Gravitational instability of a two-component medium in an expanding universe

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In the Newtonian approximation, exact new solutions to the problem of the linear gravitational instability of a two-component hydrodynamic medium in an expanding Friedmann universe are obtained for two particular cases: *a*) one component is dustlike, its sound speed vanishing; *b*) one component has an adiabatic index $\gamma = 4/3$ (in both cases the other component has arbitrary but constant γ).

1. INTRODUCTION

Galaxies and clusters of galaxies unquestionably contain material that is hidden from us, or "missing mass," hence nonluminous, dark matter evidently accounts for much of the total density in the universe. From astrophysical and cosmological arguments it appears that the dark material is not composed either of baryons or of particles having a zero rest mass. At present the most likely candidates are thought to be heavy particles that interact hardly at all with ordinary matter or each other. They would manifest themselves solely through their gravitational interaction. Neutrinos of finite rest mass are a possibility, as are more hypothetical particles such as gravitinos, photinos, and axions.

It follows that when studying how spatially inhomogeneous fluctuations would have evolved in a Friedmann model universe during the postrecombination era, one should treat matter as a two-component medium. One component would be conventional baryon matter — the recombining gas, since radiation will not contribute very significantly at this stage; the other would consist of massive collisionless particles. All velocities would now be small compared with the speed of light. The first component may be described hydrodynamically as a barotropic fluid, with a definite equation of state; but the second will in general require a kinetic description.

However, if two conditions are satisfied, namely if (a) the collisionless particles have a random-velocity dispersion small compared with the ordered velocity they acquire in the ambient gravitational field, and (b) one considers scales longer than these particles' mean free path over the whole time the universe has been expanding, then the second component may be treated as "dust" (that is, hydrodynamic material having an equation of state $p = 0$), with its own hydrodynamic velocity, different from that of the baryon component. It is well recognized that collisionless diffusion may be neglected for masses in excess of $10^{12}[m/(1\ \mathrm{keV})]^{-2}\ M_\odot$, where m denotes the rest mass of a collisionless particle (this statement refers to particles which at some prior epoch had been in thermodynamic equilibrium with the radiation.

We therefore are confronted in a natural way with the problem of linear gravitational instability in a Friedmann model universe filled with two hydrodynamic components of matter that interact with each other exclusively by gravitation. In one component the sound speed would be zero. From a theoretical viewpoint, though, it is also of interest to consider the general problem of linear gravitational instability in a multicomponent medium whose sound velocities are arbitrary, in the context of an expanding universe.

In 1946 Lifshitz[1] solved the general-relativistic problem of the gravitational instability of a one-component medium for a Friedmann model. The gravitational instability of a multicomponent medium against the background of a steady-state universe has been thoroughly investigated in the Newtonian approximation by Grishchuk and Zel'dovich[2] (see also Polyachenko and Fridman's discussion[3]). Fargion,[4] followed by Nurgaliev and one of us,[5] have treated the corresponding problem for a multicomponent medium in an expanding universe, again in the Newtonian approximation. Asymptotic expansions have been obtained[4,5] for the solutions as $k \to 0$ and $k \to \infty$ [$k = 2\pi a/\lambda$, where λ is the perturbation wavelength and $a(t)$ is the scale factor of the Friedmann model], as well as exact solutions for the cases of identical components and components whose adiabatic index $\gamma = d(\ln p)/d(\ln \rho) = 4/3$.

Extending the analysis, we shall establish exact solutions in this paper for the evolution of small fluctuations in the density of a two-component medium against the background of an expanding Friedmann universe in the Newtonian approximation, for two cases: the situation of greatest interest, where one component has a sound speed of zero, and the case where neither sound speed vanishes but one component has an adiabatic index $\gamma = 4/3$.

2. DENSITY PERTURBATIONS IN TWO-COMPONENT MEDIUM

Consider a two-component hydrodynamic medium having the equations of state $p_i = p_i(\rho_i)$ ($i = 1, 2$), with $p_i \ll \varepsilon_i \approx \rho_i$, where the ε_i denotes the components' energy densities (throughout we set the velocity of light $c = 1$). The dimensionless sound speeds $\beta_i = (dp_i/d\varepsilon_i)^{1/2}$ will be much smaller than 1. Define $\delta_i = (\delta\varepsilon/\varepsilon)_i \approx (\delta\rho/\rho)_i$. Then in the Newtonian approximation the evolution of the δ_i for one mode, that having a spatial dependence $\exp(i\mathbf{k}\cdot\mathbf{r})$, in an expanding Friedmann model with scale factor $a(t)$ will be described by the equations (see, for example, Zel'dovich and Novikov[6])

$$\ddot{\delta}_1 + 2\frac{\dot{a}}{a}\dot{\delta}_1 + \frac{k^2}{a^2}\beta_1^2\delta_1 = 4\pi G(\rho_1\delta_1 + \rho_2\delta_2),$$

$$\ddot{\delta}_i+2\frac{\dot{a}}{a}\dot{\delta}_i+\frac{k^2}{a^2}\beta_i^2\delta_i=4\pi G(\rho_1\delta_1+\rho_2\delta_2), \tag{1}$$

where $k = |\mathbf{k}| =$ const.

Equations (1) will hold provided $|\delta_i| \ll 1$, $\beta_i \ll 1$. One need not impose the constraint $k/a \gg \dot{a}/a$ (or $\lambda \ll t$). To demonstrate this fact we derive the general-relativistic analogs of Eqs. (1) in the Appendix; they differ only in having extra terms that contain quantities of order $\beta_i^2\delta_i$.

Let us take the stage of expansion when the radiation energy density has become negligible. Assume zero spatial curvature for the Friedmann model, and a zero cosmological constant. Then

$$a(t)\propto t^{2/3},\quad \rho_i=\Omega_i/6\pi Gt^2,\quad \Omega_i=\text{const},\quad \Omega_1+\Omega_2=1. \tag{2}$$

Suppose now that the components have equations of state $p_i \propto \rho_i^{\gamma_i}$, with the $\gamma_i =$ const. Accordingly $\beta_i^2 \propto \rho_i^{\gamma_i - 1} \propto t^{2(1-\gamma_i)}$. Define

$$\varkappa_i=\frac{k}{a(t)}\beta_i t^{\gamma_i-1/3}=\text{const}. \tag{3}$$

Equations (1) will then take the form

$$\ddot{\delta}_1+\frac{4}{3t}\dot{\delta}_1+\left(\varkappa_1^2t^{2\left(\frac{1}{3}-\gamma_1\right)}-\frac{2}{3}\frac{\Omega_1}{t^2}\right)\delta_1=\frac{2}{3}\frac{\Omega_2}{t^2}\delta_2,$$
$$\ddot{\delta}_2+\frac{4}{3t}\dot{\delta}_2+\left(\varkappa_2^2t^{2\left(\frac{1}{3}-\gamma_2\right)}-\frac{2}{3}\frac{\Omega_2}{t^2}\right)\delta_2=\frac{2}{3}\frac{\Omega_1}{t^2}\delta_1. \tag{4}$$

On eliminating δ_2 from Eqs. (4) we obtain for δ_1 the fourth-order equation

$$\delta_1^{\mathrm{IV}}+\frac{20}{3t}\delta_1^{\mathrm{III}}+\left[\varkappa_1^2t^{2\left(\frac{1}{3}-\gamma_1\right)}+\varkappa_2^2t^{2\left(\frac{1}{3}-\gamma_2\right)}+\frac{76}{9t^2}\right]\ddot{\delta}_1$$
$$+\left[4\varkappa_1^2\left(\frac{5}{3}-\gamma_1\right)t^{-\left(\frac{1}{3}+2\gamma_1\right)}+\frac{4}{3}\varkappa_2^2t^{-\left(\frac{1}{3}+2\gamma_2\right)}+\frac{8}{9t^3}\right]\dot{\delta}_1$$
$$+\left[\varkappa_1^2\varkappa_2^2t^{2\left(\frac{2}{3}-\gamma_1-\gamma_2\right)}\right.$$
$$+\varkappa_1^2\left(\left(\frac{8}{3}-2\gamma_1\right)(3-2\gamma_1)-\frac{2}{3}\Omega_2\right)t^{-\left(\frac{4}{3}+2\gamma_1\right)}$$
$$\left.-\frac{2}{3}\varkappa_2^2\Omega_1t^{-\left(\frac{4}{3}+2\gamma_2\right)}\right]\delta_1=0. \tag{5}$$

If either $k = 0$ or $\beta_1 = \beta_2 = 0$, the quantities $\varkappa_1 = \varkappa_2 = 0$; Eqs. (4), (5) will then have the solution

$$\delta_1=B_1t^{2/3}+B_2t^{-1}+B_3+B_4t^{-1/3},$$
$$\delta_2=B_1t^{2/3}+B_2t^{-1}+B_5+B_6t^{-1/3},$$
$$\Omega_1B_3+\Omega_2B_5=\Omega_1B_4+\Omega_2B_6=0. \tag{6}$$

First obtained by Wasserman,[7] this solution will remain valid in the multicomponent, general-relativistic case if the pressures $p_i = 0$. Only one mode here is a growing mode.

For $k \neq 0$, $\varkappa_1^2 + \varkappa_2^2 \neq 0$, and $\Omega_1\Omega_2 \neq 0$, the general solution of Eq. (5) can be expressed in terms of standard transcendental functions in four particular cases: 1) $\gamma_1 = \gamma_2$, $\varkappa_1 = \varkappa_2$; 2) $\gamma_1 = \gamma_2 = 4/3$, $\varkappa_1 \neq \varkappa_2$; 3) $\varkappa_2 = 0$, γ_1 arbitrary; 4) $\gamma_2 = 4/3$, with γ_1, $\varkappa_1$, $\varkappa_2$ arbitrary. Cases 1 and 2 were first discussed by Fargion[4]; the solutions are given in terms of power functions and Bessel functions. In Sec. 3 we present solutions for cases 3 and 4.

3. NEW EXACT SOLUTIONS FOR DENSITY PERTURBATIONS

a. Case 3: $\beta_2 = 0$. By the definition (3), the sound speed $\beta_2 = 0$, so the second component will be dustlike. As mentioned in Sec. 1, this is the most interesting case physically. We introduce the new variable

$$x=\varkappa_1^2\alpha^{-2}t^{-\alpha}=\left(\frac{t_s}{t}\right)^{\alpha},\quad \alpha=2\gamma_1-\frac{8}{3},\quad t_s=\left(\frac{\varkappa_1}{|\alpha|}\right)^{2/\alpha}. \tag{7}$$

Equation (5) will then become (with an operator $\Delta \equiv x\cdot d/dx$)

$$\left[\left(\Delta+\frac{2}{3\alpha}\right)\Delta\left(\Delta-\frac{1}{3\alpha}\right)\left(\Delta-\frac{1}{\alpha}\right)+x\left(\Delta^2+\frac{2\alpha-1/3}{\alpha}\Delta+\frac{1}{\alpha^2}\left(\alpha\left(\alpha-\frac{1}{3}\right)-\frac{2}{3}\Omega_2\right)\right)\right]\delta_1=0. \tag{8}$$

Its general solution can be represented in terms of the Meijer G-functions[8]:

$$\delta_1=C_1G_{24}^{41}\left(x\,\middle|\,\begin{matrix}a_1a_2\\b_1b_2b_3b_4\end{matrix}\right)+C_2G_{24}^{41}\left(x\,\middle|\,\begin{matrix}a_2a_1\\b_1b_2b_3b_4\end{matrix}\right)$$
$$+C_3G_{24}^{40}\left(xe^{i\pi}\,\middle|\,\begin{matrix}a_1a_2\\b_1b_2b_3b_4\end{matrix}\right)+C_4G_{24}^{40}\left(xe^{-i\pi}\,\middle|\,\begin{matrix}a_1a_2\\b_1b_2b_3b_4\end{matrix}\right),$$
$$b_1=-\frac{2}{3\alpha},\quad b_2=0,\quad b_3=\frac{1}{3\alpha},\quad b_4=\frac{1}{\alpha},$$
$$a_{1,2}=\frac{1}{6\alpha}\left(1\mp\sqrt{1+24\Omega_2}\right). \tag{9}$$

The solution for δ_2 has the same functional form, but with different values for the constants a_i:

$$a_{1,2}=1+\frac{1}{6\alpha}\left(1\mp\sqrt{1+24\Omega_2}\right). \tag{10}$$

The first Meijer G-function can be written as a Mellin–Burns integral in the following way[8]:

$$G_{24}^{41}\left(x\,\middle|\,\begin{matrix}a_1a_2\\b_1b_2b_3b_4\end{matrix}\right)=\frac{1}{2\pi i}\int_{e_1-i\infty}^{e_1+i\infty}\frac{\Gamma(1-a_1+s)\prod_{n=1}^{4}(b_n-s)}{\Gamma(a_2-s)}x^s\,ds, \tag{11}$$

where the e_i are arbitrary constants. The contour of integration in Eq. (11) circuits all the poles of the functions $\Gamma(b_n - s)$ ($n = 1, 2, 3, 4$) from the left (that is, clockwise), and all the poles of the function $\Gamma(1 - a_1 + s)$ from the right (all these poles lie along the real axis, $\mathrm{Im}\, s = 0$). Moreover the function

$$G_{24}^{40}\left(xe^{\pm i\pi}\,\middle|\,\begin{matrix}a_1a_2\\b_1b_2b_3b_4\end{matrix}\right)$$
$$=\frac{1}{2\pi i}\int_{e_2-i\infty}^{e_2+i\infty}\frac{\prod_{n=1}^{4}\Gamma(b_n-s)}{\Gamma(a_1-s)\Gamma(a_2-s)}x^se^{\pm i\pi s}\,ds, \tag{12}$$

where the upper sign in the exponent corresponds to $e_1 > \frac{1}{2}\left(\sum_{n=1}^{4} b_n - a_1 - a_2\right) = \frac{1}{6\alpha}$ arbitrary; the lower sign, to $e_2 > 1/6\alpha$, e_1 arbitrary. The integration contour circuits all the poles of the functions $\Gamma(b_n - s) \cdot (n = 1, 2, 3, 4)$ from the left.

Asymptotic expressions for the Meijer G-functions will be found in the literature,[8,9] or can be obtained directly from the integral representations (11), (12). For definiteness let us take $\gamma_1 > 4/3, \alpha > 0$; then the range $x \gg 1$ will corresponds to the range $t \ll t_J$.

As $x \to \infty$ one may shift the integration contour in Eq. (11) leftward in the complex s-plane, taking the e_i to be large negative numbers. Then the integral (11) will represent a sum of residues at the poles of the function $\Gamma(1 - a_1 + s)$ [in general the poles of $\Gamma(1 - a_1 + s)$ and $\Gamma(b_n - s)$ will not coincide]. The main contribution will come from the pole on the far right, $s = a_1 - 1$. Hence as $x \to \infty$ we may write

$$G_{24}^{41}\left(x \,\middle|\, \begin{matrix} a_1 a_2 \\ b_1 b_2 b_3 b_4 \end{matrix}\right) = \frac{\prod_{n=1}^{4} \Gamma(b_n - a_1 + 1)}{\Gamma(a_2 - a_1 + 1)} x^{a_1 - 1}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{13}$$

In the case of Eq. (12) one cannot shift the integration contour indefinitely leftward, since either e_1 or e_2 should exceed $1/6\alpha$. However, as $x \to \infty$ the integral can be evaluated by the method of steepest descent.[9] By expanding the gamma-function in the integrand of the expression (12) for large values of the argument, one readily finds that the integral for $G(xe^{i\pi})$ has a saddle point at $s = -i\sqrt{x}$, and the contour of integration ought to go through that point in the direction of steepest descent, at a 135° angle to the real axis (Im s = 0). Similarly, the integral for $G(xe^{-i\pi})$ has a saddle point at $s = i\sqrt{x}$, which the integration contour should cross at a 45° angle to the real axis. Consequently as $x \to \infty$ we will have

$$G_{24}^{40}(xe^{\pm i\pi}) = \sqrt{\pi}\,(xe^{\pm i\pi})^{\frac{d}{2} - \frac{1}{4}} \exp(\mp 2i\sqrt{x})(1 + O(x^{-1/2})),$$
$$d = \sum_{n=1}^{4} b_n - a_1 - a_2 = \frac{1}{3\alpha}. \tag{14}$$

Substituting now the expressions (13), (14) into Eq. (9) and reverting to the variable t, we arrive at the asymptotic form of δ_1 as $t \to 0$. In light of Eq. (10) we see that as $t \to 0$ $(x \to \infty)$ the first two terms in the expression (9) will describe growing and decaying monotonic perturbations, respectively, residing predominantly in the dustlike component:

$$\delta_2 \propto t^{\frac{1}{6}(\pm\sqrt{1+24\Omega_2} - 1)}, \quad \delta_1 \propto \delta_2 \left(\frac{t}{t_J}\right)^{2\gamma_1 - \frac{8}{3}}, \quad |\delta_1| \ll |\delta_2|, \tag{15}$$

while the last two terms will represent traveling acoustic waves in the finite-pressure component:

$$\delta_1 \propto t^{\frac{1}{2}\left(\gamma_1 - \frac{5}{3}\right)} \exp\left(\mp i \frac{\varkappa_1}{\gamma_1 - \frac{4}{3}} t^{\frac{4}{3} - \gamma_1}\right),$$

$$\delta_2 \propto \delta_1 \left(\frac{t}{t_J}\right)^{2\gamma_1 - \frac{8}{3}}, \quad |\delta_2| \ll |\delta_1|. \tag{16}$$

These asymptotics coincide with those determined earlier in Refs. 4 and 5.

By analogy to the one-component case we can determine the Jeans wavelength for each component separately: $\lambda_{Ji} \sim \beta_i t$. In our present case $\lambda_{J2} = 0$, while throughout the region $x \gg 1$ $(t \ll t_J)$ the perturbation wavelength $\lambda \ll \lambda_{J1}$. This region, according to the relations (15), (16), contains just one growing monotonic mode [upper sign in expression (15)], which in the lead approximation corresponds to ordinary Jeans instability in component 2, taken by itself. The gravitational coupling between the two components is manifested by the fact that the density perturbation δ_1 of the first component also develops a growing mode. Initially δ_1 has a much smaller amplitude than δ_2, but it grows rapidly with time, becoming comparable to δ_2 in absolute value at $x \sim 1$ $(t \sim t_J)$, that is, when $\lambda \sim \lambda_{J1}$.

The amplitude (16) of the acoustic waves in component 1 will increase with time if $\gamma_1 > 5/3$, remain constant if $\gamma_1 = 5/3$, and diminish if $\gamma_1 < 5/3$. The gravitational coupling of the two components serves to produce a small oscillating mode in δ_2. For $x \sim 1$ the oscillations in δ_1, δ_2 will achieve comparable amplitudes, but when $x \ll 1$ both these oscillations will be quenched.

For values $x \lesssim 1$, when λ exceeds the Jeans wavelengths of all components, the gravitational coupling will be strong and the δ_i will have amplitudes of the same order in the natural modes. To determine the asymptotic behavior of δ_1 as $x \to 0$ $(t \to \infty)$ we have to shift the integration contour in Eqs. (11), (12) to the right, taking the e_i to be large positive numbers. Then the integrals (11), (12) will represent a sum of residues at the poles of the functions $\Gamma(b_n - s)$ (n = 1, 2, 3, 4), and we will arrive at the standard representation[8] of the Meijer G-functions (11), (12) as a sum of four generalized hypergeometric series $_2F_3(x)$ multiplied by the quantities x^{b_n}. This representation is convenient as well for numerical evaluation of the G-functions when x is finite.

When $x \ll 1$ $(t \gg t_J)$, if we retain only the leading term (unity) in each of the generalized hypergeometric series we will asymptotically obtain expressions of the form (6). Of special interest here is the coefficient B_1 for the growing perturbation mode; it is given by

$$B_1 = t_J^{-b_1}\Gamma\left(\frac{2}{3\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{5}{3\alpha}\right)\left[C_1 \frac{\Gamma(1 - a_1 + b_1)}{\Gamma(a_2 - b_1)} + C_2 \frac{\Gamma(1 - a_2 + b_1)}{\Gamma(a_1 - b_1)} + \frac{1}{\Gamma(a_1 - b_1)\Gamma(a_2 - b_1)}\left(C_3 e^{-i\frac{2\pi}{3\alpha}} + C_4 e^{i\frac{2\pi}{3\alpha}}\right)\right]. \tag{17}$$

Clearly each term in the expression (9), even the second term (which describes what initially is a purely decaying mode), will in general make a nonzero contribution to B_1, that is, to the mode that grows as $t \to \infty$. If the coefficients C_n (n = 1, 2, 3, 4) are statistically independent, then $B_1 \neq 0$. Only if the C_n are correlated in some manner can B_1 vanish.

Thus the exact solution (9) enables us to relate the asymptotic expressions (15), (16) for δ_1 and δ_2 when $t \ll$

t_J to the asymptotes (6) when $t \gg t_J$, and to calculate transitional coefficients.

b. Case 4: $\gamma_2 = 4/3$. Allowing γ_1, $\varkappa_1$, $\varkappa_2$ to be arbitrary, we make the change of variables (7) in Eq. (5). It turns out that the solution of the resulting equation, as well as of the analogous equation for δ_2, can again be expressed in the form (9), but now with different coefficients a_n, b_n:

$$b_{1,2,3,4} = \frac{1}{6\alpha}\left[1 \mp \left(13 - 18\varkappa_2^2 \pm 12\sqrt{1 + (6\Omega_1 - 3)\varkappa_2^2 + \frac{9}{4}\varkappa_2^4}\right)^{1/2}\right],$$

$$a_{1,2} = \frac{1}{6\alpha}\left(1 \mp \sqrt{1 + 24\Omega_2 - 36\varkappa_2^2}\right) \quad \text{for} \quad \delta_1,$$

$$a_{1,2} = 1 + \frac{1}{6\alpha}\left(1 \mp \sqrt{1 + 24\Omega_2 - 36\varkappa_2^2}\right) \quad \text{for} \quad \delta_2. \tag{18}$$

The asymptotic relations (13), (14) will continue to hold true.

As before, $t = t_J$ is the epoch at which $\lambda \sim \lambda_{J1}$. Interpreted physically, the constant $\varkappa_2$ is of order λ_{J2}/λ (this ratio will be independent of t for a medium with $\gamma = 4/3$).

In the range $x \gg 1$ ($t \ll t_J$), unlike the preceding case, a monotonically growing mode for δ_2 and for the total density perturbation $\delta = \delta_1\Omega_1 + \delta_2\Omega_2$ will not exist only in the event that $\varkappa_2^2 < (2/3)\Omega_2$, that is, for large enough perturbation wavelengths λ. This criterion gives a critical λ-value that coincides with the Jeans wavelength for a medium having $\gamma = 4/3$, provided the components do not interact gravitationally. For $t \ll t_J$ and $\varkappa_2 \gg 1$ the perturbations of both components will represent two types of acoustic waves.

Analysis of the expressions (18) shows that just one value of each quantity b_n (that with the upper signs) can be negative, so that as $t \to \infty$ only one growing mode will exist. This result bears out the findings of Grishchuk and Zel'dovich.[2] For $\varkappa_2 \gg 1$ this mode will grow as t^q, with $q = \frac{1}{6}(\sqrt{1 + 24\Omega_1} - 1)$, while the perturbation $\delta_2 \simeq (2\Omega_1/3\varkappa_2^2)_1$, so that $|\delta_2| \ll |\delta_1|$. In other words the growing perturbation will reside predominantly in the first component. For $\varkappa_2 \ll 1$, component 2 will be essentially dustlike; Eqs. (18) will reduce to Eqs. (9), and we will be back to the case worked out before.

APPENDIX: GENERAL-RELATIVISTIC DERIVATION OF EQS. (1)

Suppose that a Friedmann model, expressed in a synchronous reference frame

$$ds^2 = dt^2 - a^2(t)\,(\delta_{\nu\sigma} + h_{\nu\sigma})\,dx^\nu dx^\sigma, \quad \nu, \sigma = 1, 2, 3, \tag{A.1}$$

and subject to small perturbations, is filled with two hydrodynamic fluids that interact with each other only by gravitation. The indices ν, σ will be raised and lowered by means of the Cartesian metric $\delta_{\nu\sigma}$. We will stipulate a spatial dependence $\exp(ik_\nu x^\nu)$ for the perturbations, with $k^2 = k_\nu k^\nu$. Scalar perturbations, as Lifshitz showed,[1] can be represented in the form

$$h_\nu^\sigma = \frac{1}{3}(\lambda + \mu)\delta_\nu^\sigma - \lambda\frac{k_\nu k^\sigma}{k^2}, \tag{A.2}$$

where μ, λ are functions of k_ν and t.

We introduce the following gauge-invariant quantities, which will vanish in fictitious perturbations of the metric:

$$U = \frac{k^2}{3}(\lambda + \mu) - a\dot{a}\dot{\lambda},$$
$$W = \frac{k^2}{3}(\lambda + \mu) + a(a\dot{\lambda})^{\cdot}. \tag{A.3}$$

The quantity W has a simple physical meaning: it is directly related to the invariant of the conformal Weyl tensor characterizing the departure from a conformally flat metric, since $C_{klmn}C^{klmn} = W^2/3a^4$. By virtue of the ($\nu$, σ) components ($\nu \neq \sigma$) of the Einstein equations for the problem at hand, $W = 2U$, so U has the same physical significance as W.

Now let v_i denote the velocity potentials of fluids $i = 1, 2$, representing the 3-components of the covariant 4-velocity $u_\nu = v_{,\nu} = ik_\nu v$. As gauge-invariant quantities to characterize each fluid we may take

$$\bar{v} = v - \frac{\dot{\lambda}a^2}{2k^2},$$
$$\overline{\delta\varepsilon} = \delta\varepsilon + 3(\varepsilon + p)\frac{\dot{a}}{a}v, \tag{A.4}$$

where $\delta\varepsilon$ denotes the energy-density perturbation in the synchronous frame. The quantity $\overline{\delta\varepsilon}$ coincides with the energy-density perturbation in a comoving reference frame whose three-dimensional hypersurfaces of constant time are orthogonal to the 4-velocity.

The quantities we have introduced are related very simply to the gauge-invariant variables of Bardeen[10]:

$$\Phi_A = \frac{1}{2k^2}(U - W), \quad \Phi_H = \frac{U}{2k^2}, \quad v_S = k\bar{v}, \quad e_m = \frac{\overline{\delta\varepsilon}}{\varepsilon}. \tag{A.5}$$

The Lukash scalar[11] q is expressed in terms of them by $q = 3(\bar{v}\dot{a}/a - U/2k^2)$.

On rewriting the $(0, \nu)$ components of the Einstein equations and of the conservation law $T^m_{lm} = 0$ for each separate component in terms of the gauge-invariant quantities, we obtain the system of equations

$$\dot{U} + \frac{\dot{a}}{a}U = -8\pi Gk^2\sum_i(\varepsilon + p)_i\bar{v}_i,$$

$$\left(\frac{\dot{\bar{v}}_i}{\beta_i^2}\right)^{\cdot} - 3\frac{\dot{a}}{a}\dot{\bar{v}}_i - 3\left(\frac{\dot{a}}{a}\right)^{\cdot}\bar{v}_i + \frac{k^2}{a^2}\bar{v}_i + \frac{1}{2k^2}\left[3\dot{U} + \left(\frac{U}{\beta_i^2}\right)^{\cdot}\right] = 0,$$

$$\beta_i^2\bar{\delta}_i = \frac{U}{2k^2} + \bar{v}_i, \qquad \delta_i \equiv \delta\varepsilon_i/(\varepsilon + p)_i, \tag{A.6}$$

where the quantities $\beta_i^2 = (dp/d\varepsilon)_i$ may depend on the ε_i. Equations (A.6) will in fact hold for an arbitrary number of hydrodynamic components that mutually interact only by gravitation.

The undisturbed Friedmann solution will conform to the relations

$$\left(\frac{\dot{a}}{a}\right)^{\cdot} = -4\pi G\sum_i(\varepsilon + p)_i, \qquad (\varepsilon + p)_i^{\cdot} = -3\frac{\dot{a}}{a}(1 + \beta_i^2)(\varepsilon + p)_i. \tag{A.7}$$

In the two-fluid case we can eliminate U from Eqs. (A.6) and use the first Eq. (A.7) to obtain

$$\dot{\bar{\delta}}_1 - 3\frac{\dot{a}}{a}\beta_1^2\bar{\delta}_1 + \frac{k^2}{a^2}\bar{v}_1 - 12\pi G(\varepsilon_2 + p_2)(\bar{v}_2 - \bar{v}_1) = 0 \qquad (A.8)$$

with a similar equation for $\bar{\delta}_2$. Multiplying Eq. (A.8) by a^2, differentiating with respect to t, and using the second Eq. (A.7), we finally get

$$\begin{aligned}\ddot{\bar{\delta}}_1 &+ 2\frac{\dot{a}}{a}\dot{\bar{\delta}}_1 + \frac{k^2}{a^2}\beta_1^2\bar{\delta}_1 - 4\pi G[(\varepsilon_1 + p_1)\bar{\delta}_1 + (\varepsilon_2 + p_2)\bar{\delta}_2] \\ &= \frac{3}{a^2}(a\dot{a}\beta_1^2\bar{\delta}_1)^{\cdot} + 12\pi G(\varepsilon_2 + p_2)(\beta_2^2\bar{\delta}_2 - \beta_1^2\bar{\delta}_1) \\ &- 12\pi G\frac{\dot{a}}{a}(1 + 3\beta_2^2)(\varepsilon_2 + p_2)\int^t(\beta_2^2\bar{\delta}_2 - \beta_1^2\bar{\delta}_1)\,dt'. \end{aligned} \qquad (A.9)$$

By interchanging subscripts 1, 2 in Eq. (A.9) we obtain the parallel equation for $\bar{\delta}_2$.

If the $\beta_i \ll 1$, all the terms in the right-hand member of Eq. (A.9) will contain a small extra factor of order β_i^2 compared with the leading terms on the left in Eq. (A.9). In principle the integral in the last term on the right can give a large factor of order t, but it is offset[1] by the coefficient $\dot{a}/a \sim t^{-1}$[1]. Hence in the Newtonian approximation ($p_i \ll \varepsilon_i$, $\beta_i \ll 1$) we are justified in omitting the whole right-hand member of Eq. (9) as well as the pressures p_i on the left-hand side. Further, we will then have $\bar{\delta}_i \approx \delta_i$, and Eq. (A.9) and its counterpart will reduce to Eq. (1).

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Relativistic kinetics of baryon production in the big bang

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The baryogenesis process in the early hot universe is investigated by means of relativistic kinetic theory. An exact solution to the kinetic equations for supermassive bosons serves to refine previous results: the optimum baryon-production domain $m_X > \alpha_X m_{Pl}\sqrt{N}$ is now complemented by bosons of low mass, $m_X < \alpha_X m_{Pl}\sqrt{N}$, thus removing the cosmological lower bound that had limited the mass of superheavy bosons.

Sakharov[1] and Kuz'min[2] have suggested that the baryon asymmetry which we observe in the universe ($n_B/n_\gamma \sim 10^{-9}$) is an artifact of CP-noninvariant processes that violated the conservation of baryon charge. One such process would be the decay of supermassive X bosons, a consequence of grand unification models:

$$X \rightleftharpoons \bar{q} + \bar{q}; \quad X \rightleftharpoons q + l \qquad (1)$$

(q designates a quark, l a lepton). But as Okun' and Zel'dovich[3] have pointed out, if these processes occurred under LTE conditions (in Lorentz retarded time), then not even CP noninvariance and baryon-charge nonconservation would suffice to create an excess of baryons over antibaryons. In addition, LTE would have to break down for the reactions (1) in order for a net baryon charge to result, and the X bosons would need time to withdraw from a state of statistical equilibrium. Such a circumstance, calling for a boson decay time $\tau_X \gg t$, could arise[4] if

$$m_X > \alpha_X m_{Pl}\sqrt{N} \qquad (2)$$

[m_X denotes the mass of an X boson, $m_{Pl} = (\hbar c/G)^{1/2}$ is the Planck mass, α_X is the interaction constant and N species of particles are present], setting a stringent lower bound on the X-boson mass.

Several authors have sought to estimate the baryon asymmetry of the universe on the premise that LTE was indeed violated.[4–8] The following value has been obtained for the ratio of the baryon number density n_B to the total entropy density s:

$$\frac{n_B}{s} = \frac{45\zeta(3)}{4\pi^4}\frac{N_X}{N}\Delta r, \qquad (3)$$

where N_X is the number of supermassive-boson species and Δr denotes the difference between the relative decay probabilities by the routes $X \to q + l$ and $\bar{X} \to \bar{q} + l$ that results from the CP noninvariance (the Boltzmann constant k is set equal to 1). Applying relativistic kinetic theory, Fry et al.[9] have performed numerical calculations of the n_B/s ratio, largely confirming the estimates.[4–8]

These investigations, however, have a salient draw-