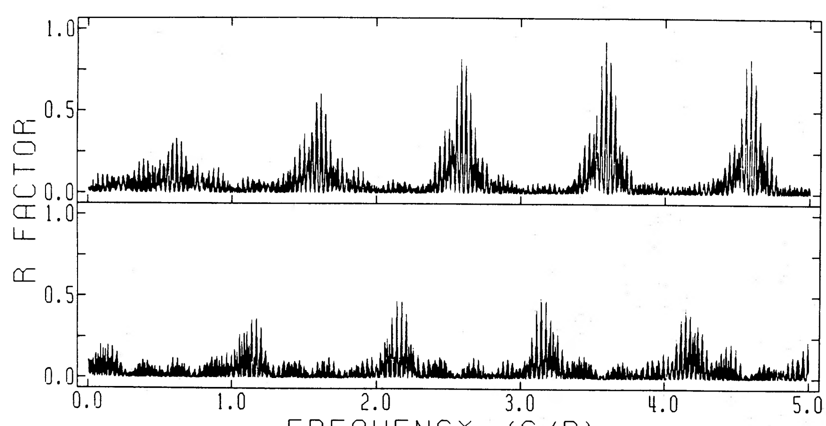

FIG. 13.—Same as in Fig. 9, but for V19 in NGC 6171

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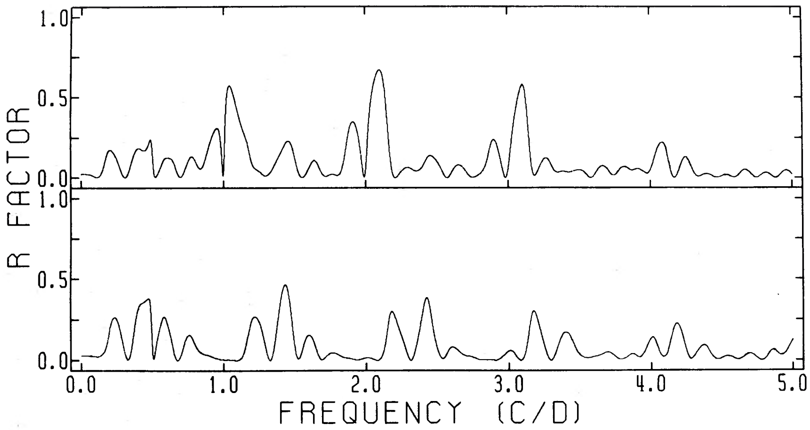

FIG. 14.—Same as in Fig. 9, but for V6 in NGC 6723, analyzed between JD 2,438,993.9070–2,439,001.9248

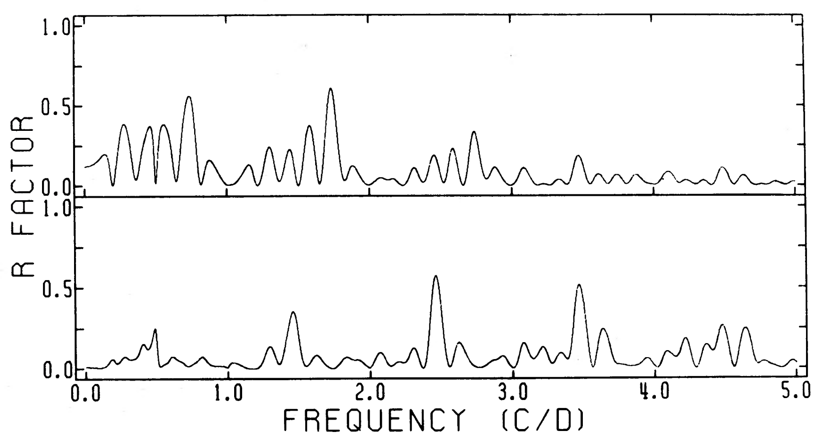

FIG. 15.—Same as in Fig. 14, but for V5 in NGC 6723

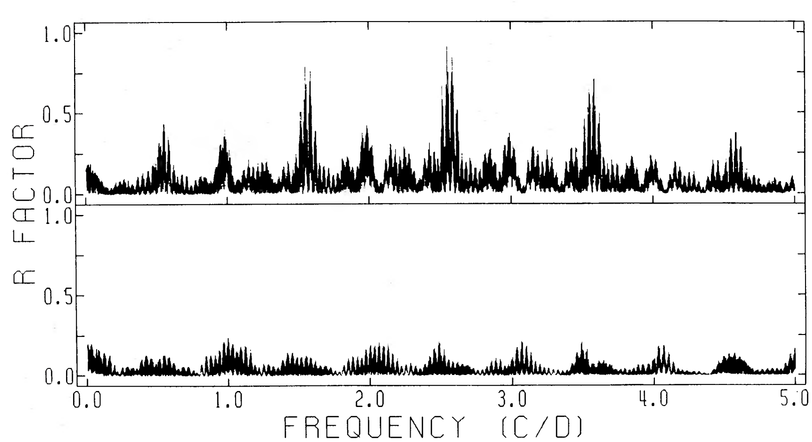

FIG. 16.—Same as in Fig. 9, but for V41 of SKS data

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TABLE 5

COMPARISON OF B LIGHT CURVE CHARACTERISTICS OF THE DOUBLE-MODE RR LYRAE STARS IN THE SKS AND THE BCDP DATA

STAR NO.	P_0 (DAYS)	P_1 (DAYS)	P_1/P_0	A_0	A_1	A_1/A_0	A_0^2 $\sigma(A_0^2)$	A_1^2 $\sigma(A_1^2)$	σ_{FIT}
17	0.5756016	0.4288622	0.7451	0.093	0.188	2.02	0.0086 (13)	0.0353 (26)	0.037
	0.5755956	0.4288917	0.7451	0.110	0.237	2.15	0.0121 (22)	0.0562 (47)	0.046
30	0.5431293	0.4059851	0.7475	0.118	0.234	1.98	0.0139 (19)	0.0548 (38)	0.045
	0.5430685	0.4059793	0.7476	0.120	0.231	1.92	0.0144 (43)	0.0534 (83)	0.084
31	0.5479709	0.4081937	0.7449	0.124	0.245	1.98	0.0154 (22)	0.0600 (44)	0.050
	0.5480056	0.4081766	0.7448	0.100	0.200	2.00	0.0100 (17)	0.0400 (33)	0.038
39	0.5229811	0.3895879	0.7449	0.112	0.240	2.14	0.0125 (17)	0.0576 (36)	0.042
	0.5229508	0.3895680	0.7449	0.087	0.234	2.69	0.0076 (25)	0.0548 (67)	0.065
53	0.5560824	0.4141556	0.7448	0.084	0.258	3.07	0.0071 (12)	0.0666 (37)	0.040
	0.5553952	0.4141325	0.7456	0.106	0.228	2.15	0.0112 (29)	0.0520 (61)	0.061
58	0.5466898	0.4072623	0.7450	0.130	0.211	1.62	0.0169 (46)	0.0445 (74)	0.097
61	0.5361452	0.3996296	0.7454	0.134	0.274	2.04	0.0180 (28)	0.0751 (57)	0.057
67	0.5423840	0.4045909	0.7459	0.114	0.215	1.89	0.0130 (15)	0.0462 (29)	0.038
	0.5423754	0.4046021	0.7460	0.128	0.231	1.80	0.0164 (67)	0.0534 (119)	0.120

NOTE.—First entry, SKS data; second entry, BCDP data. Amplitudes are measured in magnitude.

We also did not include the further possible candidates V51 and V54, as mentioned by HC, because their frequency spectra for the SKS data did not indicate any possible second component of considerable amplitude. Finally, we omitted our new double-mode suspects for reasons to be discussed in § V.

Our results are summarized in Table 5. We fitted the data with a second-order Fourier sum, i.e., besides the non-oscillating constant, six amplitudes and phases were fitted with the frequencies $f_0, f_1, 2f_0, 2f_1, f_0 + f_1, f_1 - f_0$. A higher order fit resulted in spurious instabilities because of the paucity of data and their unfavorable distribution. Amplitudes corresponding to the frequencies f_0 and f_1 are denoted by A_0 and A_1, respectively. The standard deviations of the squares of the amplitudes are denoted by $\sigma(A_0^2)$ and $\sigma(A_1^2)$, respectively. They were calculated by formula (A15), with $\sigma = \sigma_{FIT}$, where σ_{FIT} is the square root of the averaged squared deviation between the fit and the data. Amplitudes belonging to the first harmonic and interaction frequencies were in general very small, typically between one-third and one-tenth of the A_0, A_1 amplitudes. A more detailed description of the frequency analysis and its results follows:

1. *V17*.—In the case of the SKS data the main peak in the unprewhitened data appeared at $P_1 = 0.4288622$ d. After a prewhitening of the data by this period, the greatest peak in the 0–5 c/d band was at a period of 0.5747031 d, if we disregarded the slightly greater 1 day alias toward longer period. We actually opted for the 1 yr alias of 0.5747031 d, namely $P_0 = 0.5756016$ d, because it was closer to one of the possible frequencies in the BCDP data; in any case the difference in the two P_0 values had a very small effect on the amplitudes and the quality of the fit. (With $P_0 = 0.5747031$ d and P_1 we obtained $A_0 = 0.085$, $A_1 = 0.185$, $\sigma_{FIT} = 0.038$, to be compared to the value in Table 5.) The standard search for the BCDP data led to $P_1 = 04288917$ d and 0.5764962 d ($R = 0.491$); however, because of symmetry considerations and the closeness to the corresponding period of the SKS data, we finally picked its 1 yr alias, i.e. $P_0 = 0.5755956$ d ($R = 0.489$). Again, the two possible fits are practically identical. We were unable to improve the fit by iterative prewhitening.

2. *V30*.—In the case of the SKS data the standard search led to the periods shown in the table. Again, because of symmetry considerations and the closeness of the periods, we took the 1 yr alias of the main peak of the prewhitened BCDP data as shown in Table 5. Taking other possible combinations of periods or applying iterative prewhitening caused no appreciable change or improvement in the fit in either data set. As a representative case for double-mode RR Lyrae stars with reliable period determination, the frequency spectra of V30 is shown in Figure 17.

3. *V31*.—A standard search with the SKS data led to $P_0 = 0.5471864$ d, $P_1 = 0.4081896$, with $A_0 = 0.132$, $A_1 = 0.248$, and $\sigma_{FIT} = 0.053$. However, in the prewhitened spectrum, the 1 yr alias of P_0 had a similar amplitude, and actually it was closer to the symmetry center of the peak structure. The iterative prewhitening gave finally $P_0 = 0.5479709$ d, $P_1 = 0.4081937$ d. In the case of the BCDP data the standard search

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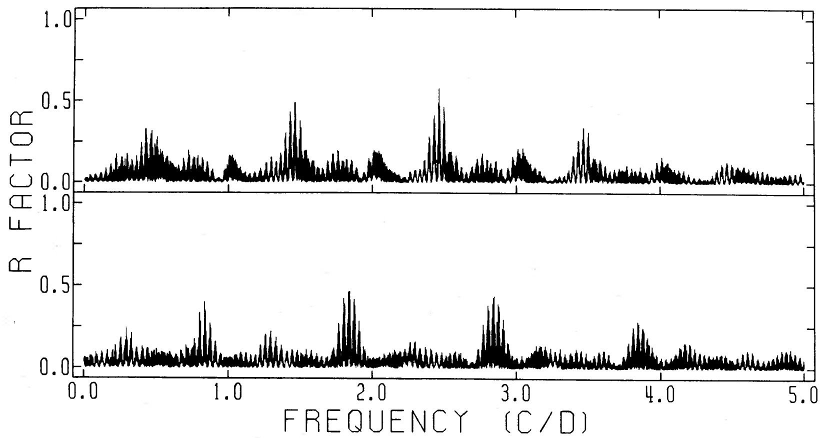

FIG. 17.—Same as in Fig. 9, but for V30 for SKS data

gave the periods shown in the table. The alias of the fundamental, i.e. 0.5471772 d, yielded a larger scatter of the residuals, $\sigma_{FIT} = 0.045$.

4. *V39*.—In the case of the SKS data the prewhitened spectrum was not symmetric, and an iterative period search did not lead to any improvement. The standard search yielded $P_0 = 0.5215845$ d, $P_1 = 0.3895845$ d, with $A_0 = 0.111$, $A_1 = 0.238$, and $\sigma_{FIT} = 0.0406$. However, because an alias of P_0 of comparable height was very close to the P_0 of the BCDP data, we finally accepted the fit resulting from the iterative search with the alias period. In the BCDP data the unprewhitened spectrum had its main peak at $P_1 = 0.3891654$ d. Because of symmetry considerations and the closeness to the SKS period, we kept the consistent result of the iterative analysis with the alias of P_1, as shown in Table 5.

5. *V53*.—Because of symmetry considerations, we performed an iterative search with the 1 yr alias of 0.5552570 d in the case of SKS data. This gave the consistent result shown in Table 5. However, the spectra of the BCDP data remained asymmetric, even when iterative prewhitening was applied. In the end we opted for the periods obtained by the standard search.

6. *V67*.—The standard search with the SKS data gave $P_0 = 0.5416829$ d, $P_1 = 0.3990909$ d, with $A_0 = 0.093$, $A_1 = 0.193$, $\sigma_{FIT} = 0.061$. However, the iterative search with the 1 yr alias of P_1 led to a consistent result with a much better fit, as shown in Table 5. The standard search applied to the BCDP data gave the periods listed in the table, which are indeed very close to those of the SKS data. The reason for the very large σ_{FIT} value for the BCDP data is unknown.

7. *V58*, *V61*.—These stars are absent in the data of BCDP. The results of the standard search are shown in Table 5. The large value of σ_{FIT} for V58 is probably caused by a long-term variability. The iterative period search did not improve the fit in either case.

We note that seven of the eight double-mode stars have period ratios falling in the very narrow range (0.7448–0.7460). Aside from this remarkable property and that six out of the eight amplitude ratios, A_1/A_0, are close to 2, the data do not seem to show any systematic correlation or trends among the coefficients (Table 5).

We can conclude from Table 5 that the modal content of the double-mode RR Lyrae stars in M15 remained constant within the noise level during the time ($\sim$20 yr) elapsed between the observations of SKS and BCDP.

V. DISCUSSION

In the previous sections we presented a detailed description of our Fourier analysis and the results for the photographic *B* data of the three globular clusters M15, NGC 6171, and NGC 6723. Here we discuss the implications of the various properties of the Fourier coefficients and the questions relating to double-mode RR Lyrae stars.

A summary of the results of our Fourier-decompositions is presented in Table 6. In the calculation of the averages and of the standard deviations we have included all stars except those listed as double mode by CHC.

It is seen that, except perhaps for the larger dispersion of the BCDP parameters, there is a very good agreement among the

TABLE 6

AVERAGE VALUES AND STANDARD DEVIATIONS OF FOURIER PARAMETERS FOR FUNDAMENTAL (F) AND FIRST OVERTONE (O) RR LYRAE STARS

CLUSTER	$\overline{A_1}$		$\sigma(A_1)$		$\overline{A_2}$		$\sigma(A_2)$		$\overline{R}_{21}$		$\sigma(R_{21})$		$\overline{\Phi}_{21}$		$\sigma(\Phi_{21})$	
	F	O	F	O	F	O	F	O	F	O	F	O	F	O	F	O
M15 (SKS)	0.364	0.305	0.083	0.032	0.155	0.060	0.042	0.020	0.430	0.196	0.090	0.058	2.24	2.93	0.28	0.36
M15 (BCDP)	0.364	0.297	0.079	0.038	0.153	0.059	0.052	0.025	0.419	0.198	0.107	0.080	2.35	2.88	0.32	0.42
NGC 6171	0.429	0.307	0.118	0.022	0.201	0.048	0.058	0.013	0.466	0.159	0.069	0.066	2.40	2.91	0.20	0.43
NGC 6723	0.477	0.265	0.142	0.026	0.200	0.033	0.075	0.016	0.402	0.252	0.128	0.206	2.38	3.82	0.34	0.94

NOTE.—Double-mode stars listed by CHC have been omitted.

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light curve parameters for the SKS and BCDP data sets. An F-test for the variances showed that they did not differ from each other within the 95% confidence limit. The agreement between the means was even more significant. A Student test for the differences indicated that they were in general well within the 50% confidence intervals. However, it is important to remark that while overall properties might agree quite well in the two data sets, some stars show quite diverse behavior. Stars V2, V7, V15, V65, V66 (and some others) show much larger scatter in BCDP than in SKS. Our own period analysis did not (or did just slightly) improve the smoothness of the phase curves of these stars. It is not possible on the basis of the available data to decide whether in these cases the data have a higher noise level (of unknown origin), or whether our interpretation of the light variations as singly periodic with constant amplitudes may be inappropriate.

Similarly, except perhaps for the first overtone variables, differences between M15 and the other two clusters were not significant, at least at the 95% confidence level. However, the differences tend to have the same sign in both Osterhoff I–type clusters, indicating that light curve properties might be slightly different in the two types of globular clusters.

Higher order phase differences can be useful for differentiating between the first overtone and fundamental pulsators. Interrelations of these phase-differences (Figs. 2, 4, 6) for the fundamental pulsators are similar to (although less pronounced than) those of the Galactic Cepheids (Simon and Moffett 1985). The double-mode RR Lyrae stars (*when analyzed as singly periodic* with the frequency of the major peak) did not occupy a unique position in any of the diagrams plotted, showing that a more sophisticated analysis of the Fourier spectrum is necessary to identify them.

We suspect that the coincidence of these double-mode stars with the single mode stars may be due to the small amplitude of the fundamental, as evidenced by the double-period analysis (Table 5). There is no reason to believe that a single-period analysis of the double-mode RR Lyrae stars will, in general, yield the same ϕ_{21} as a proper double-period analysis. There is no reason either that the corresponding ϕ_{21} behaves the same way through a resonance as it would in a singly periodic star.

Petersen (1984) has suggested that in ω Cen the RR*ab* stars follow a Cepheid-like progression, while for the RR*c* stars he postulates a new type of bump progression. For the clusters studied in this paper all trends are much less pronounced than those in ω Cen. The most visible among these trends is the decrease of A_1 and R_{21} (and a mild increase of ϕ_{21}) toward longer periods for the first overtone variables in M15. Although similar trends can also be suspected in the other two clusters, the number of first overtone variables therein is too small to be decisive at this point. It is therefore possible that the trend seen for the RR Lyrae stars in ω Cen (Petersen 1984) might be a result of their analysis as singly periodic rather than as double mode, as discussed above. It should be noted that if one disregards the double-mode stars, the trends become much less conclusive. We recall that Bump progression sequences are easily understandable with the hypothesis of a 2:1 resonance (Klapp, Goupil, and Buchler 1985; Buchler and Kovács 1986). However, first overtone pulsator models with a resonance near 0.45 d in ω Cen might pose a mass deficiency problem (Buchler and Kovács 1986). We feel that further analysis of more recent data on ω Cen is highly desirable. We conclude that the variation of the Fourier parameters with the period for the globular cluster RR Lyrae stars studied in this paper is relatively featureless, in accordance with the models of Kovács (1985), who found no close resonances in the relevant period region.

We have shown that the double-mode RR Lyrae stars in M15 have not changed their modal content (within the noise level). This is at variance with HC who claim that their Fourier analysis for the SKS and Filippenko and Simon (1981) data indicate in some cases probable changes in the modal content. We think that the very high scatter of the data of Filippenko and Simon preclude a resolution of this question. The quality of the BCDP data is somewhat better; still, there are many sources of errors (inhomogeneity of data, fewer and unfortunately distributed observations). In any case, the theoretical switching rates from stellar models are too large to explain the large number of double-mode RR Lyrae stars. It seems remarkable that, except perhaps for V39 and V53, the A_1/A_0 ratios for the other six double-mode stars are very close to 2.0 and as noted earlier, the period ratios P_1/P_0 fall in a very narrow range.

The new suspected double-mode stars pose some problems. In the case of V21 in M15, the derived periods $P_0 = 0.6487698$ d, $P_1 = 0.4887865$ d give $P_1/P_0 = 0.753$; if they are interpreted as fundamental and first overtone periods, they suggest a mass of $\gtrsim 0.85\ M_\odot$. Similarly, V62 (also in M15), with $P_0 = 0.5159805$ d, $P_1 = 0.3773201$ d gives $P_0/P_1 = 0.731$, indicating a mass of $\lesssim 0.40\ M_\odot$. In the case of V19 in NGC 6171, we obtain period ratios of 0.876 or 0.600, depending on which alias we pick; neither of these is consistent with the radial fundamental and first overtone. Also in the case of V5 in NGC 6723, the period ratio 0.706 is not reconcilable with normal RR Lyrae masses. It is clear tha further observations and subsequent analysis of these stars are necessary.

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APPENDIX

STATISTICAL PROPERTIES OF THE FOURIER AMPLITUDES

Let us assume that the data (x_n) to be fitted by least-squares to the Fourier sum of order M consists of a deterministic component (f_n) and a stochastic one (η_n) in the form of

$$x_n = f_n + \eta_n \ , \tag{A1}$$

where η_n is δ-correlated Gaussian noise, i.e., the expectation values of their products are given by

$$E(\eta_n) = 0 \ , \qquad E(\eta_i \eta_j) = \sigma^2 \delta_{ij} \ , \qquad E(\eta_i \eta_j \eta_k) = 0 \ , \qquad E(\eta_i^2 \eta_j^2) = \sigma^4(2\delta_{ij} + 1) \ , \tag{A2}$$

and, of course, all other fourth-order moments vanish.

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Assuming that we fit our time series to the sum $\sum_{k=1}^{k} A_k a(k, n)$, we obtain for the solution of the least-squares problem

$$A_j = \sum_{k=1}^{K} g(j, k) T_k \,, \tag{A3}$$

with $K = 2M + 1$, where A_j is the *phase dependent* amplitude, i.e., $A_2 = \bar{a}_1 \sin \phi_1$, $A_3 = \bar{a}_1 \cos \phi_1$, $A_4 = \bar{a}_2 \sin \phi_2$, $A_5 = \bar{a}_2 \cos \phi_2$, ..., but $A_1 = \bar{a}_0$, the zero frequency constant. The function $g(j, k)$ is the inverse of the normal matrix, and

$$T_k = \sum_{n=1}^{N} x_n a(k, n) \,, \tag{A4}$$

where N is the number of data and where the $a(k, n)$ denote sine or cosine functions. We write, equation (A3) in the following form:

$$A_j = \sum_{n=1}^{N} b(j, n) x_n \,, \tag{A5}$$

where

$$b(j, n) = \sum_{k=1}^{K} g(j, k) a(k, n) \,. \tag{A6}$$

We easily get

$$E(A_j) = \sum_{n=1}^{N} b(j, n) f_n \,, \tag{A7}$$

and

$$\begin{aligned} E(A_j^2) &= \sigma^2 \left[\sum_{k_1 k_2 = 1}^{N} g(j, k_1) g(j, k_2) \right] \left[\sum_{n=1}^{N} a(k_1, n) a(k_2, n) \right] + E^2(A_j) \\ &= \sigma^2 \sum_{k_1 k_2 = 1}^{M} g(j, k_1) g(j, k_2) G(k_1, k_2) + E^2(A_j) \,, \end{aligned} \tag{A8}$$

where $G(k_1, k_2)$ is the normal matrix. Because g is the inverse of G, we get

$$\sigma^2(A_j) \equiv E(A_j^2) - E^2(A_j) = \sigma^2 g(j, j) \,. \tag{A9}$$

It is clear from equation (A5) that the A_j amplitudes are Gaussian variables, with the mean and variance given by equations (A7) and (A8), respectively (see also Kuhn 1982). However, the amplitudes are not statistically independent with the correlation between A_j and A_k:

$$E(A_j A_k) = E(A_j) E(A_k) + \sigma^2 g(j, k) \,. \tag{A10}$$

The phase-dependent amplitudes are thus independent only for some specific cases, like equally spaced data with periods which are integer fractions of the total time span (FFT for example).

Specifically, the more commonly used *phased-independent amplitudes*, defined by

$$a = (A_j^2 + A_{j+1}^2)^{1/2} \,, \tag{A11}$$

where A_j and A_{j+1} (j even), denote some of the consecutive phase-dependent amplitudes (A_2 and A_3, or A_4 and A_5, ...). The variable a has a complicated statistical behavior in general. However, the expectation value and variance of a^2 are readily calculated. It follows from equation (A8) that

$$E(a^2) = A_s^2 + A_c^2 + \sigma^2(g_{ss} + g_{cc}) \,, \tag{A12}$$

where $g_{ss} = g(j, j)$, $g_{sc} = g(j, j + 1)$, $g_{cc} = g(j + 1, j + 1)$ $A_s = E(A_i)$, and $A_i = E(A_{j+1})$. With the help of equations (A2), (A8), and (A12), and we obtain for the variance of a^2

$$\begin{aligned} \sigma^2(a^2) \equiv E(a^4) - E^2(a^2) &= 4\sigma^2[A_s^2 g_{ss} + A_c^2 g_{cc} + 2A_s A_c g_{sc}] \\ &+ \sigma^4 \left\{ 3 \sum_{n=1}^{N} [b^2(j, n) + b^2(j + 1, n)]^2 + 2g_{ss}^2 + 2g_{cc}^2 + 4g_{sc}^2 \right\} . \end{aligned} \tag{A13}$$

In the common situation (as in the case of the RR Lyrae stars studied in this paper) g_{sc} is very small (typically a few percent) compared to g_{ss} and g_{cc}, which are $\sim 2/N$. In addition, since

$$\sum_{n=1}^{N} [b^2(j, n) + b^2(j + 1, n)]^2 \sim 1/N^3 \,,$$

we get the desired expressions

$$E(a^2) = \alpha^2 + \frac{4}{N} \sigma^2 \,, \tag{A14}$$

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$$\sigma^2(a^2) = \frac{8}{N}\,\alpha^2\sigma^2 + \frac{16}{N^2}\,\sigma^4\,, \tag{A15}$$

where $\alpha^2 = A_s^2 + A_c^2$, the noise-free amplitude. In the case where $g_{\rm sc} = 0$, the square of the corresponding phase-independent amplitude follows the same distribution as discussed by Groth (1975).

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