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EVOLUTIONARY PERIOD CHANGES IN VARIABLE HELIUM-RICH WHITE DWARFS

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ABSTRACT

In this study, we investigate the rates of pulsation period change in realistic, theoretical evolutionary models representing the transition from a helium-rich planetary nebula nucleus to a cool white dwarf. We apply our results to the study of the pulsating DB white dwarf (DBV) stars and their presumed progenitors, the pulsating PG 1159 (DOV) stars. We have computed the expected rates of period change by solving the equations of linear, adiabatic, nonradial oscillation for nonrotating evolutionary models of hydrogen-deficient white dwarfs. The value of $d(\ln P)/dt$ varies over the range $(3–5) \times 10^{-14}$ s^{-1} for models appropriate to the DOV stars. The expected value of $d(\ln P)/dt$ for DBV stars ranges from 3×10^{-16} s^{-1} at $T_{\rm eff} = 30{,}000$ K to 5×10^{-17} s^{-1} at $T_{\rm eff} = 20{,}000$ K. Period changes of this order should be measurable in DBV stars over two or three observing seasons. We have examined the effects of varying stellar mass and chemical composition on the $d(\ln P)/dt–T_{\rm eff}$ relationship, and show that measurements of the rates of period change in the DBV stars may provide important independent constraints on the effective temperatures of these stars. These period changes can be used to study the composition and structure of the outer layers, place constraints on the mass of the star, and provide a direct test of plasmon neutrino emission rates.

Subject headings: stars: evolution — stars: interiors — stars: pulsation — stars: white dwarfs

I. INTRODUCTION

There are three classes of variable white dwarf stars. We refer to these three classes of variable star using the classification scheme of Sion *et al.* (1983): the DAVs, or ZZ Ceti stars; the DBVs, or pulsating DB white dwarfs; and the DOVs, or pulsating PG 1159−035 stars. Two of these classes, the DOVs and DBVs, are of particular interest here: the DOVs occupy a somewhat uncertain region in the H-R diagram around $\log (L/L_\odot) = 2.0$ and $T_{\rm eff} = 100{,}000$ K, and the DBVs lie in a region near $\log (L/L_\odot) = -1.3$ and $T_{\rm eff} = 25{,}000$ K. The relative location of the DOVs and DBVs, as well as their helium-rich surface composition, is increasingly suggestive of a direct evolutionary connection between the two classes of objects (Sion, Liebert, and Starrfield 1985).

The locations of the DBV and DOV stars in the H-R diagram are not precisely known, and thus do not place very tight constraints on the theoretical models (Winget *et al.* 1983). The introduction of an additional observational constraint, through the determination of rates of period change, may help to relieve this situation. The purpose of this paper is to investigate this constraint in the context of the DOV and DBV stars, and their possible evolutionary connection.

An initial estimate of the effective temperature of the prototype DBV, GD 358, was made by Koester, Weidemann, and Vauclair (1983) using the low-resolution spectrophotometer of the *International Ultraviolet Explorer* (*IUE*) satellite; they obtained $T_{\rm eff} = (26 \pm 2) \times 10^3$ K. Another estimate for the effective temperature was made by Oke, Weidemann, and Koester (1984) on the basis of optical multichannel spectrophotometry using the 5-m Hale telescope. Most recently, the effective temperature has been determined by Koester *et al.* (1985). They used all the data from the two previous investigations and incorporated additional optical data, combined with new model atmosphere calculations, to arrive at $T_{\rm eff} = (24 \pm 1) \times 10^3$ K, and $\log g = 8.0 \pm 0.3$. The temperature scale for the DB stars has also been discussed by Liebert *et al.* (1986), on the basis of *IUE* observations. Their temperature determinations for the DB stars are higher by about 2000–4000 K, with the largest differences occurring at the high end of the temperature scale.

Observational measurement of dP/dt will provide a sensitive probe of the structure and evolution of DB white dwarfs. As shown by Winget, Hansen, and Van Horn (1983), a measurement of the rate of change of the pulsation period of a variable white dwarf is a direct measurement of the cooling rate in the region of the star where the period is determined. The cooling rate is a montonic function of the effective temperature. Adopting the spectroscopic determinations of $T_{\rm eff}$, the observed rate of period change can place stringent constraints on the mass of the DBVs. Alternatively, we can take advantage of the narrow mass distribution of the white dwarfs and assume a stellar mass of 0.60 $M_\odot$ (Weidemann and Koester 1983; Oke, Weidemann, and Koester 1984). By comparing the observed value of dP/dt for a DBV star with the results of pulsation calculations using evolutionary models of 0.60 $M_\odot$, we can place independent constraints on the effective temperature of the star. Throughout the DOV phase, and in the hotter parts of the DB instability strip, the plasmon neutrino energy losses are a significant fraction of the photon luminosity of the star. Thus observations of dP/dt may also put interesting limits on the theoretically determined plasmon neutrino emission rates.

In this paper we report the results of evolutionary calculations of models of hydrogen-deficient white dwarfs and present theoretical values for $d(\ln P)/dt$ over the range of interest in effective temperature. In § II we present the details of the various evolutionary sequences used for the pulsation analysis. The results for the DOV sequences are presented in § III. In § IV we present the numerical results for the DBV sequences and compare them with the simple analytical theory of white dwarf cooling, exploring how stellar mass, neutrino emission, and composition affects the numerical results. We conclude in § V with a discussion of the observability of dP/dt for DBV stars.

II. EVOLUTIONARY MODELS

In the hot DOV stars, the region of the star that is most important in establishing the adiabatic pulsation period lies within the degenerate core. As the model cools below $L = 100$ $L_\odot$, the region of maximum weight in setting the period shifts from the core out to the envelope (Kawaler, Hansen, and Winget 1985, hereafter KHW). At the effective temperatures of the DBV stars the value of the pulsation period is largely determined near the degeneracy boundary in the surface regions. The region of most importance in setting the adiabatic pulsation periods, for modes representing the observed periods, is the outer few percent, by mass, of the star. Accurate treatment of the outermost regions is therefore important for analysis of the periods and rates of change of the period in the DBVs and in the cooler DOVs. For this reason, we concentrate on a model of 0.60 $M_\odot$ with a 0.02 $M_\odot$ helium-rich envelope and a carbon-oxygen core. This sequence was evolved using as a starting model a model planetary nebula nucleus, at $\log(L/L_\odot) = 3.5$, from Iben (1984). We constructed this sequence using the evolutionary code and equation of state (including Coulomb corrections) described in detail in Iben and Tutukov (1984). For the evolution calculations, the mass included in the surface-layer integrations was reduced to 10^{-13} $M_\odot$. In this way, the region of the star of most importance in setting the pulsation period always was located in the interior of the evolutionary model. This permitted us to use just the interior of the models in the subsequent pulsation calculations. Typically, the models used as input for the pulsation calculations had about 600 zones. The largest zones ($\Delta q < 0.014$ $M_\odot$) were near $q \sim 0.20$ $M_\odot$.

The evolution of this model is similar to that of the model with a helium-rich envelope presented in that paper. We used a ratio of mixing length to pressure scale height of 1.5 in the envelope, to simulate efficient convection in the context of standard mixing-length theory. Nuclear burning of helium, although included, was not an important luminosity source at the $T_{\rm eff}$ range of the DOV stars; gravitational luminosity dominated below $\log(L/L_\odot) = 2.5$.

To judge the sensitivity of $d(\ln P)/dt$ to stellar mass, envelope composition, neutrino energy losses, and other equilibrium model quantities, we constructed four additional pure ^{12}C evolutionary sequences, with masses of 0.40, 0.60, and 0.78 $M_\odot$, to compare with the helium envelope sequence. The 0.60 $M_\odot$ and 0.78 $M_\odot$ ^{12}C sequences, described in detail in KHW, were continued down to the effective temperature range of interest for studying DBV stars. The 0.40 $M_\odot$ sequence was also evolved from an initial model in the region of the H-R diagram populated by planetary nebula nuclei (PNNs), with the same procedure used for the other pure ^{12}C sequences. The fourth sequence, for a model of 0.60 $M_\odot$, was evolved without including energy loss by neutrino emission. The starting model for this sequence was the core of a 3.0 $M_\odot$ asymptotic giant branch star in which neutrino emission was turned off at the time of helium exhaustion in the center. Otherwise, the construction of this sequence followed the same procedure as the 0.60 $M_\odot$ model in KHW.

In the DBV temperature range, the magnitude of neutrino losses decreases with increasing mass for a given effective temperature. At 30,000 K those sequences incorporating neutrinos still had a neutrino luminosity approximately equal to the photon luminosity. By the time the models had cooled to $T_{\rm eff} = 20{,}000$ K, the neutrino luminosity was at most 10% of the photon luminosity of the model.

We examined the pulsation properties of the equilibrium models using a code that solves the equations of linear, adiabatic, nonradial oscillation. This code is described in KHW. The periods of the high-order g-modes that we studied in detail are representative of the periods of pulsation for the class of variable under consideration. For the DOV stars, with periods from 500 to 800 s, we consider the $k = 25$ (25 radial nodes), $l = 1$ mode. In the DBV stars, with observed periods of 500–1000 s, the period of the $k = 25$, $l = 2$ mode is representative. To fully resolve closely spaced nodes for those modes, the equilibrium model quantities are interpolated between zones, by means of cubic splines, during the integration of the adiabatic pulsation equations (KHW). This results in 2000–3000 effective zones. We estimate the value of $d(\ln P)/dt$ by simple differencing of the periods of successive models in each evolutionary sequence. The value of $d(\ln P)/dt$ is insensitive to the values of l and k, as demonstrated by KHW. The results presented here represent the adiabatic pulsation properties of nonrotating models.

III. PERIOD CHANGES IN THE DOV MODELS

Since the 0.60 $M_\odot$ helium-rich sequence was started with a PNN model, it represents a realistic model of a cooling, hydrogen-deficient post–PNN star. Hence, we may compare the pulsation properties of PG 1159 models from this sequence with the results from the pure ^{12}C models of KHW to determine how differences in chemical composition, nuclear burning, and other input physics affect the conclusions of that paper. In addition, the effect of neutrino emission on the rate of period change in the realistic models can be investigated in a self-consistent way by following the pulsation properties of the 0.60 $M_\odot$ sequence evolved without neutrinos.

We have calculated the value of $d(\ln P)/dt$ for the $l = 1$, g_{25} mode for models in the luminosity range from 1000 to 10 $L_\odot$ for the 0.60 $M_\odot$ sequences. In the model with a helium envelope, the period of this mode ranged from 540 s at 1000 $L_\odot$ to 630 s at 10 $L_\odot$. (For comparison, the pure ^{12}C model had a period of 560 s at 1000 $L_\odot$, and 646 s at 10 $L_\odot$.) Above $\log(L/L_\odot) = 2.0$, adiabatic periods are determined in the degenerate interior. Below that transition luminosity, the periods are determined near the surface. Periods increase throughout the region of interest, with a maximum in $d(\ln P)/dt$ of 6×10^{-14} s^{-1} at $\log(L/L_\odot) = 2.6$. The magnitude and sign of $d(\ln P)/dt$, its behavior as a function of luminosity, and the weight functions were all very similar to the pure ^{12}C 0.60 $M_\odot$ model presented in KHW. As discussed in KHW, the similarity between chemically homogeneous models and stratified models in the behavior of $d(\ln P)/dt$ as a function of luminosity results from the basic similarities of the degenerate cores of the entire class of post–PNN models.

Values of $d(\ln P)/dt$ were systematically larger in the compositionally stratified sequence than in the pure ^{12}C model with neutrinos included. In the luminosity range from $\log (L/L_\odot) = 2.6$ down to $\log (L/L_\odot) = 1.0$, the period increased about twice as fast in the stratified models. The difference in $d(\ln P)/dt$ reflects the difference in composition of the region of maximum weight: the stratified models are about half carbon and half oxygen in the core. The cooling rate, and therefore the rate of period change, depend inversely on the ion specific heat of the core. Hence, the stratified model, with slightly fewer nuclei in the core, cools more quickly than the pure ^{12}C model.

As clearly demonstrated by Starrfield *et al.* (1984, 1983), the partial ionization of carbon and oxygen in the outer layers of static models in the DOV region of the H-R diagram provides enough driving to destabilize the models to high-order g-mode pulsations. Another possible destabilizing effect that may be important in these stars is the contribution of nuclear burning via the epsilon mechanism (see Unno *et al.* 1979). As mentioned previously (see also Iben and Tutukov 1984, Fig. 3) the helium-burning shell is an unimportant luminosity source below $\log (L/L_\odot) = 2.5$ and no longer affects $d(\ln P)/dt$. However, a very preliminary nonadiabatic analysis suggests that the helium-burning shell can contribute to the instability of the high-order g-modes. In the more luminous models, where the helium-burning shell is the dominant luminosity source, the epsilon mechanism may play a more important role in exciting the pulsations. We are currently investigating this phenomenon, with possible application to the pulsating nucleus of the planetary nebula K1-16 (see Starrfield *et al.* 1985).

The values of $d(\ln P)/dt$ for the homogeneous sequence with no neutrinos are much smaller than in the models that include neutrinos. At the luminosity of PG 1159−035, $\sim 100\ L_\odot$, $d(\ln P)/dt$ was $+2.4 \times 10^{-15}$ s^{-1}. This is a factor of 5 smaller than in the homogeneous model that includes neutrinos. The rate of rotational spin-up by contraction is about the same for the model without neutrinos as for the model with neutrinos. Since rotation plays an important rôle in determining the observed rate of period change for some modes (Kawaler, Winget, and Hansen 1985), observation of $d(\ln P)/dt$ alone is insufficient to constrain the neutrino rates uniquely. However, since the time baseline of photometric observations of PG 1159−035 continually lengthens, the prospects for observationally separating out the rotation effects are good (see below for the case of GD 358), in which case the constraints will become severe.

IV. PERIOD CHANGES IN THE DBV MODELS

a) *Numerical Results*

The grouping of periods in the power spectrum of GD 358 (Winget *et al.* 1982) suggests rotational splitting of several $l = 2$ modes into their 5 m components. Hence an analysis of dipole modes in DBV models is suggested by the available data. For the study of $d(\ln P)/dt$ for DBVs, we consider the $k = 25$, $l = 2$ nonradial g-mode. Periods for this mode range from 535 s at $T_{\rm eff} = 30{,}000$ K to 650 s at $T_{\rm eff} = 20{,}000$ K for the 0.60 $M_\odot$ stratified model. The period for the same mode in the 0.60 $M_\odot$ pure ^{12}C model was 20%–30% longer than in the stratified model. The longer periods reflect the fact that the mean atomic weight of the stratified models is smaller. As shown by Osaki and Hansen (1973), the periods of the g-modes are inversely proportional to the square root of the specific heat. Since the specific heat of the ions is inversely proportional to the mean atomic weight, then the periods of the g-modes are proportional to the square root of the mean atomic weight in the region of importance to setting the period.

The run of $d(\ln P)/dt$ with respect to effective temperature for DBV models is illustrated in Figure 1*a*. For the 0.60 $M_\odot$ model

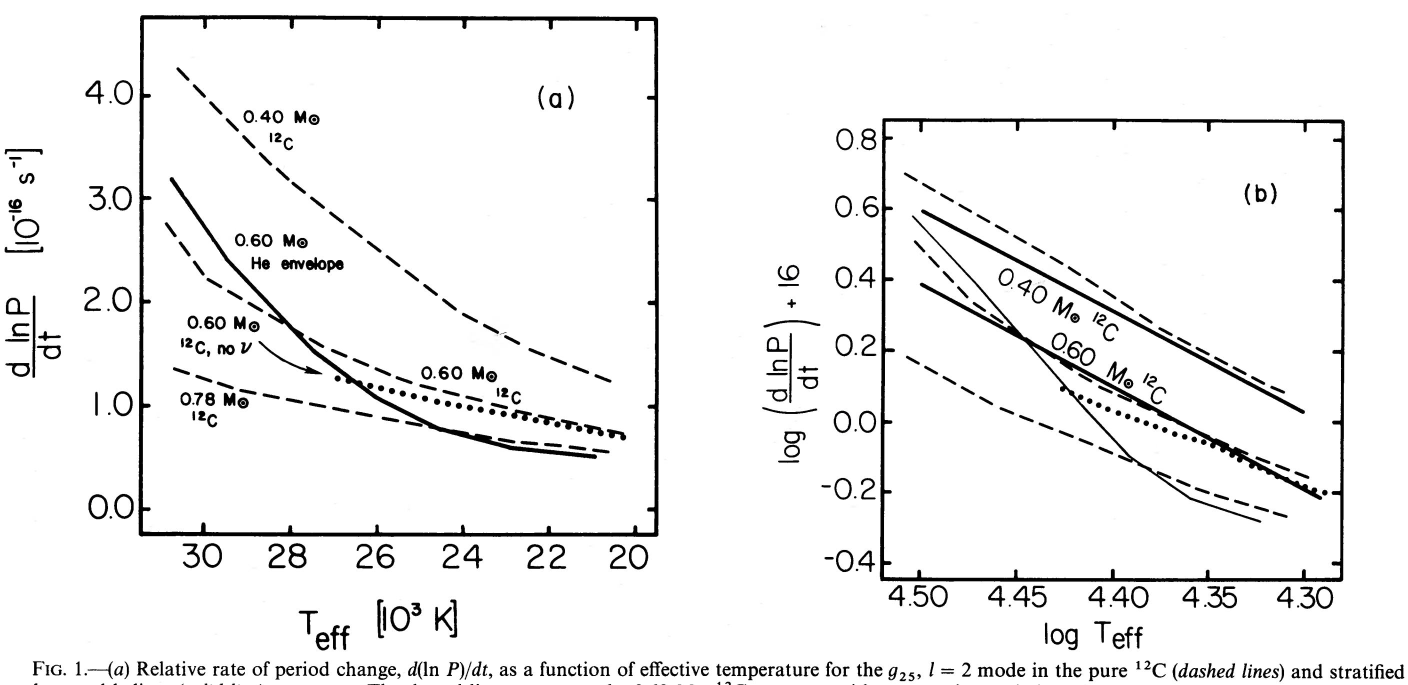

FIG. 1.—(*a*) Relative rate of period change, $d(\ln P)/dt$, as a function of effective temperature for the g_{25}, $l = 2$ mode in the pure ^{12}C (*dashed lines*) and stratified carbon and helium (*solid line*) sequences. The dotted line represents the 0.60 $M_\odot$ ^{12}C sequence without neutrino emission. (*b*) Same as (*a*) plotted on logarithmic scales. The straight solid lines are analytic determinations (eq. [2]) of $d(\ln P)/dt$ for 0.40 $M_\odot$ (*top*) and 0.60 $M_\odot$ (*bottom*) pure ^{12}C white dwarfs.

with a helium-rich envelope, we find values of $d(\ln P)/dt$ ranging from 3×10^{-16} s^{-1} at $T_{\rm eff} = 30{,}000$ K to about 5×10^{-17} s^{-1} at $T_{\rm eff} = 20{,}000$ K. This range in $d(\ln P)/dt$ is larger than in any other model that we investigated. For example, tne pure ^{12}C 0.60 $M_\odot$ sequence showed values of $d(\ln P)/dt$ of 2×10^{-16} s^{-1} at $T_{\rm eff} = 30{,}000$ K down to 7×10^{-17} s^{-1} at $T_{\rm eff} = 20{,}000$ K. This difference in the behavior of $d(\ln P)/dt$ as a function of $T_{\rm eff}$ for two models of the same mass can be understood in terms of the differences between the compositional structure of the pure ^{12}C model and the more realistic compositionally stratified model. These effects will be discussed in more detail in the next subsection.

Examination of the three ^{12}C sequences illustrates the dependence of $d(\ln P)/dt$ on stellar mass. Lower mass models show larger values of $d(\ln P)/dt$ through the range of effective temperature of DBV stars. This reflects the simple fact that, for a given effective temperature, lower mass white dwarfs are more luminuous and have a lower total heat capacity; hence the rate of leakage of thermal energy is larger than in more massive white dwarfs.

b) Comparison with a Simple Cooling Model

The cooling rate for a white dwarf can be estimated analytically using the simple model of Mestel (1952). When neutrinos and nuclear burning are unimportant luminosity sources, the Mestel cooling theory relates the photon luminosity, and therefore the cooling rate, to the temperature of the isothermal core (see the discussion of Van Horn 1971). Assuming a mass-radius relationship for white dwarfs, we can then relate the cooling rate to the effective temperature to obtain

$$\frac{\dot{T}_{\rm eff}}{T_{\rm eff}} = -5 \times 10^{-30} A \left(\frac{\mu}{\mu_e^2}\right)^{0.286} \left(\frac{M}{M_\odot}\right)^{-1.190} T_{\rm eff}^{2.857}, \quad (1)$$

where μ_e is the mean molecular weight per electron, and A is the atomic mass of the ions. The derivation of equation (1) is presented in Appendix A.

For high-order g-modes in slowly rotating ($P_{\rm puls} \ll P_{\rm rot}$) stars, the relation of Winget, Hansen, and Van Horn (1983) (their eq. [3]) becomes

$$\frac{d \ln P}{dt} = -\frac{1}{2}\frac{d \ln T}{dt} + \left(1 - b_{\rm rot}\frac{P}{P_{\rm rot}}\right)\frac{d \ln R}{dt}, \quad (2)$$

where T is the temperature in the zone of the star where the adiabatic period is determined. The rotation term, $b_{\rm rot}$, discussed in Appendix B, is of order unity. For the DBV stars, $d(\ln R)/dt \ll d(\ln T)/dt$. Hence, when a white dwarf cools in the Mestel fashion, *d(ln P)/dt* is a measure of the cooling rate of the outer layers of the star, and is uniquely determined by the effective temperature. At low effective temperatures, the values of $d(\ln P)/dt$ in the models do follow the expectations based on the simple Mestel cooling model. The values of $d(\ln P)/dt$ are the same to within a factor of 2, and sèem to follow a power law of the same slope.

Uncertainties in the calibration of the mass-radius relationship for white dwarfs, plus the complicating effects of nonideal gas contributions to the equation of state, affect the constant in equation (1). Also, we may employ the rate of change of the effective temperature in equation (2) in this discussion with appropriate adjustment to the factor of $\frac{1}{2}$ that appears there. Since the offset in $d(\ln P)/dt$ between that given by the cool ^{12}C models and that implied by equations (1) and (2) is roughly constant with mass, we can recalibrate the constants of equations (1) and (2) empirically. Setting $T = T_{\rm eff}$ and combining yields

$$\frac{d \ln P}{dt} = 2 \times 10^{-30} A \left(\frac{\mu}{\mu_e^2}\right)^{0.286} \left(\frac{M}{M_\odot}\right)^{-1.190} T_{\rm eff}^{2.857}, \quad (3)$$

which reproduces $d(\ln P)/dt$ quite well for models below $T_{\rm eff} = 24{,}000$ K. In Figure 1*b* we have replotted the numerical results of Figure 1*a* on a logarithmic scale. Also illustrated in Figure 1*b* is the analytically determined value of $d(\ln P)/dt$ (eq. [3]) for pure ^{12}C white dwarfs of 0.40 and 0.60 $M_\odot$.

At higher effective temperatures, the effects of neutrino cooling are more important. As measured by the rates of period change, the 0.60 $M_\odot$ pure carbon model cools more rapidly (along a steeper power law) than does the model with no neutrinos, until $L_\nu/L_\gamma \le 0.2$ ($T_{\rm eff} \sim 25{,}000$ K). This is apparent in Figure 1*b*. Because of the higher core temperature of the stratified model, the energy loss by plasmon and bremsstrahlung neutrino emission was about twice the rate in the pure ^{12}C, 0.60 $M_\odot$ model. Hence, in the stratified model, neutrinos remain important down to lower $T_{\rm eff}$. Stronger neutrino emission leads to a stronger dependence of $d(\ln P)/dt$ on $T_{\rm eff}$ than in the analytic case until the neutrino luminosity drops well below the photon luminosity.

The effects of composition on the cooling rates result in the different behavior of $d(\ln P)/dt$ for the model with a helium-rich envelope. In the model with a helium envelope, most of the thermal energy is released near the outer boundary of the degenerate core. In the temperature range of the DBV stars, this boundary moves outward into a region containing a substantial mass fraction of helium. At a given temperature, the higher heat capacity of helium relative to carbon slows the cooling of these models. Since the period is being formed near this degeneracy boundary, $d(\ln P)/dt$ drops below the value for the pure ^{12}C models. The major reason for this is the factor A in equation (3), which reflects the fact that the ion specific heat decreases with increasing atomic weight (as the number of ions per unit mass decreases). This is why, at the cool end of the DBV sequence, the helium envelope models show smaller values of $d(\ln P)/dt$ than the pure carbon models.

V. DISCUSSION

Because of the sensitivity of $d(\ln P)/dt$ to such quantities as stellar mass, structure, composition, and evolutionary phase, measurement of $d(\ln P)/dt$ will be a valuable probe of the properties of DOV and DBV stars. We note that, in contrast to the DOV stars (Kawaler, Winget, and Hansen 1985), the DBV models evolve at almost constant radius. Thus the radius term in equation (2) is unimportant relative to the cooling term, and the rate of period change is insensitive to the rotation rate.

Within the framework of these preliminary theoretical results, the measurements of $d(\ln P)/dt$ in DBV stars will help greatly in sorting out the remaining uncertainties in the evolutionary models. Observationally, the mass distribution of the white dwarfs is quite narrow, and is centered on 0.6 M$_\odot$ (Weidemann and Koester 1983); the mass distribution of the DB stars is almost the same (Oke, Weidemann, and Koester 1984). Hence, if we know (or assume) a mass of 0.6 $M_\odot$ for the DBV stars, $d(\ln P)/dt$ can be used as an independent temperature estimator. If the composition of the model is roughly correct, then an uncertainty in the measured value of $d(\ln P)/dt$ of 10% translates to a temperature uncertainty of about 3.5%, or about 900 K at 24,000 K.

Alternatively, with the spectroscopically determined value of $T_{\rm eff}$, the observed $d(\ln P)/dt$ will be an indicator of the mass of the star, and of the chemical composition at the degeneracy boundary. For example, consider a DBV at a temperature of 24,000 K, such as GD 358. On the basis of the results for the stratified model, we expect that the value of $d(\ln P)/dt$ should be 7.2×10^{-17} s^{-1}. If the observed period change was measured to be significantly larger than this value ($>10^{-16}$ s^{-1}), then that would strongly suggest a mass smaller than 0.6 $M_\odot$. Conversely, a smaller rate of period change ($<5 \times 10^{-17}$ s^{-1}) would indicate a mass significantly greater than 0.6 $M_\odot$. Whatever analysis is undertaken, we see that knowledge of $d(\ln P)/dt$ will provide a new tool in probing the nature of DB white dwarfs.

We can estimate the time required to observe secular changes of this magnitude using the concept of the $(O-C)$ diagram (cf. Solheim *et al.* 1984; Winget *et al.* 1985). A curve in the $(O-C)$ diagram represents the difference, in seconds, between the computed phase of a light curve and the observed phase as a function of time. This curve can be represented as an expansion about $E=0$,

$$(O-C) = \Delta t_0 + \Delta P_0 E + \tfrac{1}{2} P_0 \dot{P} E^2 + \dots , \tag{4}$$

where P_0 is the pulsation period, E is the number of cycles through which the star pulsates in a baseline time of t, and ΔP_0 is the change in period over the baseline. The first term in equation (4) represents a correction to the time of maximum (or time of zero according to arbitrary normalization). The second term represents a correction to the best period, P_0. If we assume that we have the best fit for t_0 and P_0, then the first two terms in equation (4) are zero. If we further assume that the first and second derivatives of the period with respect to time are small, then we can neglect the higher order terms, and we have

$$(O-C) = \tfrac{1}{2} P_0 \dot{P} E^2 , \tag{5}$$

or, since $E = t/P_0$,

$$t = \left[2\left(\frac{P_0}{\dot{P}}\right)(O-C)\right]^{1/2} , \tag{6}$$

Here $(O-C)$ represents the accumulated delay associated with the period change. To detect a change, clearly the accumulated $(O-C)$ must be larger than the error of measurement. Thus, assuming a typical timing accuracy of 1 s, we can tabulate the time base required to detect a rate of period change, implied by the stratified 0.60 $M_\odot$ model, as a function of effective temperature:

$$t_{30,000\,\rm K} = 2.6 \text{ yr} ,$$

$$t_{25,000\,\rm K} = 4.5 \text{ yr} ,$$

$$t_{20,000\,\rm K} = 6.3 \text{ yr} .$$

We are now in a position to evaluate our prospects for detecting a rate of period change in the two DBV stars currently being monitored. GD 358 has been under observation since its discovery in 1982 May. Although no data were obtained on the object in 1983, high-quality data are available from 1984 to the present. If it is possible to bridge back to the 1982 data, we anticipate that a detection of dP/dt for GD 358 will be made within the next one to three observing seasons, depending on its effective temperature. The second star being monitored, observed since 1984 May, is PG 1351+489 (Winget, Nather, and Hill 1986). The light curve of this star is consistent with only two periods, dominated by a single, large-amplitude peak at 489 s. For this reason, individual timings are intrinsically more accurate than for GD 358, where at least 28 modes are simultaneously present. The decreased timing errors for the large-amplitude peak in PG 1351+489 may compensate for the shorter available time baseline on the object. Hence, PG 1351+489 may become the first single DBV star for which a secular period change will be observed. Finally, we note that to determine dP/dt to an accuracy of 10% requires an accumulated phase shift of about 10 s; this will require baselines 3 times longer than those needed for detection.

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APPENDIX A

In the Mestel theory of white dwarf cooling, the luminosity of the star is derived from leakage of thermal energy of the ions in the interior. Assuming that the heat capacity of the nondegenerate ion gas is much larger than that of the electrons, and that the core is nearly isothermal, it can be shown that

$$L = -\frac{3}{2}\frac{kM}{AH}\frac{\partial T_c}{\partial t} \tag{A1}$$

(Van Horn 1971, eq. [5]). Representing the opacity in the nondegenerate envelope with Kramers's law ($K = K_0 \rho T^{-3.5}$), the equations of envelope structure can be integrated analytically to yield a relationship between the core temperature and photon luminosity (Van Horn 1971, eq. [7]). We can combine that relationship with equation (A1) to produce an equation relating the cooling rate of the core to the core temperature:

$$\frac{\dot{T}_c}{T_c} = -6.2 \times 10^{-36} A\left(\frac{\mu}{\mu_e^2}\right) T_c^{2.5} . \tag{A2}$$

Assuming a mass-radius relationship for white dwarfs of the form

$$R = 7 \times 10^8 \left(\frac{M}{M_\odot}\right)^{-1/3}, \tag{A3}$$

and noting that $L = 4\pi R^2 \sigma T^4$, we can use equation (7) of Van Horn (1971) to relate the core temperature to the effective temperature:

$$T_c = 240 \left(\frac{\mu}{\mu_e^2}\right)^{2/7} \left(\frac{M}{M_\odot}\right)^{-10/21} T_{\rm eff}^{8/7} . \tag{A4}$$

Combining equations (A2) and (A4), we obtain an expression for the core cooling rate as a function of effective temperature:

$$\frac{\dot{T}_c}{T_c} = -5.8 \times 10^{-30} A \left(\frac{\mu}{\mu_e^2}\right)^{2/7} \left(\frac{M}{M_\odot}\right)^{-25/21} T_{\rm eff}^{20/7} . \tag{A5}$$

Finally, by differentiating equation (A4) and substituting for $d(\ln T_c)/dt$ in equation (A5), we have the desired expression for the cooling of the stellar photosphere as a function of effective temperature (eq. [1]):

$$\frac{\dot{T}_{\rm eff}}{T_{\rm eff}} = -5 \times 10^{-30} A \left(\frac{\mu}{\mu_e^2}\right)^{0.286} \left(\frac{M}{M_\odot}\right)^{-1.190} T_{\rm eff}^{2.857} . \tag{A6}$$

APPENDIX B

For a slowly rotating star which conserves angular momentum, an expression for the rate of period change is

$$\frac{d \ln P_{\rm obs}}{dt} = \frac{d \ln P_0}{dt} - m \frac{P_0}{P_{\rm rot}} \left[(1 - C)\frac{\dot{I}}{I} + \dot{C}\right] \tag{B1}$$

for the case of uniform rotation (eq. [4] of Kawaler, Winget, and Hansen 1985). In equation (B1), $d(\ln P_{\rm obs})/dt$ is the rate of period change in the observer's frame, and $d(\ln P_0)/dt$ is the value in the rotating frame of the star. For white dwarfs, the rotation coefficient C is approximately $[l(l + 1)]^{-1}$ (Brickhill 1975; Kawaler, Winget, and Hansen 1985). since the moment of inertia for a uniform sphere is proportional to R^2, we have as an upper limit to the rate of change of I for a white dwarf,

$$\frac{d \ln I}{dt} \leq 2 \frac{d \ln R}{dt} . \tag{B2}$$

With the above relations, equation (B1) becomes

$$\frac{d \ln P_{\rm obs}}{dt} = \frac{d \ln P_0}{dt} - m \frac{P_0}{P_{\rm rot}} \left\{\left[2 - \frac{2}{l(l + 1)}\right] \frac{d \ln R}{dt}\right\} . \tag{B3}$$

The term in equation (B3) that rises from the rotation of the star is proportional to $d(\ln R)/dt$, and can be identified as the term $b_{\rm rot}$ in equation (2).

The value of m may be any integer between $-l$ and l, or zero. Hence,

$$|b_{\rm rot}| \leq 2 \left(\frac{l^2 + l - 1}{l + 1}\right) \tag{B4}$$

represents an upper limit to the magnitude of the rotation term $b_{\rm rot}$ in equation (2).

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