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EVOLUTIONARY PERIOD CHANGES IN ROTATING HOT PRE–WHITE DWARF STARS

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ABSTRACT

We have calculated the splitting of high order nonradial g-modes due to slow rotation in models of hot pre-white dwarf ("PWD") stars of 0.60 $M_\odot$. We have investigated the effects of rotational spin-up, produced by gravitational contraction, on the rate of evolutionary period change for the cases of uniform and differential rotation. For models in the luminosity range of PG 1159−035 ($L \sim 100\ L_\odot$), we find that rotation rates of a few thousand seconds for modes with $m \lesssim -2$ produce values of $d(\ln P)/dt$ that are consistent with the measurement of the rate of period change of the 516 second period of PG 1159−035.

Subject headings: stars: evolution — stars: pulsation — stars: rotation — stars: white dwarfs

I. INTRODUCTION

In a recent paper, Winget *et al.* (1985) reported the measurement of the rate of period change for the 516 s period of the variable star PG 1159−035. This star is the prototype of the class of variable stars believed to be post-planetary nebula nuclei (PNN) undergoing nonradial g-mode pulsation (Starrfield *et al.* 1983). The measured value of dP/dt of $-1.21 \pm 0.1 \times 10^{-11}$ s s^{-1} implies a time scale for period decrease of $\sim 10^6$ yr [$d(\ln P)/dt = -2.34 \times 10^{-14}$ s^{-1}]. This measurement gives a fundamental indication of the evolutionary time scale for PG 1159−035 (Winget, Hansen, and Van Horn 1983). As we will now show, the rate of period change reflects the evolutionary decrease in the moment of inertia of the star.

In an earlier paper we reported the results of a theoretical study of the adiabatic pulsation properties of hot pre–white dwarf (PWD) stars (Kawaler, Hansen, and Winget 1985, hereafter Paper I). That study indicated that the measured magnitude of $d(\ln P)/dt$ is consistent with the assumption that PG 1159−035 is a cooling PWD star. Although we were able to reproduce the observed magnitude, the theoretical $d(\ln P)/dt$ had the opposite sign. The periods of the theoretical models increased with time through the luminosity range appropriate to the pulsating PG 1159 stars.

The PWD models considered in Paper I were homogeneous pure ^{12}C, spherical, nonrotating models evolved from chemically inhomogeneous models of planetary nebula nuclei. Changes in the adiabatic pulsation periods of the high-order g-modes were found to be dependent on the thermal evolution of the zones within which the period was primarily formed, as measured by the "weight function" for the modes. The maximum of the weight function for high-order modes moved from within the degenerate core out to the outer envelope of a 0.60 $M_\odot$ model as cooling and contraction proceeded. Periods were found to increase with time for all models with luminosities of interest.

An observed period change, however, results not only from evolutionary changes in the "inertial" adiabatic pulsation properties of the star, but from all physical processes that contribute to the observed pulsation period. In particular, the pulsation periods are also affected by the star's rotation (Ledoux and Walraven 1958; Brickhill 1975; Cox 1984, and references therein). Rotational splitting has been observed in several pulsating white dwarfs, with rotation periods $P_{\rm rot}$ of the order of hours (Winget *et al.* 1982; McGraw and Robinson 1975; Kepler, Robinson, and Nather 1983). As the star contracts it rotates faster to conserve total angular momentum, leading to a change in the rotational splitting of nonradial g-modes with time.

In this paper, we calculate evolutionary period changes in hot PWD models in the presence of slow rotation. The numerical techniques used in calculating the rotational splitting of g-modes is briefly discussed in § II. Section III presents the results for a 0.60 $M_\odot$ sequence undergoing slow rotation. In § IV we discuss the observations in the context of these results.

II. ROTATIONAL SPLITTING OF g-MODES

The angular dependence of displacements for nonradial pulsations is usually expressed in terms of the set of spherical harmonics $Y_l^m(\Theta, \Phi)$, where l and m are integers representing the colatitudinal and azimuthal indices respectively for a given mode; the azimuthal index may range from l to $-l$. In nonrotating stars undergoing nonradial g-mode pulsation, eigenvalues corresponding to modes with the same l and radial wavenumber k but different values of m are degenerate; rotation lifts this m-degeneracy. Modes differing only in the value of m will be split in frequency by approximately the rotation frequency multiplied by the difference in m (cf. Cox 1984, and references therein).

In particular, for the case of slow ($P_{\rm rot} \gg P_{\rm puls}$) uniform rotation of frequency Ω, with rotation and pulsation axes aligned, the pulsation frequency $\sigma_{\rm obs}$ as observed in an inertial frame is given by

$$\sigma_{\rm obs} = \sigma_0 - m\Omega(1 - C)\ . \tag{1}$$

Here, σ_0 is the pulsation frequency in the rotating frame of the star, and C is a number that depends on the structure of the star and on its adiabatic pulsation properties. For higher order g-modes in white dwarfs, C approaches $[l(l+1)]^{-1}$ (Brickhill 1975).

Differentiating equation (1) with respect to time gives

$$\frac{d\sigma_{\rm obs}}{dt} = \frac{d\sigma_0}{dt} - m\,\frac{d\Omega}{dt}\,(1 - C) + m\Omega\,\frac{dC}{dt}\,. \tag{2}$$

If we assume that the total angular momentum of the star J is conserved, and that there is no redistribution of angular momentum within the star, then

$$dJ/dt = d(I\Omega)/dt = 0\;, \tag{3}$$

where I is the rotational inertia of the star which, like Ω, may change with time. Using equation (3) and assuming $\sigma_{\rm obs} \approx \sigma_0$, we can rewrite equation (2) in terms of periods and their time derivatives as

$$\frac{\dot{P}_{\rm obs}}{P_{\rm obs}} = \frac{\dot{P}_0}{P_0} - m\,\frac{P_0}{P_{\rm rot}}\left[(1 - C)\,\frac{\dot{I}}{I} + \dot{C}\right], \tag{4}$$

which is the desired expression for the case of uniform rotation. In equation (4), $P_{\rm rot}$ is the rotation period of the star, P_0 is the pulsation period in the case of no rotation, and the dots indicate time derivatives.

As the star evolves and cools, it will contract and become less centrally condensed; the rate of decrease in the moment of inertia is proportional to the contraction rate. Hence, in the early phases of PWD evolution, rotational spin-up can produce large changes in the observed rate of change of pulsation period for modes with $m \neq 0$. Prograde modes ($m < 0$) will have reduced values of $d(\ln P_{\rm obs})/dt$ compared to the nonrotating value, whereas retrograde ($m > 0$) modes will increase $d(\ln P_{\rm obs})/dt$. (These definitions of prograde and retrograde are based on our choice of phases: $e^{im\phi + i\sigma t}$ and $\Omega \propto \hat{\phi}$, where $\hat{\phi}$ is the unit azimuthal vector.) Modes with $m = 0$ are unaffected by slow rotation. We also note that the effect of rotation on $d(\ln P_{\rm obs})/dt$ decreases with increasing rotation period.

To calculate rotational splitting in the presence of differential rotation, we follow the approach of Hansen, Cox, and Van Horn (1977). Rotation is assumed to be cylindrically symmetric, with no angular momentum exchange between cylinders of fixed mass. The latter assumption precludes the possibility of momentum transfer by convection, etc. For each model in an evolutionary sequence, we used an angular momentum distribution representative of white dwarf configurations (Ostriker and Bodenheimer 1968) to derive the angular rotation velocity as a function of polar radius ϖ. We fit a quadratic of the form

$$\Omega(\varpi) = \Omega_0(1 + \Omega_1\varpi + \Omega_2\varpi^2) \tag{5}$$

to the Ostriker and Bodenheimer distribution, where Ω_0 is the angular frequency of rotation at the pole, and Ω_1 and Ω_2 are evaluated in a least-squares fit to the Ostriker and Bodenheimer rotation law.

With a rotation law of this form, the expression for rotational splitting becomes

$$\sigma_{\rm obs} = \sigma_0 - m\Omega_0(1 - C - C_1)\;, \tag{6}$$

where C is the uniform rotation coefficient and $C_1[= C_1(|m|)]$ contains the nonuniform rotation effects. The value of C_1 depends on the adiabatic pulsation properties, the equilibrium structure of the star, and the values of Ω_1 and Ω_2 (see Hansen, Cox, and Van Horn 1977, Appendix A); the expression for $C_1(3)$ for the $l = 3$ modes was evaluated for this paper using the method of Cuypers (1980). Note that C_1 introduces an asymmetry with respect to m into the splitting, since it depends on $|m|$ and not m. Differentiating equation (6) and transforming frequencies to periods, we obtain

$$\frac{\dot{P}_{\rm obs}}{P_{\rm obs}} = \frac{\dot{P}_0}{P_0} - m\,\frac{P_0}{P_{\rm rot}}\left[(\dot{C} + \dot{C}_1) - (1 - C - C_1)\,\frac{\dot{\Omega}_0}{\Omega_0}\right], \tag{7}$$

where $P_{\rm rot}$ is the rotation period at the pole. To self-consistently calculate the rate of period change for a sequence of models which conserves angular momentum, we desire an expression for $d(\Omega_0)/dt$ in terms of the structure of the equilibrium model. For this purpose, we take the limit, as ϖ goes to zero, of the Ostriker and Bodenheimer rotation law in the form presented in Hansen, Cox, and Van Horn (1977) to obtain

$$\Omega_0 = 1.1788\;J/(MR^2)f\,, \tag{8}$$

where

$$f \equiv \frac{1}{M}\int_0^M \frac{R^2}{r^2}\,dm_r \tag{9}$$

and m_r is the mass within a sphere of radius r. With this expression for Ω_0 in equation (8), we finally obtain

$$\frac{\dot{P}_{\rm obs}}{P_{\rm obs}} = \frac{\dot{P}_0}{P_0} - m\,\frac{P_0}{P_{\rm rot}}\left[(\dot{C} - \dot{C}_1) - (1 - C - C_1)\left(\frac{\dot{f}}{f} - 2\,\frac{\dot{R}}{R}\right)\right] \tag{10}$$

as the desired expression for $d(\ln P_{\rm obs})/dt$ for models with differential rotation.

III. RESULTS

We have calculated values of $d(\ln P_{\rm obs})/dt$ for the 0.60 $M_\odot$, pure ^{12}C evolutionary PWD sequence mentioned previously. These models and nonrotating values of $d(\ln P)/dt$ are described in detail in Paper I. The mode explicitly considered here is the g_{35} (35 radial nodes) mode. We chose this mode because it is representative of the general class of high-order gravity modes with periods appropriate for these stars. Also, for high-order g-modes ($k > 10$), we have found that the values of $d(\ln P_0)/dt$ and the rates of change of the rotation coefficients are very insensitive to the mode number. Table 1 gives the values of the rotation coefficients (C and C_1) and the various structure quantities used in the calculation of rotational splitting for the $l = 2$ g_{35} mode. The quantities in Table 1 were numerically differenced to yield the time derivatives found in equations (4) and (10). Figure 1 illustrates the rates of period change as a function of luminosity for the 0.60 $M_\odot$ evolutionary sequence, for a rotation period that was initially 5 times the pulsation period. For the model at log $(L/L_\odot) = 2.27$, Figure 2 shows the dependence of $d(\ln P_{\rm obs})/dt$ on the rotation rate for various values of m and l.

In consideration of the observed negative value of $d(\ln P)/dt$ for PG 1159−035, we have plotted the theoretical value of $d(\ln P_{\rm obs})/dt$ for negative values of m. Modes with positive values of m would show a larger positive value of $d(\ln P_{\rm obs})/dt$ than the nonrotating value. It is encouraging to note that full nonadiabatic calculations that implicitly include slow rotation seem to indicate that modes with negative m are slightly more unstable than those with positive m (Carroll and Hansen 1982; Hansen, Cox, and Carroll 1978).

Figure 1 shows that, as expected, high-luminosity models show large negative values of $d(\ln P_{\rm obs})/dt$, in prograde modes, resulting from the spin-up from contraction. As the model

TABLE 1

0.60 $M_\odot$ PRE-WHITE DWARF ROTATION PROPERITES

Age (yr)	log ($L/L_\odot$)	Period (s)	log R (cm)	I (g cm^2)	C	$C_1(1)$	$C_1(2)$	f
2.985(+3)......	3.002	447.526	9.403	3.1290(+50)	0.16401	0.28600	0.28664	53.442
4.267(+3)......	2.856	447.722	9.366	3.0708(+50)	0.16446	0.27498	0.27554	45.269
6.385(+3)......	2.705	448.969	9.332	3.0127(+50)	0.16493	0.25413	0.25462	38.894
1.142(+4)......	2.500	451.755	9.290	2.9315(+50)	0.16515	0.24029	0.24073	32.356
2.191(+4)......	2.273	454.330	9.250	2.8378(+50)	0.16537	0.22541	0.22579	27.230
5.313(+4)......	2.000	460.170	9.203	2.6955(+50)	0.16563	0.19689	0.19719	22.551
1.065(+5)......	1.776	466.837	9.168	2.5584(+50)	0.16574	0.17943	0.17969	19.764
2.507(+5)......	1.500	478.377	9.122	2.3299(+50)	0.16575	0.16252	0.16272	17.020
4.126(+5)......	1.304	490.818	9.092	2.1780(+50)	0.16575	0.14547	0.14563	15.542
7.429(+5)......	1.000	517.714	9.054	2.0071(+50)	0.16573	0.12932	0.12944	13.857

NOTE.—Parentheses enclose powers of factor 10.

approaches the constant radius phase, the rotational contribution to $d(\ln P_{\rm obs})/dt$ decreases. At luminosities appropriate to PG 1159−035 ($L \approx 100\ L_\odot$), rotation periods of a few thousand seconds in modes with pulsation periods of about 500 s produce rates of period change that are consistent with the observations. For slower rotation rates, the contribution of rotation to $d(\ln P_{\rm obs})/dt$ for modes with $m \neq 0$ is still significant.

Also indicated in Figure 1 and 2 is the value of $d(\ln P_{\rm obs})/dt$ for differential rotation. At luminosities below about 300 $L_\odot$, the difference in $d(\ln P_{\rm obs})/dt$ between the two rotation laws is small. Differential rotation slightly reduces the effect of rotation on $d(\ln P_{\rm obs})/dt$, compared with uniform rotation, in this luminosity range. This modification of $d(\ln P_{\rm obs})/dt$ tends to increase with l; in general, however, the period change due to rotational spin-up is not very sensitive to the form of the rotation law. We note, however, that these rotational splitting calculations originally assumed slow rotation ($P \ll P_{\rm rot}$); this condition is not entirely satisfied with the larger values of $P/P_{\rm rot}$ considered in Figure 2.

IV. CONCLUSIONS

We have demonstrated the importance of considering the effects of rotation on the interpretation of measurements of the rate of period change in the pulsating PG 1159 stars. For PG 1159−035 itself, the observed value of $d(\ln P)/dt$ for the 516 s period implies a rotation period of order 2000–4000 s for modes with $l = -m = 3$. This preliminary estimate corresponds to a rotation velocity of about 35–50 km s^{-1} and is consistent with the rotation velocities of DA white dwarfs as reported by Pilachowski and Milkey (1984).

We note also that the period ratios in another non–radially pulsating white dwarf, the ZZ Ceti variable L19-2, suggest that modes with l = 1–5 are present (O'Donoghue and Warner 1982). While mode identification in the power spectrum of PG 1159−035 is not at all clear, the period ratios are suggestive of l = 3 and 4. Dziembowski (1977, 1984) has shown that geometric cancellation effects greatly reduce the luminosity variations for large values of l. Hence it appears that the values of m suggested by this analysis are not unreasonable. Finally,

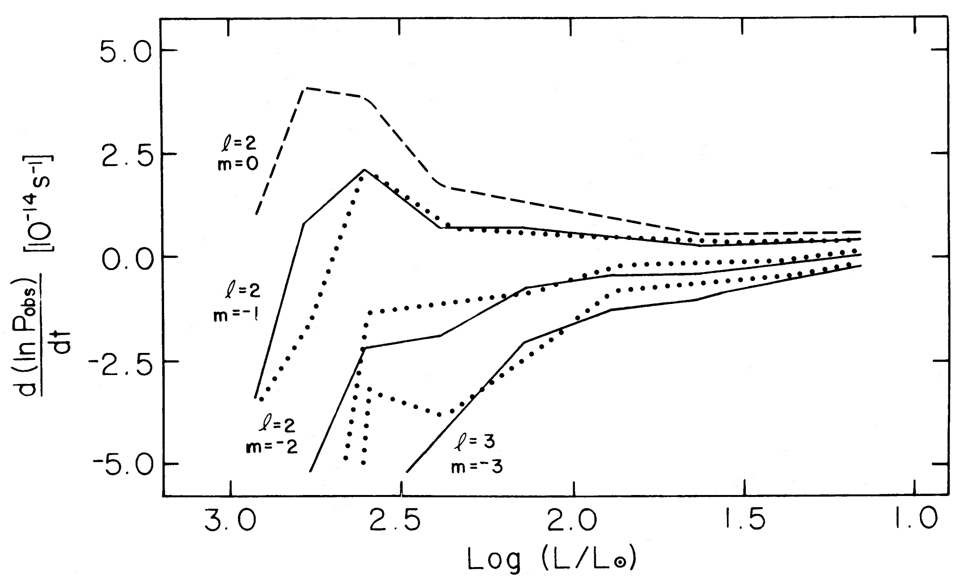

FIG. 1.—Relative rates of period change for high-order g-modes in the 0.60 $M_\odot$ model. The dashed line represents $d(\ln P_{\rm obs})/dt$ in the absence of rotation (or $m = 0$ with slow rotation). The solid lines show $d(\ln P_{\rm obs})/dt$ for the case of uniform rotation with, from top to bottom, $l = 2, m = -1$; $l = 2, m = -2$; and $l = 3, m = -3$. The dotted lines show the evolution of $d(\ln P_{\rm obs})/dt$ for the differential rotation law described in the text for the same sequence of l and m. The initial value of $P_0/P_{\rm rot}$ was 0.20 at log ($L/L_\odot$) = 3.0; the ratio had increased to 0.33 (uniform rotation) and 0.26 (differential rotation) at log ($L/L_\odot$) = 1.0 as a result of the evolutionary increase in pulsation period and decrease in rotation period.

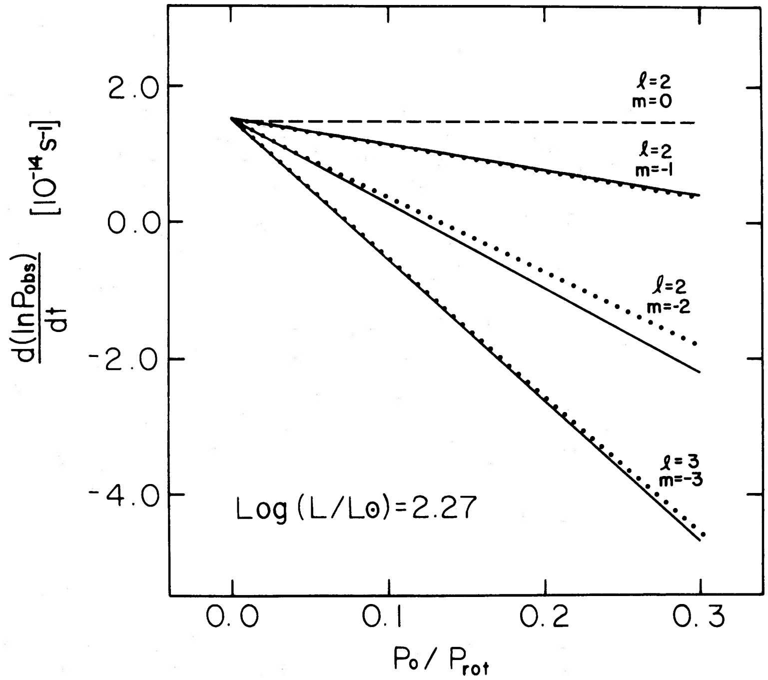

FIG. 2.—Dependence of $d(\ln P_{obs})/dt$ on rotation rate for the log $(L/L_\odot) = 2.26$ model. The horizontal dashed line gives $d(\ln P_{obs})/dt$ for the case of no rotation (or $m = 0$). Solid lines are for the case of uniform rotation with, from top to bottom, $l = 2$, $m = -1$; $l = 2$, $m = -2$; and $l = 3$, $m = -3$. The dotted lines are for differential rotation.

we note that other effects, not considered in this work, may contribute to the value of $d(\ln P_{obs})/dt$. For example, nonadiabatic contributions to the observed period may be important (as noted in Paper I).

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REFERENCES

Brickhill, A. J. 1975, *M.N.R.A.S.*, **170**, 404.
Carroll, B. W., and Hansen, C. J. 1982, *Ap. J.*, **263**, 352.
Cox, John P. 1984, *Pub. A.S.P.*, **96**, 577.
Cuypers, J. 1980, *Astr. Ap.*, **89**, 207.
Dziembowski, W. 1977, *Acta Astr.*, **27**, 203.
———. 1984, preprint.
Hansen, C. J., Cox, J. P., and Carroll, B. W. 1978, *Ap. J.*, **226**, 210.
Hansen, C. J., Cox, J. P., and Van Horn, H. M. 1977, *Ap. J.*, **217**, 151.
Kawaler, S. D., Hansen, C. J., and Winget, D. E. 1985, *Ap. J.*, **295**, 547 (Paper I).
Kepler, S. O., Robinson, E. L., and Nather, R. E. 1983, *Ap. J.*, **271**, 744.
Ledoux, P., and Walraven, T. 1958, *Handbuch der Physik*, ed. S. Flügge (Berlin: Springer-Verlag), Vol. **51**, p. 353.
McGraw, J. T., and Robinson, E. L. 1975, *Ap. J.* (*Letters*), **200**, L89.
O'Donoghue, D. E., and Warner, B. 1982, *M.N.R.A.S.*, **200**, 563.
Ostriker, J. P., and Bodenheimer, P. 1968, *Ap. J.*, **151**, 1089.
Pilachowski, C. A., and Milkey, R. W. 1984, *Pub. A.S.P.*, **96**, 821.
Starrfield, S. G., Cox, A. N., Hodson, S. W., and Pesnell, W. D. 1983, *Ap. J.* (*Letters*), **268**, L27.
Winget, D. E., Hansen, C. J., and Van Horn, H. M. 1983, *Nature*, **303**, 781.
Winget, D. E., Kepler, S. O., Robinson, E. L., Nather, R. E., and O'Donoghue, D. 1985, *Ap. J.*, **292**, 606.
Winget, D. E., Robinson, E. L., Nather, R. E., and Fontaine, G. 1982, *Ap. J.* (*Letters*), **262**, L11.

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