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ABSOLUTE MAGNITUDES AND KINEMATIC PROPERTIES OF RR LYRAE STARS

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ABSTRACT

A maximum-likelihood statistical analysis of several subgroups of the field RR Lyrae stars was performed to determine the relative solar motion, velocity ellipsoid parameters, and mean absolute visual magnitude for each group. The full sample of 159 stars was taken from a recent Chinese proper-motion survey, and new mean radial velocities were used for 46 of the stars. A geometric minimization technique known as simplex optimization was used to apply a rigorous maximum-likelihood model to the data. Our best estimate for the mean absolute visual magnitude is $\langle M_V \rangle = 0.76 \pm 0.14$ mag for the entire RR *ab*-type sample.

Subject headings: stars: luminosities — stars: RR Lyrae — stars: stellar dynamics

I. INTRODUCTION

The field RR Lyrae stars may be used to address several astrophysical problems of interest. They are primary distance indicators in our own and nearby galaxies, and, along with Cepheid variables, offer the best hope for establishing the zero point of the extragalactic distance scale (de Vaucouleurs 1978; Stothers 1983). They can be used to probe the kinematics of the Galactic halo and thus lend insight into the dynamics and evolution of the Galaxy. Also, the observed properties of RR Lyrae stars offer empirical constraints on pulsation theories and stellar evolution models.

RR Lyrae stars are easily identified by their low metallicities, short periods, and by the characteristic effect of the pulsation on their light curves. When plotted on a color-magnitude diagram, they are observed to lie within a narrow region of the instability strip, and have long been thought to be all of approximately the same age, mass, and absolute magnitude. It is natural then to view them as a homogeneous group and to use a statistical analysis to determine their mean properties.

Many such analyses have been attempted in the past (see Stothers 1983 for the most recent review of this field). Classical statistical parallax methods, culminating in the work of Hemenway (1975*b*) are fraught with uncertainty because of the many approximations needed to make the problem tractable for linear least-squares solutions. Rigal (1958) was the first to recognize these shortcomings and attempt a maximum-likelihood formulation of the model. Ideally, the maximum-likelihood model requires no approximations and allows simultaneous solution for all the parameters through some nonlinear programming technique. However, the most recent published works (Heck and Lakaye 1978; Clube and Dawe 1980) have not attained this goal. Clube and Dawe find that Heck's method is afflicted with considerable statistical bias, which is accounted for in a rather ad hoc manner. We, in turn, find that Clube and Dawe make an approximation in their expression for the probability of obtaining an observation, which is then used to form the likelihood. A second difficulty with Clube and Dawe's work is the use of a nonlinear least-squares technique to solve for the parameters of the model. Least-squares techniques are very difficult to apply to this problem because of the strong interdependence of the parameters and the indirect way in which some of the parameters are determined. This leads to serious trouble in the convergence to a solution. The discrepant results obtained by these and other authors, often when analyzing substantially the same data, led Stothers to conclude that the statistical method is currently unreliable in determining the absolute magnitude of the RR Lyrae stars.

We have sought to improve the statistical analysis in several ways. Our proper-motion data are drawn from a new Chinese survey covering 75 years of observations. The 159 star sample is the largest of its kind from a single survey, which enables us to avoid many of the difficulties associated with combining proper-motion data from several sources. In addition, we have determined new mean radial velocities for 46 of the stars in the sample (Hawley and Barnes 1985). Many of these stars had previously been observed only once, resulting in large uncertainties in their mean velocities. Third, we use a formulation of the maximum-likelihood model which is both rigorous and more straightforward than previous attempts. Finally, we use a direct minimization technique known as simplex optimization, which is very effective in solving nonlinear problems of this type.

Studies of globular cluster variables (Sandage 1981, 1982) and theoretical calculations of pulsation models (Christy 1966) suggest that there should be a dependence of the absolute magnitude and kinematic properties on such quantities as Bailey type (*ab* or *c*), length of pulsation period, and metallicity (as measured by Preston's metallicity index ΔS). We use our new data and improved analysis to investigate the existence of such dependences in the field RR Lyrae stars.

II. DATA

The data necessary for the statistical analysis are the velocity and position of each star in the sample. The velocity is found from the radial velocity and two orthogonal components of proper motion. The position is determined from the celestial coordinates, apparent magnitude, and reddening. The latter two, together with an absolute magnitude estimate, give a distance which is then used to convert the proper motions to velocities.

The proper-motion data have been previously published in the *Annals of the Shanghai Observatory* (Wan, Mao, and Ji 1981) and are reproduced in Table 1 for easier reference. The data were obtained over a 75 yr baseline. Comparison with the largest previous sample, taken from the McCormick proper-motion survey (Hemenway 1975*a*), gives for the 101 RR Lyrae stars in common the mean differences:

$$\Delta(\mu_\alpha \cos \delta) = -0''.0004 \pm 0''.0003 \text{ yr}^{-1} ,$$
$$\Delta(\mu_\delta) = +0''.0003 \pm 0''.0005 \text{ yr}^{-1} ,$$

while the proper motions themselves are typically of order $0''.015$ yr^{-1}. Discussion of the observing and measuring procedures, as well as comparison with other surveys, may be found in the original reference.

We used the metallicity, period, radial velocity, apparent magnitude, and reddening data given by Hemenway (1975*b*), and a complete list of references may be found there. The few stars which have incomplete entries in her table have data taken from the *General Catalogue of Variable Stars* (GCVS) (Kukarkin *et al.* 1969). The apparent magnitudes are mean light V magnitudes as defined by Fitch, Wisniewski, and Johnson (1966). The reddening was determined according to the color-excess relation given by Sturch (1966). We have substituted our new mean radial velocities for the 46 stars discussed in Hawley and Barnes (1985).

To account properly for the variance in the velocity residuals, the error in each velocity component must also be estimated. Table 1 includes individual error estimates for each proper-motion determination. The uncertainty in each mean radial velocity was assumed to be 10 km s^{-1}.

III. MAXIMUM-LIKELIHOOD MODEL

Qualitatively, the maximum-likelihood model may be thought of as follows. Many stars, whose absolute magnitudes are presumed the same within some cosmic dispersion, are observed within the Galaxy. Each star will have velocity components which reflect our own moving reference frame. In the two proper-motion directions, the magnitudes of these velocity components will change depending on the assumed distance. In addition to this reflex solar motion, each star has its own peculiar velocity. We assume that the stars' peculiar velocities are independent and randomly distributed, and, hence, model them with a multinormal distribution. This distribution may be thought of as a velocity ellipsoid. In the coordinate system in which the velocity ellipsoid is diagonal, the three orthogonal axes represent the directions along which the velocity distributions are independent random variables and the length of each axis corresponds to the dispersion of the Gaussian velocity distribution in that direction. In general, there are six independent components of the velocity ellipsoid. These, together with the absolute magnitude and its dispersion and the three components of the apparent solar motion, are the model parameters that are adjusted until the observations are predicted by the model with maximum likelihood.

Our quantitative formulation of the maximum-likelihood model is taken from Murray (1983, pp. 297–302), and we use his notation in our discussion. We begin by defining the likelihood of obtaining all of the observations simultaneously:

$$L = \Pi \text{ prob } (\mathbf{v}) , \tag{1}$$

where $\mathbf{v}$ is the velocity residual vector, that is, the observed velocity minus the velocity expected from the model (in this case, the reflex solar motion). The product is taken over all of the stars in the sample. Assuming the velocity residuals $\mathbf{v}$ follow a Gaussian distribution with zero mean and a covariance tensor $\mathbf{M}$, we have

$$\text{prob } (\mathbf{v}) = (2\pi)^{-3/2} |\mathbf{M}|^{-1/2} \exp\left(-\tfrac{1}{2}\mathbf{v}'\mathbf{M}^{-1}\mathbf{v}\right) , \tag{2}$$

where $|\mathbf{M}|$ denotes the determinant of $\mathbf{M}$ and $\mathbf{v}'$ is the transpose of the vector $\mathbf{v}$. Thus,

$$\begin{aligned} \ln L &= \sum \ln \text{ prob } (\mathbf{v}) \\ &= -0.5 \sum (\ln |\mathbf{M}| + \mathbf{v}'\mathbf{M}^{-1}\mathbf{v}) + \text{constant} \end{aligned} \tag{3}$$

is the logarithmic likelihood function which is to be maximized. We must now express the parameters of the model, $\mathbf{v}$ and $\mathbf{M}$, as functions of the observations and the kinematic and physical parameters we wish to determine.

The velocity residual $\mathbf{v}$ for each observation may be written as

$$\mathbf{v} = |\boldsymbol{b}_A| \mu_{\text{obs}} + \rho_{\text{obs}} \boldsymbol{r} - [(1 + k)(\mathbf{U} - \mathbf{rr}') + \mathbf{rr}'] \boldsymbol{V}_\odot . \tag{4}$$

Here ρ_{obs} is the observed radial velocity and μ_{obs} is the observed proper motion. The factor $|\boldsymbol{b}_A|$ is an assumed distance found from

$$|\boldsymbol{b}_A| = 10^{0.2(m_V - M_A + 5)} \text{ pc} , \tag{5}$$

where m_V is the apparent magnitude corrected for reddening and M_A is an assumed absolute magnitude. The calculation is done in the equatorial coordinate system where $\boldsymbol{r}$ is the unit vector pointing radially to the star. The observed proper motion, μ_{obs}, is found by projecting the measured proper motions $\mu_\alpha \cos \delta$ and μ_δ onto this coordinate system. The first two terms in equation (4) thus combine to give the observed velocity.

The predicted velocity is the apparent solar motion through the sample, $\boldsymbol{V}_\odot$. We introduce a distance scale factor k that adjusts the velocity in the plane perpendicular to the radius vector. Thus the projection matrix $\mathbf{P} = \mathbf{U} - \mathbf{rr}'$, where $\mathbf{U}$ is the unit matrix, multiplied into $\boldsymbol{V}_\odot$ gives the expected velocity in this plane, while the projection matrix $\mathbf{rr}'$ multiplied into $\boldsymbol{V}_\odot$ gives the expected radial velocity. The last term in equation (4) is therefore the predicted velocity due to the apparent solar motion.

To investigate the effects of differential galactic rotation, a second term of the form $(dV/d\boldsymbol{b}')\boldsymbol{b}$ can be added to the predicted velocity $\boldsymbol{V}_\odot$ in equation (4). The vector $\boldsymbol{b}$ points to the star and has a magnitude $|\boldsymbol{b}|$ equal to the distance to the star. This term allows two additional parameters, the Oort constants A and B, to be incorporated into the model.

The velocity residuals $\mathbf{v}$ are assumed to be due to observational errors, the velocity ellipsoid, and the cosmic dispersion in absolute magnitude of the sample. We model the covariance tensor $\mathbf{M}$ by

$$\begin{aligned} \mathbf{M} = \langle \mathbf{v}\mathbf{v}' \rangle = \mathbf{E} + \mathbf{X} + (\mathbf{U} + k\mathbf{P})\boldsymbol{\Sigma}(\mathbf{U}' + k\mathbf{P}') \\ + \sigma_k^2 \mathbf{P}(\boldsymbol{\Sigma} + \boldsymbol{V}_\odot \boldsymbol{V}'_\odot)\mathbf{P}' . \end{aligned} \tag{6}$$

Here $\mathbf{E}$ and $\mathbf{X}$ are the observational variances associated with the radial and proper motion coordinates, $\boldsymbol{\Sigma}$ is the velocity ellipsoid (i.e., the covariance tensor of the true residuals), and σ_k is the cosmic dispersion in the distance scale parameter k.

The absolute magnitude M_V and its cosmic dispersion σ_M may be found from the distance scale parameters k and σ_k by

TABLE 1

PROPER-MOTION AND RADIAL VELOCITY DATA

Name	$\mu_\alpha \cos\delta$ ("/yr)	$\sigma(\mu_\alpha \cos\delta)$ ("/yr)	μ_δ ("/yr)	$\sigma(\mu_\delta)$ ("/yr)	V_r (km/sec)
SW And	-0.0024	0.0026	-0.0221	0.0013	-22
XX And	0.0603	0.0035	-0.0361	0.0027	-1
AC And	0.0029	0.0023	0.0002	0.0014	-50
AT And	-0.0020	0.0020	0.0460	0.0020	-240
WY Ant	0.0256	0.0089	-0.0481	0.0104	208
SW Aqr	-0.0436	0.0012	-0.0521	0.0021	-62
SX Aqr	-0.0451	0.0067	-0.0464	0.0036	-178
TZ Aqr	0.0060	0.0090	-0.0110	0.0008	25
DN Aqr	0.0460	0.0080	-0.0160	0.0100	-230
AA Aql	-0.0022	0.0014	-0.0124	0.0017	-30
V341 Aql	0.0348	0.0010	-0.0240	0.0031	-77
S Ara	-0.0234	0.0021	-0.0153	0.0012	185
X Ari	0.0682	0.0030	-0.0894	0.0022	-37
RW Ari	-0.0070	0.0060	-0.0060	0.0060	-60
TZ Aur	-0.0053	0.0022	-0.0095	0.0013	45
RS Boo	0.0121	0.0055	0.0035	0.0040	-9
RU Boo	-0.0160	0.0030	-0.0030	0.0040	-60
ST Boo	-0.0148	0.0062	-0.0040	0.0048	10
SV Boo	0.0002	0.0025	-0.0176	0.0051	-160
SW Boo	-0.0238	0.0113	0.0003	0.0096	15
SZ Boo	-0.0030	0.0028	0.0033	0.0031	-45
TV Boo	-0.0104	0.0055	-0.0290	0.0024	-64
TW Boo	0.0011	0.0021	-0.0489	0.0025	-99
UU Boo	-0.0074	0.0012	-0.0514	0.0054	20
UY Boo	-0.0115	0.0024	0.0177	0.0191	145
RW Cnc	-0.0001	0.0030	-0.0360	0.0022	-86
SS Cnc	0.0042	0.0016	-0.0076	0.0021	5
TT Cnc	-0.0486	0.0009	-0.0420	0.0000	49
W CVn	-0.0274	0.0047	-0.0115	0.0023	22
Z CVn	-0.0080	0.0036	-0.0261	0.0007	20
RR CVn	-0.0216	0.0063	-0.0227	0.0082	-10
RU CVn	-0.0425	0.0033	-0.0004	0.0025	-55
RX CVn	0.0498	0.0114	-0.0273	0.0004	5
RZ CVn	-0.0443	0.0021	-0.0057	0.0045	-11
SS CVn	0.0026	0.0029	-0.0429	0.0022	-5
ST CVn	-0.0289	0.0031	-0.0175	0.0009	-123
SV CVn	-0.0200	0.0100	-0.0300	0.0100	12
RV Cap	0.0258	0.0035	-0.1110	0.0017	-110
YZ Cap	-0.0195	0.0035	-0.0149	0.0069	-88
IU Car	-0.0130	0.0040	0.0060	0.0330	307
BI Cen	-0.0076	0.0054	0.0015	0.0057	210
V499 Cen	0.0210	0.0110	-0.0025	0.0045	332
RZ Cep	0.0977	0.0022	0.1896	0.0013	0
RR Cet	0.0162	0.0024	-0.0475	0.0040	-96
RV Cet	0.0273	0.0020	-0.0213	0.0109	-94
RZ Cet	0.0181	0.0055	0.0024	0.0009	-15
RY Col	0.0360	0.0060	0.0180	0.0060	471
S Com	-0.0140	0.0025	-0.0077	0.0030	-55
U Com	-0.0457	0.0029	-0.0166	0.0010	16
V Com	-0.0119	0.0042	-0.0024	0.0019	0
Z Com	0.0150	0.0050	-0.0030	0.0050	-50
RY Com	-0.0166	0.0232	0.0133	0.0029	-28
ST Com	-0.0361	0.0045	-0.0357	0.0054	-68
RV CrB	-0.0239	0.0019	-0.0233	0.0023	-125
W Crt	-0.0190	0.0050	-0.0110	0.0050	69
X Crt	0.0078	0.0108	-0.0457	0.0077	78
SW Cru	0.0107	0.0110	0.0019	0.0082	-23
UY Cyg	0.0013	0.0020	-0.0076	0.0020	3
XZ Cyg	0.0843	0.0028	-0.0215	0.0028	-160
DM Cyg	0.0004	0.0030	-0.0143	0.0019	-49
DX Del	0.0182	0.0043	0.0083	0.0010	-45
RW Dra	-0.0080	0.0071	-0.0086	0.0037	-124
SU Dra	-0.0384	0.0022	-0.0735	0.0011	-175
SW Dra	-0.0211	0.0047	-0.0036	0.0018	-33
XZ Dra	0.0117	0.0044	0.0030	0.0066	-31
RX Eri	-0.0208	0.0060	-0.0030	0.0043	70
SV Eri	0.0216	0.0030	-0.0475	0.0005	-19
SS For	0.0430	0.0030	-0.0725	0.0035	-122
SW For	0.0125	0.0135	-0.0010	0.0030	167
RR Gem	-0.0035	0.0017	-0.0024	0.0024	60
SZ Gem	-0.0109	0.0011	-0.0298	0.0020	305
SW Her	-0.0070	0.0040	0.0130	0.0040	-130
TW Her	0.0023	0.0020	-0.0083	0.0020	-15
VX Her	-0.0353	0.0105	0.0060	0.0122	-377
VZ Her	-0.0161	0.0022	-0.0169	0.0014	-115
AF Her	-0.0165	0.0015	-0.0100	0.0040	-270
AG Her	-0.0044	0.0022	-0.0170	0.0015	-75
AR Her	-0.0578	0.0025	0.0131	0.0029	-350
CE Her	-0.0004	0.0034	-0.0025	0.0021	-258
SV Hya	-0.0551	0.0136	0.0081	0.0092	101
SZ Hya	0.0017	0.0068	-0.0465	0.0014	100
UU Hya	-0.0100	0.0040	-0.0080	0.0080	300
WZ Hya	0.0000	0.0070	-0.0290	0.0070	304
XX Hya	-0.0016	0.0072	-0.0360	0.0020	-10
DH Hya	-0.0152	0.0032	-0.0075	0.0034	368
FY Hya	-0.0412	0.0144	-0.0004	0.0072	80
V Ind	-0.0700	0.0060	-0.0900	0.0060	198
CZ Lac	-0.0040	0.0060	-0.0060	0.0060	-120
DE Lac	0.0031	0.0031	-0.0027	0.0007	0
RR Leo	-0.0113	0.0031	-0.0116	0.0010	94
RV Leo	-0.0050	0.0030	-0.0130	0.0030	0
RX Leo	0.0158	0.0023	-0.0306	0.0021	-127
SS Leo	-0.0232	0.0030	-0.0155	0.0026	180
ST Leo	-0.0140	0.0031	-0.0413	0.0041	150
SZ Leo	-0.0174	0.0009	-0.0226	0.0093	187
TV Leo	0.0138	0.0068	0.0168	0.0084	-102
V LMi	0.0237	0.0054	-0.0303	0.0031	-110
X LMi	0.0055	0.0052	-0.0200	0.0014	40
U Lep	0.0429	0.0009	-0.0552	0.0021	122
TV Lib	0.0008	0.0022	0.0076	0.0013	-52
TT Lyn	-0.0870	0.0020	-0.0506	0.0001	-63
TV Lyn	0.0000	0.0060	0.0050	0.0060	-7
Y Lyr	-0.0003	0.0023	0.0017	0.0006	-110
RR Lyr	-0.1095	0.0018	-0.1942	0.0035	-76
RZ Lyr	0.0156	0.0023	0.0238	0.0041	-231
EZ Lyr	-0.0132	0.0056	-0.0020	0.0072	-75
UV Oct	-0.0693	0.0044	-0.1230	0.0108	109
ST Oph	-0.0009	0.0017	-0.0008	0.0011	9
V445 Oph	0.0047	0.0024	0.0124	0.0021	-15
V716 Oph	-0.0242	0.0022	-0.0110	0.0017	-230
TY Pav	-0.0210	0.0110	-0.0220	0.0230	255
DN Pav	-0.0090	0.0060	-0.0300	0.0060	-95
VV Peg	0.0090	0.0016	-0.0041	0.0031	15
AV Peg	0.0154	0.0009	-0.0100	0.0040	-56
BH Peg	-0.0156	0.0052	-0.0681	0.0035	-278
CG Peg	-0.0039	0.0059	-0.0077	0.0027	2
DH Peg	0.0179	0.0033	0.0001	0.0007	-56
TU Per	0.0151	0.0008	-0.0061	0.0063	-377
AR Per	-0.0015	0.0030	0.0050	0.0130	1
RV Phe	0.0415	0.0085	-0.0185	0.0005	87
U Pic	-0.0010	0.0060	-0.0170	0.0060	14
RU Psc	0.0978	0.0035	-0.0421	0.0018	-131
RY Psc	0.0440	0.0025	0.0033	0.0015	26
SS Psc	-0.0060	0.0060	-0.0020	0.0060	5
XX Pup	-0.0313	0.0017	-0.0014	0.0013	374
BB Pup	0.0070	0.0070	0.0060	0.0070	82
V440 Sgr	-0.0020	0.0080	-0.0500	0.0090	-71
V675 Sgr	0.0000	0.0060	0.0120	0.0060	-105
V1640 Sgr	-0.0040	0.0060	0.0090	0.0060	-17
V487 Sco	-0.0132	0.0004	-0.0138	0.0016	-53
V494 Sco	-0.0034	0.0111	-0.0062	0.0102	26
RU Scl	0.0563	0.0053	-0.0204	0.0013	27
SV Scl	-0.0034	0.0023	-0.0364	0.0044	-10
VY Ser	-0.0588	0.0326	0.0294	0.0215	-140
AN Ser	-0.0060	0.0050	-0.0116	0.0005	-47
AP Ser	-0.0437	0.0033	-0.0364	0.0025	-91
AR Ser	-0.0500	0.0060	0.0170	0.0060	133
AT Ser	-0.0093	0.0010	-0.0045	0.0037	-55
AV Ser	0.0020	0.0070	-0.0010	0.0070	-55
T Sex	-0.0262	0.0015	-0.0246	0.0002	38
SS Tau	0.0106	0.0035	0.0007	0.0027	-50
U Tri	0.0070	0.0040	-0.0096	0.0027	-60
W Tuc	0.0030	0.0060	0.0020	0.0060	71
YY Tuc	0.0014	0.0040	-0.0034	0.0040	31
RV UMa	-0.0257	0.0064	-0.0433	0.0032	-178
SX UMa	-0.0743	0.0034	0.0044	0.0033	-135
TU UMa	-0.0697	0.0047	-0.0514	0.0012	87
AF Vel	0.0590	0.0060	-0.0175	0.0035	236
ST Vir	-0.0045	0.0017	-0.0192	0.0023	-35
UU Vir	-0.0296	0.0016	-0.0045	0.0046	-17
UV Vir	-0.0175	0.0012	-0.0204	0.0103	95
XX Vir	-0.0120	0.0016	-0.0129	0.0015	-55
AS Vir	-0.0087	0.0323	-0.0575	0.0020	85
AT Vir	-0.0563	0.0056	-0.0282	0.0028	358
AU Vir	-0.0101	0.0069	-0.0182	0.0035	118
AV Vir	0.0160	0.0040	-0.0360	0.0030	154
BB Vir	-0.0444	0.0009	0.0104	0.0048	-5
BC Vir	0.0129	0.0218	-0.0333	0.0010	0
BN Vul	-0.0485	0.0025	-0.0380	0.0020	-267

assuming that the RR Lyrae luminosity function is Gaussian with dispersion σ_M. Then,

$$M_V = 5 \log_{10} (1 + k) + M_A - 0.1 \ln 10\sigma_M^2 \qquad (7)$$

and

$$\sigma_M^2 = \log_{10} [1 + \sigma_k^2/(1 + k)^2]/0.04 \ln 10 \,. \qquad (8)$$

IV. SIMPLEX OPTIMIZATION

To maximize the likelihood in this model we use a geometric minimization technique known as simplex optimization. This technique was first introduced by Nedler and Mead (1965). Our application is taken from Daniels (1978, pp. 183–190). Briefly, a simplex is the geometric figure formed by $n + 1$ noncollinear points in an n-dimensional space. Thus in two dimensions a simplex is a triangle. Each vertex of the simplex represents a set of parameter values and an error associated with the set. For example, in the two-parameter case, all values of the two parameters define a plane at each point on which the parameters combine through the model to produce an error value. The initial simplex is calculated by changing each parameter in turn from the initially adopted value and finding the error associated with the new parameter set. Depending on which vertices of the simplex have the lowest and highest errors, the parameters are then varied according to a set of geometric rules to define a new simplex. The technique thus follows a simple search-and-move algorithm to explore methodically the parameter space and find the set of parameter values that gives the smallest value of the error function. Although slower than derivative methods, e.g., least-squares, simplex optimization is very stable. It is eminently suitable to problems of this type where there are many interdependent quantities which make least-squares methods complicated and difficult to follow to convergence.

We take the error function S to be the negative of twice the logarithmic likelihood from equation (3):

$$S = -2 \ln (L) = \sum \ln |\mathbf{M}| + \mathbf{v}'\mathbf{M}^{-1}\mathbf{v} \,, \qquad (9)$$

where we have dropped the constant. Minimizing S will maximize the likelihood as desired.

The uncertainties in the final values of the parameters are found by numerically computing the derivatives. To find the uncertainty in parameter X_i, we fix X_i at the value $X_{i0} + d_i$, where X_{i0} is the final value of X_i at the converged solution, and d_i is a small, usually 1%, deviation from X_{i0}. We then allow the other parameters to converge to a new minimum solution having an error S. If the error at the true solution is S_0, the variance in X_i is given by

$$\sigma_i^2 = d_i^2/(S - S_0) \,. \qquad (10)$$

Clearly, S must always be larger than S_0, so σ_i^2 is positive. We note that this is a biased but consistent estimator of the variance. The covariance between two parameters X_i and X_j is computed in a similar, although more complicated, fashion.

V. RESULTS

The results of our analysis for seven subgroups of the sample are shown in Table 2. Six of the subgroups were formed according to type (*ab* or *c*), metallicity ($\Delta S < 5$, $\Delta S \geq 5$) and period ($P < 0.52$ days, $P \geq 0.52$ days). The last subgroup was suggested by de Vaucouleurs (1978) as being representative of the globular cluster RR Lyrae stars. It includes stars with low metallicity ($\Delta S > 3$) in a limited period range (0.42 days $< P <$ 0.60 days). The number of stars in each group is given in the second column. The next three columns show the three components of reflex solar motion in galactocentric cylindrical coordinates (V_ϖ, V_θ, V_z). Although halo stars are expected to have velocity ellipsoids which are aligned along a spherically symmetric coordinate system, we found that the velocity ellipsoid was practically diagonal in this cylindrical coordinate system for each of the six large groups. The stars in our sample are close enough to the disk that the difference is negligible. When the principal axes of the velocity ellipsoid were forced to be the galactocentric cylindrical coordinate axes, there was no change in the resulting values for the other parameters. The quantities σ_ϖ, σ_θ, and σ_z are therefore the Gaussian velocity dispersions along these axes. The final column gives the absolute visual magnitude for each group. All parameters have uncertainty estimates which were found as described above.

We also investigated smaller subgroups based on metallicity and period. However, small sample size often made the likelihood function ill-conditioned. We found that only groups with more than 50 stars gave meaningful solutions. Even with these larger groups, the cosmic dispersion in absolute magnitude was not well determined. For half of the groups, it converged to a value of zero, while for the other half it converged to unrealistically large values. This significantly affected the determination of the other parameters. To avoid this problem we set the cosmic dispersion to a fixed value and did not allow it to vary while the other parameters converged to a solution. The results shown in Table 2 are those found when σ_M was taken to be zero. The effect of a nonzero σ_M is always to make the mean absolute magnitude brighter. For example, when σ_M was set equal to 0.1, the absolute magnitude became brighter by 0.01–0.02 mag for the six large groups, while a value $\sigma_M = 0.3$ changed it by 0.05–0.07 mag. The velocity parameters were essentially the same for all values of σ_M.

The *c*-type group required a further approximation because

TABLE 2

RESULTS OF ANALYSIS

Group	Number of Stars	V_ϖ	V_θ	V_z	σ_ϖ	σ_θ	σ_z	M_V
RR *ab*	142	-10 ± 13	-155 ± 12	-9 ± 8	150 ± 59	120 ± 47	87 ± 33	0.76 ± 0.14
RR *c*	17	-26 ± 25	-124 ± 25	-6 ± 13	101 ± 57	71	51	1.09 ± 0.38
RR *ab* $\Delta S < 5$	65	$+5 \pm 17$	-120 ± 17	-14 ± 10	128 ± 66	120 ± 57	78 ± 36	0.79 ± 0.21
RR *ab* $\Delta S \geq 5$	77	-21 ± 19	-184 ± 17	-4 ± 11	166 ± 76	114 ± 52	91 ± 40	0.73 ± 0.18
RR *ab* $P < 0.52$	73	-37 ± 16	-120 ± 17	-9 ± 10	135 ± 61	119 ± 57	81 ± 37	0.88 ± 0.21
RR *ab* $P \geq 0.52$	69	$+20 \pm 20$	-185 ± 17	-9 ± 11	161 ± 78	106 ± 49	89 ± 39	0.71 ± 0.20
RR *ab* $\Delta S > 3$, $0.42 < P < 0.60$	55	-18 ± 24	-195 ± 20	-9 ± 12	174 ± 86	114 ± 56	86 ± 41	0.83 ± 0.21

of the small size of the sample (17 stars!). We forced the velocity ellipsoid to be diagonal in the galactocentric cylindrical coordinate system, and the ratio of the dispersions along the coordinate axes was taken to be that found for the *ab*-type group. Thus only one velocity ellipsoid parameter was independently varied. These results are clearly more uncertain than those for the larger groups and are presented only for completeness.

Mean light $B-V$ magnitudes from Fitch, Wisniewski, and Johnson (1966) were available for 84 of the stars in the full RR *ab* sample. The mean absolute B magnitude was found to be 1.17 ± 0.18 for this subgroup. For comparison, the mean absolute V magnitude for the same subgroup was 0.91 ± 0.18 mag. This reflects the fact that most RR Lyrae *ab*-type stars have a mean $(B-V)_0$ color of ~ 0.26.

Finally, we investigated the effect of differential galactic rotation on the sample. The largest subset, the *ab*-type group, gave $A = 1.6 \pm 2.5$ km s^{-1} kpc^{-1}, $B = 2.3 \pm 2.5$ km s^{-1} kpc^{-1}, as compared to the usual Population I values $A = 15$ km s^{-1} kpc^{-1} and $B = -10$ km s^{-1} kpc^{-1} (Mihalas and Binney 1982). The other parameters showed no significant change from their previous values. When the Oort constants were forced to have their Population I values, the derived velocity and distance parameters were still unchanged. Differential galactic rotation clearly does not have an important effect on the results for these halo stars.

VI. DISCUSSION

Several important points are evident from the results shown in Table 2. The absolute magnitude for the largest and best determined group, the RR *ab*-type sample, is 0.76 ± 0.14. From the discussion above, this implies a mean absolute B magnitude of 1.02 ± 0.14. The uncertainty of 0.14 mag is substantially less than that obtained in any previous statistical analysis, while the value of 0.76 for the absolute magnitude is in agreement at the 1 σ level with the mean value of 0.61 ± 0.15 found by combining many previous determinations from several different methods of analysis (see Stothers 1983, Table 3). We stress, however, that our accurate data and rigorous analysis allow our determination to stand alone as the best current estimate of the RR *ab*-type absolute magnitude. Agreement with previous results serves merely as assurance that this important astrophysical quantity can now be used to determine distances with an uncertainty of only 6%.

The smaller subgroups based on metallicity and period show suggestive trends but no definitive results. Sandage (1981, 1982) found for globular cluster RR Lyrae stars a relationship between period and absolute magnitude in the sense that stars with longer periods are brighter. Because a strong correlation exists between metallicity and period, this leads to the result that stars with lower metallicity are also brighter. Pulsation model calculations dating back to Christy (1966) also predict these correlations. We find this trend in our data, but the larger uncertainties for these smaller samples make the result inconclusive. In fact, all of our subgroups have the same absolute magnitude at the 1 σ level. Also, again because of the limitation of sample size, we are not able to divide the stars into distinct period and metallicity groups to truly test the hypothesis. When four period groups and three metallicity groups were attempted, the convergence was found to be unsatisfactory (Hawley 1984).

At the suggestion of the referee we applied Sandage's period-luminosity-amplitude relation (Sandage 1981) to the $\Delta S \geq 5$ sample. The hope was that correcting for a possible *P-L-A* effect in the field RR Lyrae stars would reduce the uncertainty in the computed absolute magnitude. We used period and B-magnitude amplitude data from the GCVS to adjust the apparent magnitude of each star. Our resulting mean absolute visual magnitude for this adjusted sample was 0.66 ± 0.18. Neither the absolute magnitude nor the uncertainty is significantly different from the results for the original $\Delta S \geq 5$ group, indicating that failure to account for a *P-L-A* relation is not the principal source of scatter in our data. We conclude that considerably more data are needed before the statistical analysis can be usefully applied to the problem of absolute magnitude correlations with metallicity and/or period.

The kinematic results are consistent with the accepted picture of the Galaxy as an axisymmetric rotating disk surrounded by a more slowly rotating halo population. If the solar rotational velocity is taken to be ~ 250 km s^{-1} with respect to the galactic center (Mihalas and Binney 1982), the full RR *ab*-type sample has a rotation velocity of ~ 100 km s^{-1} relative to the galactic center. The subgroups which are perhaps most representative of the globular cluster population, namely, the low-metallicity and long-period groups and the group suggested by de Vaucouleurs, show a significantly larger reflex solar motion and, hence, a smaller intrinsic rotational velocity of ~ 55–65 km s^{-1}. This is in good agreement with the rotation velocity of 60 ± 20 km s^{-1} found for the globular clusters by Frenk and White (1980), although they use a somewhat smaller solar rotation velocity. We find no evidence for a radial or vertical component to the systemic motion, with only the short-period group having a radial motion significant at the 2 σ level.

The velocity ellipsoid parameters are very uncertain, but two general points may be made. The principal axes were found to be close to the galactocentric cylindrical coordinate axes in all cases, and the velocity dispersions were found to be ordered in the sense $\sigma_\varpi > \sigma_\theta > \sigma_z$, as expected from simple dynamical arguments in a steady state, somewhat flattened, axisymmetric system. The globular clusterlike groups again show a trend of increasing velocity dispersion in accord with Frenk and White's results, but the uncertainties are too large to give a conclusive result.

Finally, although the uncertainties in the absolute magnitudes do not warrant a subdivision of the data on that basis alone, the reflex solar motion results do seem to indicate that the low-metallicity, long-period groups are kinematically distinct and perhaps follow the globular cluster population most closely.

VII. CONCLUSIONS

Our statistical analysis of the field RR Lyrae stars using an improved data sample and a rigorous maximum likelihood model has given more accurate results than have been previously obtained. In particular, our absolute magnitude estimate is now the most accurate available for the RR *ab*-type stars, and is in agreement with previous, more uncertain, determinations. Subgroups of the data chosen by metallicity and period show suggestive trends in absolute magnitude and in the velocity dispersion parameters, but only the systemic motion is significantly different within the uncertainties. Based on this kinematic parameter, the long-period and low-metallicity groups are most representative of the halo population as defined by the globular clusters.

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