General-relativistic equations of binary motion for extended bodies, with conservative corrections and radiation damping

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Equations of motion are derived in the weak-field, slow-motion approximation for a binary system whose components are extended and spherically symmetric. The post-Newtonian approximation (PNA) method is used; the equations include all the conservative pre-radiative terms as well as the first $[\sim(v/c)^5]$ relativistic correction for radiation damping. Analysis of the residual terms yields constraints on the small PNA parameters such that the equations of motion will remain applicable. Each binary component is characterized by a single parameter, the Tolman mass. Since the equations hold true for bodies of arbitrary compactness, they may be applied even to binary neutron stars or black holes.

1. INTRODUCTION

A complete solution to the problem of two bodies moving in mutual orbit, including the conservative correction terms and the gravitational-wave reaction, would be useful in practice and of fundamental value. In the wake of the 1974 discovery[1] of PSR 1913 + 16, a pulsar belonging to a binary system in which relativistic effects are particularly strong (the ratio $v/c \sim 10^{-3}$, where v is the characteristic orbital velocity and c is the speed of light), many investigators have been working on this subject.

Among the results have been the first neutron-star mass determination,[2,3] measurements of radio-signal delay in a gravitational field[3] (an effect hitherto observed only through radar experiments in the solar system), and the discovery that the PSR 1913 + 16 orbital period is steadily growing shorter,[3] presumably because of energy loss to gravitational waves. This last conclusion relies on the agreement between the pulsar observations and theoretical calculations of how the orbit parameters should evolve, as based on the expressions for the gravitational-wave flux[4] or on an extrapolation of the wave solution to the near zone.[5,6]

These methods, however, are incapable of fully describing the binary motion, since they wholly ignore the conservative relativistic corrections, and even in the radiative approximation they provide information only on the secular terms. Furthermore, the approaches previously adopted all involve extra assumptions, which often are open to doubt. What is needed here is a method for deriving the equations of motion of isolated bodies directly from the Einstein equations.

Three different procedures are currently available for obtaining the equations of motion: the Einstein–Infeld–Hoffman (EIH) method,[7] the matched asymptotic-expansion approach,[8–10] and the post-Newtonian approximation (PNA) introduced by Fok.[11] Systematic expositions of the PNA method will be found in the literature (Refs. 12-14). This method has several drawbacks; in particular, divergent integrals come into play[15] because the near-zone metric does not have an analytic dependence on the small parameter occurring in the expansion.[16] A suitable expansion with respect to the parameter can eliminate the divergent terms.[17]

In this paper, the author extends recent PNA analysis of the equations of motion of isolated bodies,[18] originally undertaken in order to obtain the radiation damping force and to integrate the equations of motion including terms of the order $(v/c)^5$. Equations of motion will be derived for a pair of spherically symmetric extended bodies, with full allowance for all the conservative pre-radiative $[\sim (v/c)^2, (v/c)^4]$ terms as well as the first $[\sim(v/c)^5]$ correction for braking by gravitational radiation. The calculations presuppose slow motions and a weak field. Estimates will be given for the residuals, and constraints will be placed on the parameters of binary systems to which our equations of motion are applicable. We shall find that the equations depend neither on the internal structure nor the compactness of the orbiting bodies; the sole parameter that enters is the mass. This is an important result, for it suggests that the equations of motion we shall obtain can be applied to study the behavior of objects ranging all the way from conventional stars and planets to black holes.

The PNA method employed in this investigation has certain advantages over the EIH and asymptotic-matching methods, since it does enable us to deduce the metric and the equations of motion directly from the Einstein equations. Computational difficulties are a notorious feature of the EIH method: one has to invoke various regularization techniques[19–22] in order to make meaningful any functions that become infinite along the world lines. As yet it is unclear whether or not these regularization procedures yield the same equations of motion in the $(v/c)^4$ and $(v/c)^5$ approximations.[23] Still, despite its shortcomings the EIH approach (as well as the asymptotic-matching method) is probably superior to the PNA method in accurately describing the motion of black holes.[8]

The paper is organized as follows. In Sec. 2 we shall outline the principles of the PNA method and shall specify what assumptions and limitations on the binary-system parameters are entailed by our premise that the bodies are spherically symmetric and nonrotating, and by our neglect of terms in the equations of motion that are of the same order as the ratio of the bodies' diameter to the size of the whole system. In Sec. 3 we determine the center of inertia of each body as well as its velocity and acceleration. The virial theorem will be proved to order $(v/c)^2$ for the case of two isolated bodies, and its implications will be set forth. We have also thought it advisable to define and to evaluate in Sec. 3 the Tolman mass, for this is the sole parameter that appears in the equations of motion.

In Sec. 4 a procedure will be described for establishing the equations of motion. To unify the picture and to make it complete we give explicitly the results obtained in the zeroth-order post-Newtonian approximation [$\sim(v/c)^0$, designated PNA 0] and in PNA 1 [$\sim(v/c)^2$]. Corrections to the equations of motion for PNA 2 [$\sim(v/c)^4$] are calculated in Sec. 5, and the explicit form of the force is obtained. Section 6 presents a derivation of the radiation damping force, generalizing the expression given in the latter by Frishchuk and the author[18] to the case of unbounded orbits.

2. POST-NEWTONIAN APPROXIMATION

Let the matter energy-momentum tensor have the form appropriate to an ideal fluid[1)]:

$$T^{\mu\nu}=(\rho c^2+p)u^\mu u^\nu-pg^{\mu\nu} \tag{2.1}$$

$$\rho=\mu(1+\Pi/c^2), \tag{2.2}$$

where μ denotes the energy density in a comoving reference frame, the isotropic pressure p is related to μ by the equation of state $p = p(\mu)$, $u^\mu = dx^\mu/ds$ represents the 4-velocity of a fluid element, and the internal energy Π of that element is related to p, μ by the equation

$$d\Pi+pd\left(\frac{1}{\mu}\right)=0. \tag{2.3}$$

Now introduce the auxiliary quantities

$$\gamma^{\mu\nu}\equiv\sqrt{-g}g^{\mu\nu}-\eta^{\mu\nu}, \tag{2.4}$$

where $g^{\mu\nu}$ is the metric tensor and $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ represents the Minkowski metric. The Einstein equations are solvable in a harmonic coordinate system, as specified by the condition

$$\gamma^{\mu\nu}_{,\nu}=0 \tag{2.5}$$

In this coordinate system the Einstein equations take the form[14]

$$\Box\gamma^{\mu\nu}=\frac{16\pi G}{c^4}\Lambda^{\mu\nu}, \tag{2.6}$$

where $\Lambda^{\mu\nu}=\theta^{\mu\nu}-\dfrac{c^4}{16\pi G}(\gamma^{\mu\nu}\gamma^{\alpha\beta}-\gamma^{\mu\alpha}\gamma^{\nu\beta})_{,\alpha\beta}$,

$$\theta^{\mu\nu}=(-g)(T^{\mu\nu}+t^{\mu\nu}),\quad \Box\equiv\eta^{\mu\nu}\partial_\mu\partial_\nu,$$

$t^{\mu\nu}$ representing the Landau–Lifshitz pseudotensor.[24]

Equation (2.6) may be written as an integral equation: $\gamma^{\mu\nu} = (16\pi G/c^4)\Box^{-1}\Lambda^{\mu\nu}$, where $\Box^{-1}$ signifies the retarded integral operator in Minkowski space. Upon expanding the retarded time with respect to the small parameter v/c, we obtain the following formal integro-differential equation from this integral equation:

$$\gamma^{\mu\nu}(x,t)=\frac{4G}{c^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}I_{n-1}(\partial_t{}^n\Lambda^{\mu\nu}), \tag{2.7}$$

$$I_n(f)\equiv\int_{R^3}d^3y f(y,t)\,|x-y|^n. \tag{2.8}$$

In Eq. (2.7) the time derivatives have been left beneath the integral sign, as suggested by Ehlers.[25] As far as the $(v/c)^6$ approximation, this practice does not introduce any divergent integrals.[17,26]

Commencing our evaluation of the metric with the quantities $\overset{(0)}{\gamma}{}^{\mu\nu} = 0$ and performing three successive iterations in Eq. (2.7),

$$g_{00}=1-\frac{2U}{c^2}+\frac{1}{c^4}(2U^2-4\Phi+\chi_{,tt})+\frac{1}{c^5}\overset{(5)}{A}_{00}+\frac{1}{c^6}\underset{6}{g_{00}}+\frac{1}{c^7}\underset{7}{g_{00}}+o(c^{-8}), \tag{2.9}$$

$$g_{0i}=\frac{4W_i}{c^3}+\frac{1}{c^5}\overset{(5)}{A}_{0i}+\frac{1}{c^6}\overset{(6)}{A}_{0i}+o(c^{-7}), \tag{2.10}$$

$$g_{ik}=-\left(1+\frac{2U}{c^2}\right)\delta_{ik}+\frac{1}{c^4}\overset{(4)}{A}_{ik}+\frac{1}{c^5}\overset{(5)}{A}_{ik}+o(c^{-6}), \tag{2.11}$$

where $U = GI_{-1}(\rho)$, $\Phi = GI_{-1}(\rho v^2 + \rho\bar{U} + {}^3/_2 p)$,

$$\chi=\frac{1}{2\pi}I_{-1}(U),\qquad W_i=GI_{-1}(\rho v_i), \tag{2.12}$$

$$\overset{(5)}{A}_{00}=\frac{4G}{3}\frac{d^3I_{kk}}{dt^3},\qquad \overset{(6)}{A}_{0i}=\frac{2G}{3}x^k\frac{d^4I_{ki}}{dt^4}-\frac{2G}{3}\frac{d^3}{dt^3}\int\rho v_i x^2\,d^3x,$$

$$\overset{(5)}{A}_{ik}=2G\frac{d^3I_{ik}}{dt^3}-\frac{2G}{3}\delta_{ik}\frac{d^3I_{kk}}{dt^3},\qquad I_{ik}=\int_{R^3}\rho x^i x^k\,d^3x,$$

$$v^i=cu^i/u^0. \tag{2.13}$$

Anderson and Decanio[14] give expressions for the other tensor components. Equations (2.9)–(2.11), however, do not yet constitute solutions of the Einstein relations; for that purpose we have to substitute the metric (2.9)–(2.11) into the equations of motion and to determine $\rho(x, t)$, $v^i(x, t)$ as functions of time.

The equations of motion are

$$0=T^{\mu\nu}{}_{;\nu}=T^{\mu\nu}{}_{,\nu}+\Gamma_{\alpha\beta}{}^{\mu}T^{\alpha\beta}+\Gamma_{\beta\alpha}{}^{\alpha}T^{\mu\beta}. \tag{2.14}$$

When the Christoffel symbols and the energy-momentum tensor components are introduced, Eqs. (2.14) become (Refs. 12, 13, 27, and 28)

$$\rho_{,t}=-(\rho v_k)_{,k}+\frac{1}{c^2}k+\frac{1}{c^4}q+\frac{1}{c^5}s+o(c^{-6}), \tag{2.15}$$

$$(\rho v_i)_{,t}=-\vartheta_{ij,j}+\frac{1}{c^2}k_i+\frac{1}{c^4}q_i+\frac{1}{c^5}s_i+o(c^{-6}), \tag{2.16}$$

$$\vartheta_{ij}=\rho v_i v_j+p\sigma_{ij}+\frac{1}{8\pi G}[2U_{,i}U_{,j}-\delta_{ij}(\nabla U)^2]. \tag{2.17}$$

The quantities k, q, s, k_i, q_i, s_i are functionals of ρ and the v^k and their derivatives.

We shall be applying Eqs. (2.9)–(2.17) to trace the motion of two isolated bodies that form a binary system, have a similar mass M and characteristic size R, and are separated by a distance L. Accordingly, along with the original parameter $\varepsilon \equiv \varepsilon_1 \sim v/c \sim (GM/c^2L)^{1/2} \ll 1$, the problem will contain another small characteristic quantity, the "compactness" parameter $\varepsilon_2 \sim (GM/c^2R)^{1/2} \ll 1$, which measures how closely the size of the body approximates its gravitational radius $R_g = 2GM/c^2$. We shall assume that $\delta \equiv R/L \ll 1$; consequently $\varepsilon_1 \ll \varepsilon_2$. The parameter δ describes how nearly the two bodies resemble point masses.

To facilitate the calculations we should like to treat the bodies as spherically symmetric, and nonrotating in a comoving reference frame. Whether this simplification is justified has to be decided by special arguments: mutual tides will distort the spherical shape of the bodies, imposing a further constraint on the parameter δ, already bounded below by the value $R_g/L \sim \varepsilon_1^2$; and secondly, one cannot regard the angular rotation speed as arbitrarily small, for it must be at least as large[29] as $\nu \sim \varepsilon_1^2(GM/L^3)^{1/2}$.

As for the tides,[30] the distortion

$$\left(\frac{\Delta R}{R}\right)_{\rm tid} \sim \varkappa\left(\frac{R}{L}\right)^3 \tag{2.18}$$

in the contour of the bodies will produce a tidal force which in the PNA 0 case is of order

$$F_{\rm tid} \sim \varkappa\left(\frac{R}{L}\right)^5 \frac{GM^2}{L^2}, \tag{2.19}$$

the Love number $\varkappa$ depending on the density distribution inside the bodies[30] ($\varkappa \approx 0.001$-0.1 for conventional stars). In order for a spherically symmetric configuration not to be tidally distorted, this tidal force should be much smaller than the radiation force, which may be estimated from Eq. (2.16):

$$F_{\rm rad} \sim \left(\frac{v}{c}\right)^5 \frac{GM^2}{L^2}, \tag{2.20}$$

The requirement that $F_{\rm tid} \ll F_{\rm rad}$ therefore imposes on δ the additional constraint

$$\delta \ll \varkappa^{-1/5}\frac{v}{c}, \tag{2.21}$$

which is consistent with our previous lower bound on δ.

The existence of a minimum angular rotation speed ν will also cause the bodies to depart from spherical symmetry[30]:

$$\left(\frac{\Delta R}{R}\right)_{\rm rot} \sim \frac{\nu^2 R^3}{GM} \sim \left(\frac{v}{c}\right)^2\left(\frac{R}{L}\right)^3. \tag{2.22}$$

In PNA 0 the corresponding force is of order[30]

$$F_{\rm rot} \sim \left(\frac{\Delta R}{R}\right)_{\rm rot}\left(\frac{R}{L}\right)^2 \frac{GM^2}{L^2}. \tag{2.23}$$

From the estimates (2.21), (2.22) one can readily show that $F_{\rm curl} \ll F_{\rm rad}$. Furthermore, the angular velocity ν will generate a force[29] $F \sim (\nu R/cL)^2 GM^2$ in the PNA 1 case, and an estimate for ν indicates that $F \ll F_{\rm rad}$.

Thus if the inequality (2.21) is satisfied, we may regard the bodies as retaining spherical symmetry to the requisite accuracy, and as not rotating. But these properties will not ensure that the bodies may be treated as mass points – that terms of order R/L or higher in the equations may be ignored. Actually the multipole expansions of the metric coefficients in the equations of motion will produce extra terms of order $\varepsilon_1^2\delta^2 GM^2/L^2$ in PNA 1 and $\sim \varepsilon_1^4\delta GM^2/L^2$ in PNA 2. The condition (2.21) is not enough to make these terms small compared with the radiation force $F_{\rm rad}$; for that purpose we need the stronger inequality

$$\delta \ll \left(\frac{v}{c}\right)^{4/3}. \tag{2.24}$$

We shall in fact assume that the parameter δ satisfies this last inequality, and accordingly we shall ignore all terms in the equations of motion that are $\sim$ (R/L) or higher.

Equations describing the motion of the bodies as a whole can now be obtained from Eq. (2.16) by integrating over the spherically symmetric volumes which the nonrotating bodies occupy. In so doing, we regard the coordinates x^1, x^2, x^3 as Cartesian, since the background spacetime is taken to be flat. In this background spacetime we specify 10 potentials $\gamma^{\mu\nu}$ related by the gauge conditions (2.5), a density ρ(x, t) and three velocity components v^i(x, t). These quantities are determined by Eqs. (2.6) and (2.14). Such an approach to relativity theory is more reminiscent of ordinary celestial mechanics than of field theory. Grishchuk et al.[31] have recently developed the methods systematically, generalizing them to the case of a curved spacetime.

3. TOLMAN MASS, CENTER OF INERTIA, VIRIAL THEOREM

Henceforth an important role in our discussion will be played by the Tolman mass, for in deriving the equations of motion we find it arising in a natural way as the sole parameter that describes a member of the binary system. Let us therefore briefly review its properties.

The Tolman mass of an isolated static source is defined by the expression[24]

$$M = \frac{2}{c^2}\int_{R^3} d^3x\sqrt{-g_s}\left(g_{00}{}^s T_s{}^{00} - \frac{1}{2}T_{s\alpha}{}^{\alpha}\right), \tag{3.1}$$

where s stands for "static." Substituting into Eq. (3.1) the components (2.9)-(2.11) and (2.1) of the metric and energy-momentum tensors, we obtain

$$M = \int d^3x\left[\rho + \frac{1}{c^2}(2\rho U + 3p) + \frac{1}{c^4}7\rho U^2\right] + o(c^{-6}). \tag{3.2}$$

In place of the density ρ we will later find it convenient to use the invariant density $\rho^* = \mu u^0\sqrt{-g}$. The density ρ^* satisfies the exact equation of continuity[11]

$$\rho^*_{,t} + (\rho^* v_k)_{,k} = 0. \tag{3.3}$$

Furthermore, for any sufficiently smooth function f(x, t) we may write the equation

$$\frac{\partial}{\partial t}\int_{(a)} \rho^* f(x,t)\, d^3x = \int_{(a)} \rho^* \frac{df}{\partial t} d^3x, \tag{3.4}$$

where $d/dt = \partial/\partial t + v^k(\partial/\partial x^k)$, while (a) means that the integral extends over the volume of body a. The properties of ρ^* make it helpful in calculations, so we shall convert all ρ-containing quantities to ρ^*, using the expression[29]

$$\rho = \rho^*\left(1 + \frac{\Pi}{c^2}\right)(u^0\sqrt{-g})^{-1} = \rho^*\left\{1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U - \Pi\right) - \frac{1}{c^4}\left(\frac{1}{8}v^4 + \frac{1}{2}v^2 U - \frac{15}{2}U^2 - 4v^k W_k - \frac{1}{2}A^{(4)}_{kk}\right)\right\} + o(c^{-6}). \tag{3.5}$$

Here the quantities

$$\begin{aligned} U&=\underset{0}{U}+c^{-2}\underset{2}{U}+c^{-4}\underset{4}{U}+o(c^{-6}),\\ W_i&=\underset{0}{W}+c^{-2}\underset{2}{W_i}+o(c^{-4}),\\ \Phi&=\underset{0}{\Phi}+c^{-2}\underset{2}{\Phi}+o(c^{-4}),\\ \chi&=\underset{0}{\chi}+c^{-2}\underset{2}{\chi}+o(c^{-4}),\\ \Pi&=\underset{0}{\Pi}+c^{-2}\underset{2}{\Pi}+o(c^{-4}), \end{aligned} \tag{3.6}$$

where

$$\underset{0}{\Pi}=\Pi(\rho^*),\ \text{a}\ \underset{2}{\Pi}=-\left(\frac{1}{2}v^2+3\underset{0}{U}\right)p/\rho^*\ [29].$$

The remaining terms in the expansions (3.6) can easily be derived from Eqs. (2.12) and (3.5).

We now define the rest mass of body a by the expression[11]

$$m_a=\int\limits_{(a)}\rho^*\,d^3x. \tag{3.7}$$

Since Eq. (3.4) holds, the mass m_a remains constant throughout the motion. We can therefore define the center of inertia $x_a{}^i$ of body a by means of the formula

$$m_a x_a{}^i=\int\limits_{(a)}\rho^* x^i d^3x. \tag{3.8}$$

On differentiating with respect to time and applying Eq. (3.4), we readily obtain the velocity $v_a{}^i$ and acceleration $a_a{}^i$ of the center of inertia:

$$m_a v_a{}^i=\int\limits_{(a)}\rho^* v^i d^3x, \tag{3.9}$$

$$m_a a_a{}^i=\int\limits_{(a)}\rho^*\frac{dv^i}{dt}d^3x, \tag{3.10}$$

Note that the velocity of any fluid element of a may be written as $v^i = v_a{}^i + \bar{v}^i$, where $v_a{}^i$ is the velocity of the center of inertia of a [Eq. (3.9)] and $\bar{v}^i$ denotes the velocity of a fluid element relative to the center of inertia. A direct calculation indicates that since we can neglect the effects of tidal and rotational forces on the orbital motion, we need not include the relative velocity $\bar{v}^i$ in the calculations, but may set $v^i = v_a{}^i$. Accordingly v^i may be taken outside the integration sign.

We now proceed to establish the virial theorem to terms of order $(v/c)^2$. For this purpose we use Eq. (3.5) to replace the density ρ in Eq. (2.16) by ρ^*, and we multiply that equation scalarly by the quantity $r_a{}^i = x^i - x_a{}^i$. We then integrate the result over the volume of body a. In view of the constraints on the binary-system parameters we obtain

$$\begin{aligned} &\delta_{ik}I_0{}^a(p)+\frac{1}{8\pi G}\int\limits_{(a)}[2\underset{0}{U}_{a,i}\underset{0}{U}_{a,k}-\delta_{ik}(\nabla_0 U_a)^2]d^3x\\ &+\frac{1}{c^2}\Big\{\delta_{ik}I_0{}^a(p\underset{0}{U}_a)+\frac{1}{3}\delta_{ik}I_0{}^a(\rho^*\underset{0}{U}_a{}^2)-\frac{1}{3}\delta_{ik}I_0{}^a(\rho^*\underset{0}{\Pi}\underset{0}{U}_a)\\ &+\frac{2}{3}\delta_{ik}\frac{Gm_b}{r}I_0{}^a(\rho^*\underset{0}{U}_a)+\frac{2}{15}\delta_{ik}v_a{}^2 I_0{}^a(\rho^*\underset{0}{U}_a)-\frac{1}{15}v_a{}^i v_a{}^k I_0{}^a(\rho^*\underset{0}{U}_a)\Big\}, \end{aligned} \tag{3.11}$$

where $r\equiv(r_k r^k)^{1/2}$, $r^k\equiv r_{ab}{}^k=x_a{}^k-x_b{}^k$, $I_0{}^a(f)\equiv\int\limits_{(a)} f d^3x$.

If we contract over the spatial indices in Eq. (3.11) we find

$$\begin{aligned} &\frac{1}{8\pi G}\int\limits_{(a)}(\nabla\underset{0}{U}_a)^2 d^3x=\frac{1}{2}I_0{}^a(\rho^*\underset{0}{U}_a)=3I_0{}^a(p)+\frac{1}{c^2}\Big\{3I_0{}^a(p\underset{0}{U}_a)\\ &+I_0{}^a(\rho^*\underset{0}{U}_a{}^2)-I_0{}^a(\rho^*\underset{0}{\Pi}\underset{0}{U}_a)+2\frac{Gm_b}{r}I_0{}^a(\rho^*\underset{0}{U}_a)+\frac{1}{3}v_a{}^2 I_0{}^a(\rho^*\underset{0}{U}_a)\Big\}. \end{aligned} \tag{3.12}$$

On differentiating Eqs. (3.11), (3.12) with respect to time, we arrive at the relations

$$\begin{aligned} \frac{\partial}{\partial t}I_0{}^a(p)=\frac{1}{c^2}\Big[&\frac{2}{3}\frac{Gm_b}{r^3}(r^k v_a{}^k)\\ &-\frac{2}{3}\frac{Gm_b}{r^3}(r^k v_b{}^k)-\frac{2}{9}(v_a{}^k a_a{}^k)\Big]I_0{}^a(\rho^*\underset{0}{U}_a), \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{4\pi G}\int\limits_{(a)}\underset{0}{U}_{a,i}\underset{0}{U}_{a,k}d^3x\right)=\frac{1}{c^2}\Big[&\frac{1}{15}a_a{}^i v_a{}^k\\ &+\frac{1}{15}v_a{}^i a_a{}^k-\frac{2}{45}\delta_{ik}(v_a{}^l a_a{}^l)\Big]I_0{}^a(\rho^*\underset{0}{U}_a). \end{aligned} \tag{3.14}$$

Equations (3.11)-(3.14) will be used in the next section, when we perform PNA 1 and PNA 2 calculations.

When Eq. (3.2) for the Tolman mass is reexpressed in terms of ρ^* by means of the static virial theorem, we obtain

$$\begin{aligned} M=\int d^3x\Big\{&\rho^*+\frac{1}{c^2}\left(\rho^*\underset{0}{\Pi}-\frac{1}{2}\rho^*\underset{0}{U}\right)\\ &+\frac{1}{c^4}\left(\frac{1}{2}\rho^*\underset{0}{U}^2-\rho^*\underset{0}{\Pi}\underset{0}{U}+\rho^*\underset{2}{\Pi}_s\right)\Big\}+o(c^{-6}) \end{aligned} \tag{3.15}$$

As will be shown below, the expression (3.15) follows directly from the equations of motion, representing the sole parameter that characterizes a given body. Direct computation indicates that the mass M_a, M_b will remain constant throughout the orbital motion – a corollary of Eqs. (2.16), (3.4) and the constraints on the binary-system parameters.

4. NEWTONIAN AND POST-NEWTONIAN EQUATIONS OF MOTION

Although this section presents few new results, it is needed to round out the picture and to illustrate the procedure for deriving the equations of motion.

We begin by substituting ρ^* for the ρ in Eqs. (2.16), using Eqs. (3.5), (3.6). As a result we obtain

$$(\rho^* v^i)_{,t}+(\rho^* v^i v^k)_{,k}+p_{,i}-\rho^*\underset{0}{U}_{,i}+\frac{1}{c^2}k_i{}^*+\frac{1}{c^4}q_i{}^*+\frac{1}{c^5}s_i{}^*=o(c^{-6}), \tag{4.1}$$

where the c^{-n} terms ($n = 0, 2, 4, 5$) combine small quantities of the same order, the coefficients $k_i{}^*$, $q_i{}^*$, $s_i{}^*$ being functionals of ρ^*, the v^i, and their derivatives.

To establish the equations of motion for the center of inertia of body a, we integrate Eqs. (4.1) over the volume of a. In doing so we have to distinguish between the inner part of the potentials U, W_i, Φ, χ (the part due to the body a itself) and the outer part[11,29] (resulting from body b). We break down the outer part of the potentials into multipole series; in view of the constraint (2.24) on the parameter δ, we retain in the expansions just as many terms as are needed to ensure that the equations of motion gain no terms of order R/L or higher.

Next we evaluate all the integrals explicitly, using the definitions (3.7)–(3.10), Eq. (3.4), and various transformations of the integrands, to be described below. Some of these transformations give rise to additional expansions in inverse powers of c. We group the terms occurring in those expansions together with the terms of corresponding order in Eqs. (4.1).

Once the integrals have been calculated we can reduce the terms of this kind and use the virial theorem [Eq. (3.12)] to transform the terms that contain the integral $I_0(p)$. As a result we find that although static terms, which describe the bodies' internal structure and depend on the compactness parameter ε_2, do occur in the equations of motion in various combinations (depending on what analytic coefficient contains them), they are so grouped that they can be written as products of the bodies' Tolman masses. Thus the Tolman mass emerges naturally in the equations of motion through direct calculation, making it the natural parameter to characterize the binary motion.

The equations obtained in this manner demonstrate that the relativistic corrections to the Newtonian equations of motion depend not only on the coordinates of the bodies' centers of inertia but also on their derivatives, up to the fifth order. We can lower the order of the higher derivatives by using the post-Newtonian equations of motion in the $(v/c)^2$ terms and the Newtonian equations in the $(v/c)^4$, $(v/c)^5$ terms. With this step we will have completed our derivation of the equations of motion.

Now let us examine the calculation process more carefully. Integration of the Newtonian terms in Eqs. (4.1) yields the Newtonian equations of motion for the bodies' centers of inertia:

$$m_a a_a^{\,i} = -\frac{Gm_a m_b}{r^3} r^i. \tag{4.2}$$

The integral of the $p_{,i}$ will vanish because the pressure is zero on the surface of the bodies, while the integral of the inner part of the $\rho^* \underset{0}{U}_{,i}$ term will vanish due to the antisymmetry of the integrand.

For the $k_i{}^*$ term, integration yields the equations of motion in the first post-Newtonian approximation. In performing the integration we transform the $\int\limits_{(a)} p\underset{0}{U}_{,i} d^3x$ term occurring in $k_i{}^*$ according to the relation[28]

$$\int\limits_{(a)} p\underset{0}{U}_{,i} d^3x = -3\int\limits_{(a)} p\underset{0}{U}_{,i} d^3x - 4\int\limits_{(a)} p_{,i}\underset{0}{U} d^3x, \tag{4.3}$$

the $p_{,i}$ term then being replaced by means of Eq. (4.1), carried to order $(v/c)^2$. The extra term $4/c^2 \int\limits_{(a)} k_i{}^* \underset{0}{U} d^3x$ that arises in this way appears in the PNA 2 equations of motion. The transformation (4.3) serves to reduce the static terms in the PNA 1, 2 equations of motion to a form symmetric with regard to interchange of the indices a, b. As a consequence, all the integrals that depend only on the inner parts of the potentials U, W_i, Φ, χ will cancel out in the corresponding approximations.

After having evaluated the integrals of the $k_i{}^*$, we can use the virial theorem (3.12) to reduce the static terms (describing the bodies' internal structure) to the form

$$\begin{aligned}
&\frac{G}{c^2}\left[m_a I_0{}^b\left(\rho^*\underset{0}{\Pi} - \frac{1}{2}\rho^*\underset{0}{U}_b\right) + m_b I_0{}^a\left(\rho^*\underset{0}{\Pi} - \frac{1}{2}\rho^*\underset{0}{U}_a\right)\right]\frac{r_i}{r^3} \\
&\quad - \frac{1}{c^4}\left[\frac{Gm_a}{r^3} r^i I_0{}^b (3p\underset{0}{U}_b + \rho^*\underset{0}{\Pi}\underset{0}{U}_b + \rho^*\underset{0}{U}_b{}^2) + \left(\frac{2G^2 m_a{}^2}{r^4} r^i \right.\right. \\
&\qquad \left.\left. + \frac{1}{3}\frac{Gm_a}{r^3} v_a{}^2 r^i\right) I_0{}^b(\rho^*\underset{0}{U}_b) + (a \leftrightarrow b)\right],
\end{aligned} \tag{4.4}$$

where the symbol $(a \leftrightarrow b)$ designates terms of order $(v/c)^4$ obtained from the preceding terms by interchanging a and b. Upon applying Eqs. (3.12)–(3.14) we find that the dynamic terms depending on the bodies' internal structure become

$$\frac{1}{c^2} a_a{}^i I_0{}^a\left(\rho^*\underset{0}{\Pi} - \frac{1}{2}\rho^*\underset{0}{U}_a\right) + \frac{1}{c^4}\left[\frac{2}{45} v_a{}^i (v_a{}^k a_a{}^k) - \frac{4}{45} v_a{}^2 a_a{}^i\right] I_0{}^a(\rho^*\underset{0}{U}_a). \tag{4.5}$$

The $(v/c)^4$ terms in Eqs. (4.4), (4.5) enter into the PNA 2 equations.

By combining the Newtonian terms in Eq. (4.2) with the post-Newtonian terms in Eqs. (4.4), (4.5) we can refine the value of the mass occurring in the PNA 1 equations of motion, by carrying the Tolman-mass definition (3.15) to terms of order $(v/c)^2$, inclusive. As a result the compactness parameter will be eliminated from the PNA 1 equations. After the order of the second derivatives has been lowered, the PNA 0, 1 equations of motion will take the form[7,9,11,19,20–22,29]

$$M_a a_a{}^i = -\frac{GM_a M_b}{r^3} r^i + \frac{1}{c^2} \underset{(2)}{F}_a{}^i (r^k, v_a{}^k, v_b{}^k), \tag{4.6}$$

$$\begin{aligned}
\underset{(2)}{F}_a{}^i = \frac{Gm_a m_b}{r^2}\Bigg[&\left(4v_a{}^k v_b{}^k - v_a{}^2 - 2v_b{}^2 + \frac{3}{2}(n^k v_b{}^k)^2 \right. \\
&\left. + 5\frac{Gm_a}{r} + 4\frac{Gm_b}{r}\right) n^i + (4(n^k v_a{}^k) - 3(n^k v_b{}^k))(v_a{}^i - v_b{}^i)\Bigg],
\end{aligned} \tag{4.7}$$

where $n^i \equiv r^i/r$.

5. POST-POST-NEWTONIAN EQUATIONS OF MOTION

To derive the equations of motion in the post-post-Newtonian approximation is a most complicated job: not only do the equations contain more terms, but one has to evaluate the nonlinear terms that describe the self-action of the gravitational field.

The calculations for the PNA 2 case are carried out in much the same way as for PNA 0 and 1. In PNA 2 we may consider that

$$\underset{0}{U}_a = 4\pi c \int\limits_{r_a}^{\infty} \rho^* \xi\, d\xi + \frac{Gm_a(r_a)}{r_a}, \quad m_a(r_a) = 4\pi \int\limits_0^{r_a} \rho^* \xi^2 d\xi, \tag{5.1}$$

$$\underset{0}{W}_a{}^i = 4\pi G v_a{}^i \int\limits_{r_a}^{\infty} \rho^* \xi\, d\xi + \frac{Gm_a(r_a)}{r_a} v_a{}^i, \tag{5.2}$$

$$\underset{0}{\chi_a} = -Gm_a(r_a)r_a - 4\pi G\int_{r_a}^{\infty}\rho^*\xi^3\,d\xi$$
$$-\frac{4\pi G}{3}\frac{1}{r_a}\int_0^{r_a}\rho^*\xi^4\,d\xi - \frac{4\pi G}{3}r_a^2\int_{r_a}^{\infty}\rho^*\xi\,d\xi \tag{5.3}$$

and that the relations

$$\int_{(a)}\zeta(r_a)r_a^i\,d^3x = \int_a \zeta(r_a)r_a^i r_a^k r_a^n\,d^3x = 0, \tag{5.4}$$

$$\int_{(a)}\zeta(r_a)r_a^i r_a^k\,d^3x = \frac{1}{3}\delta_{ik}\int_{(a)}\zeta(r_a)r_a^2\,d^3x, \tag{5.5}$$

$$\int_a \zeta(r_a)r_a^i r_a^k r_a^n r_a^l\,d^3x = \frac{1}{15}(\delta_{ik}\delta_{nl}+\delta_{in}\delta_{kl}+\delta_{il}\delta_{kn})\int_{(a)}\zeta(r_a)r_a^4\,d^3x \tag{5.6}$$

are satisfied, where $\zeta(r_a)$ represents a certain smooth function depending on r_a alone. Incidentally, in the preceding approximations these formulas are not exact, because they omit the $(v/c)^2$ terms which result from the Lorentzian and gravitational contractions of the bodies.

In our calculations we shall need the following quantity, appearing in the $A_{ij}^{(4)}$:

$$\frac{1}{4\pi}I_{-1}(\underset{0}{U}_{,i}\underset{0}{U}_{,j}) = \frac{1}{4\pi}I_{-1}(\underset{0}{U}_{a,i}\underset{0}{U}_{a,j}) + \frac{1}{4\pi}I_{-1}(\underset{0}{U}_{b,i}\underset{0}{U}_{b,j})$$
$$+\frac{1}{4\pi}I_{-1}(\underset{0}{U}_{a,i}\underset{0}{U}_{b,j}) + \frac{1}{4\pi}I_{-1}(\underset{0}{U}_{a,j}\underset{0}{U}_{b,i}). \tag{5.7}$$

The first and second integrals in Eq. (5.7) were evaluated by Fok[11]:

$$-\frac{1}{4\pi}I_{-1}(\underset{0}{U}_{a,i}\underset{0}{U}_{a,j}) = \left(r_a^i r_a^j - \frac{1}{3}\delta^{ij}r_a^2\right)q_2(r_a) + \delta^{ij}q_0(r_a), \tag{5.8}$$

$$q_0(r_a) = \frac{1}{6}\frac{G^2m_a^2(r_a)}{r_a^2} - \frac{8\pi G^2}{3}\frac{1}{r_a}\int_0^{r_a}\rho^* m_a(\xi)\,\xi\,d\xi - \frac{4\pi G^2}{3}$$
$$\times\int_{r_a}^{\infty}\rho^* m_a(\xi)\,d\xi, \tag{5.9}$$

$$q_2(r_a) = -\frac{1}{4}\frac{G^2m_a^2(r_a)}{r_a^4} + \frac{8\pi G^2}{5}\frac{1}{r_a^5}\int_0^{r_a}\rho^* m_a(\xi)\,\xi^3\,d\xi$$
$$-\frac{2\pi G^2}{5}\int_{r_a}^{\infty}\rho^*\frac{m_a(\xi)}{\xi^2}\,d\xi. \tag{5.10}$$

On computing the third and fourth integrals in Eq. (5.7) we obtain

$$V_{ij}^{ab} \equiv -\frac{1}{4\pi}I_{-1}(\underset{0}{U}_{a,i}\underset{0}{U}_{b,j}) = \frac{1}{2}\frac{Gm_b}{r^3}n_a^i r^j(\chi_a' + Gm_a)$$
$$-\frac{1}{2}\frac{Gm_a}{r^3}r^i n_b^j(\chi_b' + Gm_b) + G^2 m_a m_b\frac{\partial^2 \ln s}{\partial x_a^i\,\partial x_b^j}, \tag{5.11}$$

where $s = r_a + r_b + r_{ab}$, $n_a^i = r_a^i/r_a$, $\chi_a' \equiv \partial\chi_a/\partial r_a$, and $\partial/\partial x_a^i$ signifies differentiation with respect to the parameter x_a^i. Only the last term in the expression (5.11) is evaluated in Fok's book, because he regarded the first two terms in Eq. (5.11), resulting from the integration over the bodies' volumes, as small compared with the third term (obtained from integration over the space outside the bodies) in the event that $R/L \ll 1$. From the exact solution (5.11) we see, however, that this will not be so, and we can easily understand why not. In the integration of the third term the integrand function is of order $1/L^5$, whereas the volume of integration is $\sim L^3$; hence the integral we obtain will be $\sim 1/L^2$. On the other hand, when we integrate the first and second terms in Eq. (5.11) the integrand function will be $\sim 1/R^2L^2$, while the volume of integration is R^3, so the integrals will be quantities of order $1/L^2$, the same as for the third integral.

The most difficult integral to calculate is the quantity

$$\varphi^i \equiv -\frac{1}{4\pi^2}\int_{(a)}\rho^* I_{-1,i}[\underset{0}{U}_{,kn}I_{-1}(\underset{0}{U}_{,k}\underset{0}{U}_n)]\,d^3x$$
$$= -\frac{1}{4\pi^2 G}\int_{R^3}\underset{0}{U}_{,kn}I_{-1}(\underset{0}{U}_{,k}\underset{0}{U}_{,n})\underset{0}{U}_{a,i}\,d^3x. \tag{5.12}$$

To evaluate it we have used the relations

$$\frac{r_a^k r_a^n}{r_a^2}\frac{\partial^2 \ln s}{\partial x_a^k\,\partial x_b^n} = -\frac{1}{2}\frac{r_a^n}{r}\left(\frac{r_b^n}{r_a^2 r_b} - \frac{r_b r^n}{r_a^2 r^2} + \frac{r_n}{r_a r^2}\right), \tag{5.13}$$

$$\frac{r_b^k r_b^n}{r_b^2}\frac{\partial^2 \ln s}{\partial x_a^k\,\partial x_b^n} = -\frac{1}{2}\frac{r_b^n}{r}\left(\frac{r_a^n}{r_b^2 r_a} + \frac{r_a r_n}{r_b^2 r^2} - \frac{r^n}{r_b r^2}\right), \tag{5.14}$$

together with the rule for differentiating homogeneous functions $\Phi(x)$ of equal degree $-k-2$ in 3-space[32]:

$$\frac{\partial\Phi}{\partial x^i} = \left(\frac{\partial\Phi}{\partial x^i}\right)_g + \frac{(-1)^k}{k!}\frac{\partial^k\delta(\mathbf{r})}{\partial x^{n_1}\partial x^{n_2}\ldots\partial x^{n_j}}\oint_\sigma \Phi(x)\,x^{n_1}x^{n_2}\ldots x^{n_j}\,dS_i, \tag{5.15}$$

where $(\partial\Phi/\partial x^i)_g$ represents the derivative of $\Phi(x)$ computed by the conventional rules, σ designates a surface surrounding the singular point, and $\delta(\mathbf{r})$ is the Dirac delta-function.

With Eqs. (5.13)-(5.15) we can reduce all the terms in Eq. (5.12) not describing the bodies' internal structure to the form $\int_{R^3} d^3x\, r_a^B r_b^C$, where B, C are certain integers. We evaluate all the integrals of this kind by means of the following Fourier representations[22] of the functions r_a^B, r_b^C:

$$\frac{1}{4\pi}\int_{R^3} d^3x\, r_a^B r_b^C = \frac{\pi^{1/2}}{4}\frac{\Gamma\left(\frac{B+3}{2}\right)\Gamma\left(\frac{C+3}{2}\right)\Gamma\left(-\frac{B+C+3}{2}\right)}{\Gamma(-B/2)\Gamma(-C/2)\Gamma((B+C+6)/2)}r^{B+C+3}. \tag{5.16}$$

The integrals in Eq. (5.12) that do characterize the internal structure can be calculated from Eqs. (5.4)-(5.6). Ultimately we find that the expression (5.12) becomes

$$\varphi^i = \frac{G^3 m_a m_b^3}{r^5}r^i + 8\frac{G^3 m_a^2 m_b^2}{r^5}r^i + \frac{2}{3}\frac{Gm_b}{r^3}r^i I_0^a(\rho^*\underset{0}{U}_a^2)$$
$$+\frac{2}{9}I_0^a(\rho^*\underset{0}{U}_a)I_0^b(\rho^*\underset{0}{U}_b)\frac{Gr^i}{r^3} + \frac{Gm_a}{r^3}r^i[I_0^b(\rho^*\underset{0}{U}_b^2) - 4I_0^b(p\underset{0}{U}_b)]. \tag{5.17}$$

It is interesting to observe that in evaluating the φ^i we gain terms of the form $(\ln r_{ab}/r_{ab}^5)r_{ab}^i$, only to find that they cancel out in the final expression. These terms appear because of the behavior of the expression (5.16) at points where the functions r_a^B, r_b^C have poles.

In just the same way as for the integral (5.12), we can determine a number of other integrals in the PNA 2 equations of motion. Some of these may conveniently be evaluated from the relations

$$-\frac{1}{4\pi}I_{-1}(g_{,k}h_{,k}) = \frac{1}{8\pi}I_{-1}(g\Delta h + h\Delta g) + \frac{1}{2}gh, \tag{5.18}$$

where g, h represent certain smooth functions and the I_{-1} are given by

$$GI^{a}_{-1}(\rho^{*}\underset{0}{U_b})(x\in V_a)$$
$$=\sum_{n=0}^{\infty} f_n{}^{a}(r_a)\frac{1}{n!}\,\frac{\partial^n \underset{0}{U_b}(x_a)}{\partial x^{p_1}\partial x^{p_2}\ldots\partial x^{p_n}}\, r_a^{p_1} r_a^{p_2}\ldots r_a^{p_n}, \tag{5.19}$$

$$f_n{}^{a}(r_a)=\frac{4\pi G}{2n+1}\left(\int_{r_a}^{\infty}\rho^{*}\xi\,d\xi+\frac{1}{r_a^{2n+1}}\int_0^{r_a}\rho^{*}\xi^{2(n+1)}d\xi\right) \tag{5.20}$$

$$GI_{-1}{}^{a}(\rho^{*}\underset{0}{U_b})(x\in V_b)=\frac{G^2 m_a m_b}{r_a r}+O\left(\frac{G^2M^2}{L^2}\delta^2\right). \tag{5.21}$$

The remaining integrals of the outer parts of the potentials U, W_i, Φ, χ occurring in the PNA 2 equations of motion can be evaluated by a multipole-expansion procedure similar to that used in calculating the PNA 1 integrals. All the integrals of the inner parts of the potentials mutually cancel. In the PNA 2 case we also have to allow for the terms of order $(v/c)^4$ in Eqs. (4.3)-(4.5).

By combining the PNA 2 terms that describe the bodies' internal structure, we can refine the mass values occurring in Eqs. (4.6), (4.7) so as to include the $(v/c)^4$ terms in the Tolman-mass definition (3.15). The compactness parameter will thereby be eliminated from the PNA 2 equations of motion. After lowering the order of the second, third, and fourth derivatives in the $(v/c)^4$ terms by means of the Newtonian equations (4.2), and allowing for the extra terms that result from lowering the order of the second derivatives in the $(v/c)^2$ terms by means of the post-Newtonian equations of motion (4.6), we find that the PNA 2 equations of motion take the form

$$M_a a_a{}^{i}=-\frac{GM_aM_b}{r^3}r^i+\frac{1}{c^2}\underset{(2)}{F_a{}^{i}}(r^k,v_a{}^k,v_b{}^k)+\frac{1}{c^4}\underset{(4)}{F_a{}^{i}}(r^k,v_a{}^k,v_b{}^k), \tag{5.22}$$

where the mass M_a is determined according to formula (3.15), $\underset{(2)}{F^i_a}$ is taken from (4.7) where, instead of mass m_a there appears M_a, and $\underset{(4)}{F^i_a}$ has the form:

$$\begin{aligned}
\underset{(4)}{F_a{}^{i}}={}&\frac{Gm_am_b}{r^3}r^i\Big[-2v_b{}^4-\frac{15}{8}(n^kv_b{}^k)^4+\frac{9}{2}(n^kv_b{}^k)^2v_b{}^2+\frac{3}{2}(n^kv_b{}^k)^2v_a{}^2\\
&-2(v_a{}^kv_b{}^k)^2+4(v_a{}^kv_b{}^k)v_b{}^2-6(v_a{}^kv_b{}^k)(n^kv_b{}^k)^2\Big]+\frac{G^2m_a{}^2m_b}{r^4}r^i\Big[\frac{5}{4}v_b{}^2\\
&+\frac{17}{2}(n^kv_b{}^k)^2-39(n^kv_a{}^k)(n^pv_b{}^p)-\frac{5}{2}(v_a{}^kv_b{}^k)+\frac{39}{2}(n^kv_a{}^k)^2-\frac{15}{4}v_a{}^2\Big]\\
&+\frac{G^2m_am_b{}^2}{r^4}r^i[4v_b{}^2-6(n^kv_b{}^k)^2-4(n^kv_a{}^k)(n^pv_b{}^p)-8(v_a{}^kv_b{}^k)+2(n^kv_a{}^k)^2]\\
&+\frac{G^3m_am_b}{r^5}r^i\left(-\frac{57}{4}m_a{}^2-\frac{69}{2}m_am_b-9m_b{}^2\right)+\frac{Gm_am_b}{r^3}(v_a{}^i-v_b{}^i)\\
&\times\Big[-5(r^kv_b{}^k)v_b{}^2+(r^kv_b{}^k)v_a{}^2+4(r^kv_a{}^k)v_b{}^2\\
&+4(r^kv_b{}^k)(v_a{}^pv_b{}^p)-6(n^kv_b{}^k)^2(r^pv_a{}^p)\\
&-4(r^kv_a{}^k)(v_a{}^pv_b{}^p)+\frac{9}{2}(n^kv_b{}^k)^2(r^pv_b{}^p)\Big]\\
&+\frac{G^2m_a{}^2m_b}{r^4}(v_a{}^i-v_b{}^i)\Big[\frac{55}{4}(r^kv_b{}^k)\\
&-\frac{63}{4}(r^kv_a{}^k)\Big]+\frac{G^2m_am_b{}^2}{r^4}(v_a{}^i-v_b{}^i)[-2(r^kr_b{}^k)-2(r^kv_a{}^k)].
\end{aligned} \tag{5.23}$$

Damour's calculations[22] by the EIH method yield an expression for the $\underset{(4)}{F^i_a}$ in full agreement with ours. This circumstance proves the legitimacy of Damour's regularization procedure. We would point out, however, that Damour established only reduced equations of motion (5.23), that is, equations in which the order of the higher derivatives was lowered by using the equations of motion for the preceding approximations. Our application of the PNA method has enabled us to derive the complete equations, including the higher derivatives. As a result we are able to obtain the complete Lagrangian of the two-body problem in the PNA 2 case, unlike Damour, who evaluated only part of the full Lagrangian.[22]

The left-hand member of Eqs. (5.22) and the first term on the right-hand side of the same equations have also been determined through the PNA method by Breuer and Rudolph.[28]

6. RADIATIVE APPROXIMATION; RADIATION FRICTION

We conclude by amplifying the account of the radiation damping force given in our preliminary latter,[18] and by supplying some pertinent details of the calculations.

When ρ is replaced by ρ^*, no extra terms are introduced in the s_i in Eqs. (4.1). The reason is that Eq. (3.5) contains no terms[13] or order $(v/c)^3$ or $(v/c)^5$. Accordingly $s_i{}^{*}\equiv s_i = s^i$. Several authors have obtained expressions[13,27,33] for the radiative force density s^i; a comparison we have made indicates that they are all equivalent. We shall use Chandrasekhar and Esposito's expression[13] for s^i, which becomes, after some straightforward transformations.[18]

$$\begin{aligned}
s_i={}&\frac{2G}{3}\rho^{*}\Big[2I_{kk}^{(3)}\dot v^i+I_{kk}^{(4)}v^i+\frac{d}{dt}(D_{ki}^{(3)}v_k)\\
&+\frac{1}{2}D_{kl}^{(3)}\frac{\partial B_{kl}}{\partial x^i}+\frac{3}{10}D_{ki}^{(5)}x_k\\
&+\frac{1}{20}I_{kki}^{(5)}+\frac{1}{2}\frac{d^3}{dt^3}\int\vartheta_{kk}x^i\,d^3x-\frac{d^4}{dt^4}\int\rho^{*}x^2v^i\,d^3x\Big],
\end{aligned} \tag{6.1}$$

where an expression for the I_{ik} is included in Eqs. (2.13),

$$I_{kki}=\int\rho^{*}x^2x^i\,d^3x,\ D_{ik}=3I_{ik}-\delta_{ik}I_{pp},\ B_{kl}$$
$$=G\int\rho^{*}(y)(x^k-y^k)(x^l-y^l)/|x-y|^3\,d^3y,\quad I^{(p)}_{ik\ldots n}=\frac{d^p}{dt^p}I_{ik\ldots n},$$

ϑ_{kk} is given by Eq. (2.17), and a dot connotes a time derivative.

It is not hard to evaluate $I_{ik\ldots n}$; to requisite accuracy we have

$$I_{kl}=\sum_{a=1}^{2}m_ax_a{}^kx_a{}^l,\quad I_{kki}=\sum_{a=1}^{2}m_ax_a{}^2x_a{}^i. \tag{6.2}$$

Similarly

$$\int\rho^{*}x^2v^i\,d^3x=\sum_{a=1}^{2}m_ax_a{}^2v_a{}^i. \tag{6.3}$$

By integrating the s^i over the volume of body a we obtain the gravitational-radiation reaction force $\underset{(5)}{F_a{}^{i}}$ on that body. The calculation is facilitated by dividing

$\underset{(5)}{F}_a{}^i$ into two parts:

$$\underset{(5)}{F}_a{}^i=\frac{m_a}{m_a+m_b}\underset{(5)}{\mathcal{F}}{}^i+\frac{m_a m_b}{m_a+m_b}\underset{(5)}{F}{}^i, \tag{6.4}$$

$$\underset{(5)}{\mathcal{F}}{}^i=\int_{(a)} s_i\, d^3x+\int_{(b)} s^i d^3x, \tag{6.5}$$

$$\underset{(5)}{F}{}^i=\frac{1}{m_a}\int_{(a)} s^i\, d^3x-\frac{1}{m_b}\int_{(b)} s_i\, d^3x. \tag{6.6}$$

First let us evaluate $\underset{(5)}{\mathcal{F}}{}^i$. One can easily see that

$$\underset{(5)}{\mathcal{F}}=-\frac{1}{3}G(m_a+m_b)\left[\frac{1}{10}I^{(5)}_{hhi}+\frac{d^3}{dt^3}\int\vartheta_{hk}x^i\, d^3x-2\frac{d^4}{dt^4}\int\rho^* x^2 v^i\, d^3x\right]. \tag{6.7}$$

The second term in the expression (6.7) contains the integral $\int (\nabla \underset{0}{U})^2 x^i\, d^3x$. On integrating it by parts we can transform it to read $4\pi G\int \rho^* \underset{0}{U} x^i\, d^3x$. We then find without difficulty that

$$\int\vartheta_{hk}x^i\, d^3x=\sum_{a=1}^{2}m_a x_a{}^i\left(v_a{}^2-\frac{1}{2}\frac{Gm_b}{r}\right), \tag{6.8}$$

where we have made use of the virial theorem (3.12), whereby the terms depending on the bodies' internal structure cancel out. An evaluation of the derivatives in Eq. (6.7) shows that the $\underset{(5)}{\mathcal{F}}{}^i$ depend on the coordinates and derivatives of the centers of inertia, up to fifth order. On lowering the order of the derivatives in the $\underset{(5)}{\mathcal{F}}{}^i$, using the Newtonian equations of motion, we obtain

$$\underset{(5)}{\mathcal{F}}{}^i=\frac{4}{5}\frac{Gm_a m_b(m_a-m_b)}{m_a+m_b}\left(\dot r_k{}^2-\frac{2Gm_0}{r}\right)\dddot{r}^i, \tag{6.9}$$

where $m_0 = m_a + m_b$.

To calculate the force $\underset{(5)}{F}{}^i$ we have to calculate the integral $\int_{(a)}\rho^*\frac{\partial B_{kl}}{\partial x^i}d^3x$. Performing some straightforward operations and applying the Newtonian equations of motion, we find that

$$\int_{(a)}\rho^*\frac{\partial B_{kl}}{\partial x^i}d^3x=3m_a n^k n^l \ddot x_a{}^i-m_a\delta^{ik}\ddot x_a{}^l-m_a\delta^{il}\ddot x_a{}^k. \tag{6.10}$$

If we substitute the expression (6.10) into Eq. (6.6), we readily see that

$$\underset{(5)}{F}{}^i=G\left[\frac{1}{5}D^{(5)}_{ki}r_k+2I^{(4)}_{ki}\dot r_k+\frac{10}{3}I^{(3)}_{kk}\ddot r^i\right]. \tag{6.11}$$

This expression for $\underset{(5)}{F}{}^i$ has been obtained independently by Linet.[34] Upon evaluating the derivatives in Eq. (6.11) we arrive at an expression depending on the coordinates of the bodies' centers of inertia and their derivatives of all orders through the fifth. Lowering the order by means of the Newtonian equations, we find:

$$\underset{(5)}{F}{}^i=-\frac{G^3m_a m_b(m_a+m_b)}{r^5}\left(8r\dot r^i-\frac{56}{3}\dot r r^i\right)+\frac{8}{5}\frac{Gm_a m_b}{m_a+m_b}\left(\frac{2Gm_0}{r}-\dot r_k{}^2\right)\dddot{r}^i. \tag{6.12}$$

The total radiation damping force is now given by Eq. (6.4):

$$\underset{(5)}{F}_a{}^i=\frac{4}{5}\frac{G^2m_a{}^2m_b}{r^4}\left(2\frac{Gm_0}{r}-\dot r_k{}^2\right)(r\dot r^i-3\dot r r^i)+8\frac{G^3m_a{}^2m_b}{r^5}\left(\frac{7}{3}\dot r r^i-r\dot r^i\right). \tag{6.13}$$

This expression for the force $\underset{(5)}{F}_a{}^i$ is appropriate to any motions that satisfy the restrictions set forth in Sec. 2. For the case of bounded orbits the radiation friction has been calculated in our previous letter,[18] where we used the equation

$$\frac{Gm_0}{a_0}=\frac{2Gm_0}{r}-\dot r_k{}^2, \tag{6.14}$$

expressing the Newtonian law of conservation of orbital energy. Schafer[35] has applied the EIH method to calculate the radiative damping force in a nonharmonic coordinate system: Damour,[22] in harmonic coordinates. Damour's expression for this force agrees with Eq. (6.13), which we have derived by the PNA method.

In final form, the equations of motion of two extended bodies comprising a binary system are

$$M_a a_a{}^i=-\frac{GM_aM_b}{r^3}r^i+\frac{1}{c^2}\underset{(2)}{F}_a{}^i(r^k, v_a{}^k, v_b{}^k)+\frac{1}{c^4}\underset{(4)}{F}_a{}^i(r^k, v_a{}^k, v_b{}^k)+\frac{1}{c^5}\underset{(5)}{F}_a{}^i(r^k, \dot r^k), \tag{6.15}$$

where the terms $\underset{(n)}{F}_a{}^i$ in the force (n = 2, 4, 5) are given by Eqs. (4.7), (5.23), (6.13), respectively, with the mass m_a replaced by M_a. Equations (6.15) contain only one parameter, the Tolman mass of the binary components; there is no dependence on the compactness parameter, which automatically falls out of the equations of motion as the concept of mass is successively refined in each new approximation. As we have intimated, this is a persuasive argument that Eqs. (6.15) may be employed to describe the motion of any objects, even neutron stars or black holes.

Equations (6.15) also imply that the inertial mass of each component is equal to its gravitational mass, to within terms of order $(v/c)^6$. The absence of any $(v/c)^3$ terms in Eqs. (6.15) suggests that dipole gravitational waves do not exist in the general-relativistic context. The force $\underset{(5)}{F}_a{}^i$ will produce a secular change in the orbital period of a binary system[18,36] due to the quadrupole gravitational radiation, an effect confirmed to 4% accuracy by observations[3] of the binary pulsar 1913 + 16.

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[1]R. A. Hulse and J. H. Taylor, Astrophys. J. 195, L51 (1975).
[2]V. A. Brumberg, Ya. B. Zel'dovich, I. D. Novikov, and N. I. Shakura, Pis'ma Astron. Zh. 1, No. 1, 5 (1975) [Sov. Astron. Lett. 1, 2 (1975)].
[3]J. M. Weisberg and J. H. Taylor, Phys. Rev. Lett. 52, 1348 (1984).
[4]P. C. Peters and J. Mathews, Phys. Rev. 131, 435 (1963).
[5]W. L. Burke, J. Math. Phys. 12, 401 (1971).
[6]C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, Freeman, San Francisco (1973) [Vol. 3, Mir, Moscow (1977)].

[7]A. Einstein, L. Infeld, and B. Hoffman, Ann. Math. 39, 65 (1938).
[8]M. Demiański and L. P. Grishchuk, Gen. Rel. Grav. 5, 673 (1974).
[9]P. D. D'Eath, Phys. Rev. D 12, 2183 (1975).
[10]R. E. Kates, Phys. Rev. D 22, 1853 (1980).
[11]V. A. Fok, The Theory of Space, Time, and Gravitation, Phys. Math. Press, Moscow (1955, 1961) [Pergamon (1964)].
[12]S. Chandrasekhar and Y. Nutku, Astrophys. J. 158, 55 (1969).
[13]S. Chandrasekhar and F. P. Esposito, Astrophys. J. 160, 153 (1970).
[14]J. L. Anderson and T. C. Decanio, Gen. Rel. Grav. 6, 197 (1975).
[15]J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas, Astrophys. J. 206, L77 (1976).
[16]T. Futamase and B. F. Schutz, Phys. Rev. D 28, 2363 (1983).
[17]T. Futamase, Phys. Rev. D 28, 2373 (1983).
[18]L. P. Grishchuk and S. M. Kopeikin, Pis'ma Astron. Zh. 9, 436 (1983) [Sov. Astron. Lett. 9, 230; 10, 200 (1984)].
[19]L. Infeld and J. Plebański, Motion and Relativity, Oxford Univ. Press (1960) [Foreign Lit. Press, Moscow (1962)].
[20]M. Carmeli, Nuovo Cimento (10) 37, 842 (1965).
[21]L. Bel, T. Damour, N. Deruelle, J. Ibañez, and J. Martin, Gen. Rel. Grav. 13, 963 (1981).
[22]T. Damour, in: Gravitational Radiation (NATO Adv. Study Inst., Les Houches, June 1982), North-Holland and Elsevier (1983), p. 59.
[23]J. Ibanez, J. Martin, and E. Ruíz, Gen. Rel. Grav. 16, 225 (1984).
[24]L. D. Landau and E. M. Lifshits, The Classical Theory of Fields, 4th ed., Nauka, Moscow (1975) [Pergamon (1976)].
[25]J. Ehlers, Proc. 9th Texas Sympos. Relativistic Astrophysics (Munich, Dec. 1978), Ann. New York Acad. Sci. 336, 279 (1980).
[26]G. D. Kerlick, Gen. Rel. Grav. 12, 467, 521 (1980).
[27]R. A. Breuer and E. Rudolph, Gen. Rel. Grav. 13, 777 (1981).
[28]R. A. Breuer and E. Rudolph, Gen. Rel. Grav. 14, 181 (1982).
[29]V. A. Brumberg, Relativistic Celestial Mechanics [in Russian], Nauka, Moscow (1972).
[30]M. E. Alexander, Astrophys. Space Sci. 23, 459 (1973).
[31]L. P. Grishchuk, A. N. Petrov, and A. D. Popova, Commun. Math. Phys. 94, 379 (1984).
[32]I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. 1, Properties and Operations, Phys. Math. Press, Moscow (1959) [Academic Press (1964)]
[33]A. Papapetrou and B. Linet, Gen. Rel. Grav. 13, 335 (1981).
[34]B. Linet, C. R. Acad. Sci. (Paris) 292, 1425 (1981).
[35]G. Schäfer, Progr. Theor. Phys. 68, 2191 (1982).
[36]T. Damour, Phys. Rev. Lett. 51, 1019 (1983).

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Doppler shifting of a distant light source in a Schwarzschild gravitational field

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For three limiting cases of relative placement of a light source, a gravitational lens, and the observer, equations are established that describe the lens *chromatic* properties. The results have direct applications to experimental tests of general relativity and to the discovery of dark gravitating bodies.

1. INTRODUCTION

When a ray of light passes through a Schwarzschild gravitational field, its frequency will shift; in other words, the gravitational lens will have chromatic properties. In discussing this effect we shall regard the frequency emitted by the distant light source as unknown, and shall inquire only as to how the frequency perceived by the observer ought to depend on the position and motion of both the source and the observer in the reference frame of the gravitating body (although, of course, we allow for the difference in gravitational potential at different points of the field). If any clues about the source frequency should in fact be available, the chances of correctly interpreting the relations to be obtained would be rather better.

In the post-Newtonian approximation, the ratio of the light (or any electromagnetic vibration) frequency ν_1 measured by the observer after the ray has passed through a Schwarzschild field to the frequency ν_2 emitted by the source may be written as

$$\frac{\nu_1}{\nu_2}=\frac{1-(1/c)\,(P_1+\mu S_1+\alpha\mu R_1)}{1+(1/c)\,(P_2+\mu S_2+\alpha\mu R_2)} \tag{1}$$

where $\mu = GM/c^2$ is the gravitational radius[1)] of the spherically symmetric body that is generating the field (M is the body's mass, G is the gravitational constant, c is the speed of light); α is an arbitrary real number, whose choice serves to fix the coordinate system being used; and the quantities P_1, P_2, S_1, S_2, R_1, R_2 are functions of the coordinates and velocities of the source and the observer.

These functions are expressed by

$$P_1=\frac{1}{Q}[(k-\cos\lambda)\dot r_1-r_1\dot u_1\sin\lambda\cos I_1],$$

$$P_2=\frac{1}{Q}[(1-k\cos\lambda)\dot r_2+kr_2\dot u_2\sin\lambda\cos I_2],$$

$$S_1=\frac{2}{Q}\left[(k-1)\frac{\dot r_1}{r_1}-\frac{(k+1)\dot u_1\sin\lambda\cos I_1}{1+\cos\lambda}\right]$$

$$S_2=-\frac{2}{Q}\left[(k-1)\frac{\dot r_2}{r_2}-\frac{(k+1)\dot u_2\sin\lambda\cos I_2}{1+\cos\lambda}\right]$$

$$R_1=\frac{\sin\lambda}{Q^3}\left[k(k-1)\frac{\dot r_1}{r_1}\sin\lambda+(1-k\cos\lambda-k^2\cos\lambda+k^3)\dot u_1\cos I_1\right],$$

$$R_2=-\frac{\sin\lambda}{Q^3}\left[k(k-1)\frac{\dot r_2}{r_2}\sin\lambda+(1-k\cos\lambda-k^2\cos\lambda+k^3)\dot u_2\cos I_2\right],$$

$$Q=\sqrt{k^2+1-2k\cos\lambda},\qquad k=r_1/r_2. \tag{2}$$

In Eqs. (2) the r_i are the moduli of the observer (i = 1) and source (i = 2) radius vectors in a coordinate system centered on the gravitating body; λ denotes the angle between the two vectors; the $\dot r_i$, $\dot u_i$ are respectively the