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UPPER AND LOWER BOUNDS OF PERIODS IN VARIABLE WHITE DWARFS

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ABSTRACT

Upper and lower critical frequencies for adiabatic radial and nonradial oscillations of white dwarfs are established. It is found that acoustic modes are strongly damped by outwardly propagating running waves in the atmosphere when periods are less than 0.1–1 s. For gravity modes, a similar result holds when periods exceed $\sim 7 \times 10^3[l(l+1)]^{-1/2}$ s, where l is the colatitudinal spherical harmonic index. Applications to observed white dwarf variable periods are discussed.

Subject headings: instabilities — stars: pulsation — stars: white dwarfs

I. INTRODUCTION AND THEORY

During the last decade it has become well established that there are at least three classes of variable white dwarfs:[1] (*a*) the ZZ Ceti variables, which are DA white dwarfs of T_{eff} ranging between 11,000 K and 13,000 K with perhaps an error of no more than 1000 K on either side (Weidemann and Koester 1984); (*b*) the recently discovered "DB" variables, whose effective temperatures are not well determined but which probably lie between 25,000 K and 30,000 K (Oke, Weidemann, and Koester 1984; Koester, Weidemann, and Vauclair 1983); (*c*) the very hot "PG 1159" objects, which may constitute a homogeneous class, although even the prototype has an uncertain T_{eff} lying between approximately 80,000 K and 150,000 K. Despite the wide spread in white dwarf evolutionary cooling history which these classes represent, they still share some common characteristics. In particular, all currently published observations indicate that their periods of variability are bounded by approximately 100 s and 1200 s and, from theoretical studies, that they represent nonradial, gravity-mode pulsations of the star excited by the same mechanisms responsible for the variability of classical Cepheids and the like. Pertinent general references for this material are Winget and Fontaine (1982), Winget *et al.* (1983), Van Horn (1985), Starrfield *et al.* (1983*b*, 1984), and references therein.

The object of this brief paper is to examine, in a preliminary way, whether theoretical upper and lower bounds may, in principle, be placed on the periods of these variables.

The motivation for this study arises from two sets of pulsational stability investigations which suggest that some of the variables are fully capable of varying with periods outside the observed range. The first set (Starrfield *et al.* 1983*a*; Saio, Winget, and Robinson 1983; see also Starrfield *et al.* 1984) concerns radial oscillations and predicts that some stars on the cooling track (notably DA white dwarfs near the blue edge of the ZZ Ceti temperature range) should oscillate with periods between 0.1 and 5 s. These periods usually correspond to modes of high overtone order. On the other hand, Robinson (1984, and see references therein) has surveyed and reviewed fast photometric observations of 19 DA white dwarfs, of which six are ZZ Ceti variables, and finds no evidence of ultrashort period variability in luminosity down to periods of 0.1–0.2 s. The upper limit of relative luminosity variation was typically 10^{-3} mag, a value considerably smaller than a typical white dwarf variable. These observations signal a significant difficulty for pulsation theory, because the methods and physical modeling used to make the predictions in the radial case are those also used in the very successful nonradial calculations (cf. Cox 1984). However, the latter are not entirely immune to this sort of difficulty either. Winget (1981; see also Winget *et al.* 1982; Dolez and Vauclair 1981) finds instabilities for modes with periods longer than 1200 s in models which are cooler than the red edge of the ZZ Ceti instability strip. These periods are longer than any observed for these stars. Both the predicted short-period radial modes and very long period nonradial modes share a common feature, however; they are concentrated in overall pulsation amplitude very near the surface of the stellar model. This suggests a possible resolution of the difficulty; namely, the surface atmospheric layers may not be able to properly reflect internal waves and, hence, the energy of a self-excited pulsation may leak out through the surface and thus eventually damp the pulsation.

This idea has been applied to the Sun (as in Ando and Osaki 1975, 1977; Mihalas and Toomre 1981, 1982) but not to white dwarfs. The major effort in this paper will be to estimate those ranges of pulsation frequency where wave leakage may occur and how important that leakage might be in damping pulsation. In the remainder of this section we develop the necessary (albeit approximate) theoretical tools. In the last section we summarize our numerical results and conclusions and also discuss, briefly, some possible observational consequences, and problems.

a) Critical Pulsation Frequencies

Our takeoff point is from Unno *et al.* (1979, § 17.1, hereafter UOAS), where the appropriate surface boundary conditions for linear, adiabatic nonradial stellar oscillations are discussed. It is assumed there that the atmosphere lying above the photosphere (at T_{eff}) is quasi-isothermal, which implies that the fol-

[1] We shall exclude such pre–white dwarf objects as the variable planetary nucleus K1-16 in this paper (Grauer and Bond 1984; Starrfield *et al.* 1985).

lowing atmospheric quantities (V_g and A^*) are essentially constant:

$$V_g = -(d \ln p/d \ln r)/\Gamma_1 . \quad (1)$$

where r is radius, p is pressure, and Γ_1 is the adiabatic index to be used in the sound speed (squared), $c^2 = \Gamma_1 p/\rho$, and ρ is the density. The quantity A^* is given by

$$A^* = -rN^2/g , \quad (2)$$

where N^2 is the square of the Brunt-Väisälä frequency and g is the local gravity. The latter may also be taken as constant above $T_{\rm eff}$. If R is the radius at $T_{\rm eff}$ and $x = r/R$, then equations (1) and (2) imply that the atmospheric density distribution is given by

$$\rho(x) = \rho(1) \exp [-(A^* + V_g)(x - 1)] . \quad (3)$$

The analysis of UOAS proceeds by using the above conditions in the linearized adiabatic pulsation equations so as to reduce them to ordinary differential equations with constant coefficients. We also do so with the simplification that perturbations of the gravitational potential are ignored (the Cowling approximation). This is more than adequate for high-order modes in white dwarfs where (again) only the surface layers have an appreciable pulsation amplitude. Forming the coefficient matrix, we find the characteristic roots

$$\lambda^{\pm} = \tfrac{1}{2}[(V_g + A^* - 2) \pm \gamma^{1/2}] , \quad (4)$$

where

$$\gamma = (A^* - V_g + 4)^2 + 4[(\omega^2 - A^*)(L^2/\omega^2 - V_g)] , \quad (5)$$

which is that given in equation (17.34) of UOAS. Here, $L^2 = l(l + 1)$, and ω is the dimensionless version of the pulsation frequency σ, i.e.,

$$\omega^2 = \sigma^2 R/g . \quad (6)$$

What equation (5) implies is that if $\gamma > 0$, then $\lambda^{\pm}$ is pure real, and solutions are represented by rising or falling exponential behavior in the atmosphere. The exponentially damped solution is that chosen for most pulsation calculations because it results in global modes trapped within the star, i.e., standing waves with perfect reflection at the photosphere. For $\gamma < 0$, however, $\lambda^{\pm}$ is complex, and running waves are possible which may carry energy out through the atmosphere.

The sign of γ is determined by the value of ω^2 (for given V_g and A^*) and, hence, so is the condition for running waves. The values of ω^2 for which $\gamma = 0$ are termed the *critical frequencies*. For V_g and A^* large and positive (we exclude $A^* < 0$, which implies a convective atmosphere) compared to L^2 and unity, these critical frequencies are given approximately by

$$\omega_p^2 = (A^* + V_g)^2/(4V_g) , \quad (7a)$$

which is the "large" root of $\gamma = 0$, and

$$\omega_g^2 = 4L^2 A^*/(A^* + V_g)^2 , \quad (7b)$$

which is the second, and small, root. The subscript $p(g)$ is used to denote a pressure, or p-mode (gravity, or g-mode) as these correspond to the high- and low-frequency solutions for nonradial modes in uncomplicated stars. Thus, if ω^2 does not lie between these two extremes, the mode has a running wave character.

It is a fairly simple matter to estimate values for the critical frequencies for, say, an Eddington gray atmosphere. With $V_g = 3g\mu R/(5N_a kT_{\rm eff})$, where μ is the mean molecular weight,

$$\omega_p^2 \approx V_g/2 , \quad (8a)$$

and

$$\omega_g^2 \approx L^2/V_g . \quad (8b)$$

For a DA white dwarf (with $T_{\rm eff} \approx 10{,}000$ K, $\mu \approx 1$, $R \approx 10^9$ cm, and $g \approx 10^8$ cm s^{-2}, we find $V_g \approx 10^5$, $\omega_p^2 \approx 5 \times 10^4$, and $\omega_g^2 \approx 10^{-5} L^2$. The corresponding critical oscillation periods are $P_p \approx 0.1$ s and $P_g \approx 6000/L$ s. For moderately low $L \approx 2$–$3(l \approx 2$–$3)$, these periods are within a factor of 2 of those discussed earlier in this section.

For purely radial modes ($l = 0$), there is only one critical frequency, and this is an upper bound for reflected waves. Its value may be simply derived (Ledoux and Walraven 1958, § 68) to yield

$$\omega_r^2 = V_g \Gamma_1^2/4 . \quad (9)$$

This corresponds to a period $P_r \approx 0.1$ s for the same gray atmosphere as before. It is not at all surprising that this period is very similar to the critical p-mode frequency, since high-order p-modes are not unlike radial pressure modes.

b) Leakage Power Loss

The critical frequencies discussed above define the limits for the reflection condition to be satisfied. Thus, for all frequencies outside these bounds there will be some mechanical power loss to the system. These mechanical power losses due to running wave leakage are relatively easy to derive. However, before we do so, it would be best to define the limitations of our analysis.

Ideally, a calculation of the oscillations of a star should explicitly include the relevant heat transport mechanisms as they couple to pulsation. This is especially true in the outer stellar layers. In the pulsation vernacular, this means that we should do a complete nonadiabatic analysis of the leakage problem. However, to our knowledge, such a fully self-consistent analysis of this type has not been reported. In addition, Mihalas (1984) has pointed out a conceptual error in the usual formulation of the interaction between stellar oscillations and the atmospheric radiation field. The consequences of this error have not yet been explored, but it raises serious doubts as to the validity of extant nonadiabatic analyses involving the outermost stellar layers.

With this in mind we propose to continue with an adiabatic analysis, using it as a suggestive tool to estimate the effects of leakage. We shall quote numerical results but treat them as being qualitative.

The instantaneous radial kinetic energy flux due to pulsations is given by (UOAS, § 26) the expression $p'v$, where p' is the Eulerian pressure variation and v is the radial component of material velocity. Both these last two quantities are assumed to vary temporally as $\exp(-i\sigma t)$. The leakage luminosity for running waves is obtained by integrating $p'v$ over a sphere of radius r and then taking the time average over one pulsation period. We denote this luminosity by $\bar{L}$. In terms of the "Dziembowski" pulsation variables y_i (see UOAS, § 17), the result is

$$\bar{L} = 2\pi R^4 g\rho\sigma \,\mathrm{Re}\,(i y_1^* y_2) , \quad (10)$$

where Re denotes "real part of" and * means "complex conjugate." Here σ is regarded as purely real even though in a fully consistent analysis involving running waves it would be complex.

The general atmospheric solutions for y_i are given in UOAS (eq. [17.36]), and we do not repeat them here. The choice of particular solution depends on whether ω^2 exceeds ω_p^2 or is less than ω_g^2. In either case we seek only outgoing waves. If $\omega^2 > \omega_p^2$ (*p*-modes), then the λ^+ root of equation (4) represents outwardly running group velocities (see UOAS). On the other hand, if $\omega^2 < \omega_g^2$ (*g*-modes), then λ^- is chosen. We use $\bar{L}^{\pm}$ for the luminosity in these two instances. Performing all the indicated operations in equation (10), while normalizing y_1 to be unity at the photosphere, we find the desired result:[2]

$$\bar{L}^{\pm} = \mp \pi g R^4 \rho \sigma |\gamma|^{1/2}/(L^2/\omega^2 - V_g) . \quad (11)$$

Note that if ω equals either of the critical frequencies, then γ and hence $\bar{L}$ vanishes. If ω lies between those frequencies, then the same result obtains because there are no running waves. Expression (11) also applies to radial modes if L^2 is set to zero and γ is replaced by $V_g(\Gamma_1^2 V_g - 4\omega^2)$.

A characteristic *e*-folding time τ_L for energy dissipation may be formed by dividing $\bar{L}$ into the total, time-averaged, kinetic energy of oscillation; that is, $\tau_L = \bar{T}/\bar{L}$. From Cox (1980, § 8.11).

$$\bar{T} = \tfrac{1}{4}\sigma^2 \int |\xi|^2 dm , \quad (12)$$

where ξ is the relative Lagrangian displacement vector. The normalization of ξ must be consistent with that chosen for the y_i in equation (10). All our numerical results for $\bar{L}$ and $\bar{T}$ are based on the normalization $y_1 = \xi_r/r = 1$ at $T_{\rm eff}$.

II. NUMERICAL RESULTS AND CONCLUSIONS

One way to gauge the possible importance of wave leakage for an unstable mode is to compare τ_L to the *e*-folding time (τ_D) for nonadiabatic driving of that mode. If τ_L is short compared to τ_D, then we might presume that leakage dominates over internal driving. This comparison is difficult to effect directly for white dwarfs, because all calculations performed to date incorporate boundary conditions at the surface which assume complete reflection of the (standing) wave even when frequencies exceed, or are less than, critical values. The following representative ranges of τ_D and $\bar{T}$ for three kinds of white dwarf variables are therefore taken from the literature (already cited) as they are and without further comment on their appropriateness. One general rule of thumb to be noted, however, is that a low value of $\bar{T}$ usually corresponds to a short, rapidly driving, time scale for τ_D. Note also that calculations of *p*-modes are not available.

For longer period unstable *g*-modes in both DA and DB white dwarfs, $\tau_D = 0.01$–10^6 yr (or longer) while $\bar{T} = 10^{42}$–10^{43} ergs. The corresponding radial mode figures are $\tau_D = 0.01$–10^3 yr and $\bar{T} = 10^{41}$–10^{44} ergs. For PG 1159–type variables, $\tau_D = 0.01$–100 yr and $\bar{T} = 10^{44}$–10^{46} ergs for *g*-modes. For radial modes, Starrfield *et al.* (1983*b*) state that τ_D is at least two to three orders of magnitude shorter than for gravity modes for PG 1159 variables. No kinetic energies are given. We note that the kinetic energies for *g*-modes and radial modes are comparable, even though radial mode frequencies are much higher. The explanation is that the *g*-mode mass motions are dominated by the transverse component of displacement which is absent for radial modes, and this makes up the difference.

The ingredients required to estimate $\bar{L}$, τ_L, and the critical frequencies have been taken from our own evolutionary models for the various classes of white dwarfs (see also Kawaler, Hansen, and Winget 1985). Representative values of critical periods are listed in Table 1 for models with effective temperatures appropriate for the three types of variables. (Note that for *g*-modes we list $P_g L$, because ω_g^2 scales directly as L^2 by explicit calculation as indicated in the approximate expression [7b]).

In order to calculate $\bar{L}$, we must specify ω^2. For this purpose we have chosen *p*, *g*, and radial periods which are 5% greater or less than their respective critical values. Thus the results we obtain for $\bar{L}$ and τ_L will be representative of conditions near critical. Table 2 summarizes these results for the same models as in Table 1 and for the $\bar{T}$ quoted earlier. Here a +, −, or *r* superscript refers respectively to a *p*-, *g*-, or radial mode. The values of $\bar{L}$ quoted may be adjusted by a factor of 2 either way because they are typical results for a number of models. The quantity $\bar{L}$ for gravity modes is for $l = 1$ but does not vary strongly with *l*.

If the entries for τ_L of Table 2 are compared to the nonadiabatic driving time scales τ_D given previously, we find the following: for gravity modes, τ_L is comparable or somewhat shorter than τ_D (remembering that both quantities vary directly with $\bar{T}$). We conclude that, for these modes, wave leakage may well compete with the intrinsic driving of the system and thus help damp long-period modes. This conclusion is somewhat weak, however, because the time scales are not dramatically different and a complete, self-consistent analysis could very well tip the balance one way or the other. On the other hand, very long period modes of frequency considerably below the critical frequency should not exist in prin-

[2] This result could also have been obtained by computing the work done per cycle on material exterior to the photosphere. See, for example, Baker and Kippenhahn (1965), or Ando and Osaki (1975). The present method was chosen because it involves the energy flux directly.

TABLE 1

REPRESENTATIVE CRITICAL PERIODS (s)

Variable	$P_g L$	P_p	P_r
ZZ Ceti (DA)	7×10^3	0.1	0.07
DB	10^4	0.1	0.08
PG 1159	5×10^3	0.8	0.8

TABLE 2

REPRESENTATIVE VALUES OF $\bar{L}$ (ergs s^{-1}) AND τ_L (yr)

Variable	$\bar{L}^-$	τ_{L^-}	$\bar{L}^+$	τ_{L^+}	$\bar{L}^r$	τ_{L^r}
DA	1×10^{35}	0.3–3	1×10^{40}	NA[a]	4×10^{40}	10^{-7} to 10^{-5}
DB	5×10^{33}	6–60	4×10^{39}	NA	2×10^{40}	NA
PG 1159	2×10^{36}	2–200	3×10^{39}	NA	1×10^{40}	NA

[a] NA, not available because kinetic energies are not available.

ciple. Thus, for example, we would be very surprised to find periods much in excess of $7000/L$ s for cool DA white dwarfs.

This result may have important consequences in resolving the outstanding theoretical problem of the red edge of the ZZ Ceti instability strip (see Winget and Fontaine 1982 for a discussion of this problem). The theoretical calculations suggest that pulsations should still be found in models with effective temperatures well below the observed red edge of the instability strip. As models cool below the blue edge, the partial ionization zone moves deeper into the star, and driving is provided for modes of lower and lower frequency. This is illustrated in the calculations of Winget (1981), who shows that the thermal time scale in the partial ionization driving zone exceeds 10^4 s by the time ZZ Ceti models have cooled 1300 K below the theoretical blue edge. Modes with periods corresponding to these long time scales, however, will not be driven to observable amplitudes because of energy leakage.

Thus, the observed narrow width of the instability strip, as well as the rough upper limit (~ 1000 s) to period, promises to be comprehensible on the basis of these fairly straightforward considerations without appeal to the *deus ex machina* of poorly understood pulsation-convection interactions. A word of caution is necessary here because of the problem of observational selection effects' decreasing the chances of detection of periods much longer than 1000 s using conventional high-speed photometric observations. Problems detecting periods longer than 1000 s arise because this is roughly the time scale on which atmospheric transparency variations occur, and periods that long are a significant fraction of one night. The observed low-frequency limit needs to be firmly established—if one exists at all—using extended coverage of individual objects from observing sites at different longitudes, as well as using filtered light, two-star, observations to reduce transparency variation effects still further; an interesting (and, perhaps, necessary) alternative would be to obtain extended observations immune to transparency and sky brightness variation problems with the Hubble Space Telescope.

For radial modes the situation is quite different. Here the leakage times are several orders of magnitude shorter than those for driving. Even with the uncertainties in our procedures, this must be telling us that very short period ($P_r < 1$ s) radial modes should not be seen—and they are not; but periods of near 1 s are not seen, either. An attractive conjecture is that, even for the latter periods, running wave leakage might be crucial for their absence, but this must await further developments. If true, we may speculate on the consequences. For example, would energy deposition in the atmosphere result in chromospheric heating and the generation of a "tight" (due to high gravities) corona? Such a restructuring of the atmosphere for some effective temperatures for some classes of white dwarfs might be observable in the UV with sensitive space instruments and might explain some of the problems in the matching of theoretical white dwarf atmospheres to observation.

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