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THE OVERSHOOT REGION AT THE BOTTOM OF THE SOLAR CONVECTION ZONE

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ABSTRACT

We have investigated the extent and thermal stratification of the region of convective overshoot underneath the convection zone of the Sun. This study has been motivated by recent suggestions that this overshoot zone may be a suitable place for magnetic flux storage during the course of the solar cycle, and that doubly diffusive instabilities may act in this region as a mechanism for breaking up diffuse magnetic fields into flux ropes. Regardless of the details of the breakup of the magnetic field, any mechanism to release magnetic flux from this region into the convection zone (and eventually to the surface) crucially depends on the strength of the thermal stratification in this overshoot layer. We have adopted a semitheoretical approach originally developed to describe the release of buoyant material in the laboratory and in the Earth's atmosphere, and we derive a set of differential equations suitable for the study of convective overshoot in an astrophysical context. We present numerical solutions of these equations and obtain simple scaling relations for the ensuing overshoot in terms of the initial velocity and the filling factor. Our results indicate the formation of an almost adiabatically stratified region underneath the convection zone proper; the region is of only moderate extent, i.e., a few tenths of a pressure scale height, if initial velocities predicted by mixing length theory are chosen. In addition, we demonstrate the existence of a stabilizing boundary layer between adiabatically and nonadiabatically stratified regions and compute an upper bound of 500 km to its thickness.

Subject headings: hydrodynamics — Sun: interior

I. INTRODUCTION

The cyclic nature of surface activity on the Sun and similar late-type stars is usually discussed in the framework of so-called mean field dynamo theories, in which the relevant field quantity is regarded as homogeneous on scales of the observed turbulent flows. High spatial resolution observations reveal, however, that at the photospheric level most and probably all magnetic flux is highly intermittent (Zwaan 1981). This phenomenon can be readily understood from magnetohydrodynamic numerical experiments, which show that in the presence of convective flows, magnetic flux tends to be swept into intercellular regions (cf. Weiss 1966, 1981*a*, *b*); thus the convection zone's storage capability for magnetic fields seems to be quite limited. Further observational and theoretical arguments have been marshaled within the past several years to support the view that the inhomogeneous magnetic flux observed on the Sun at photospheric levels is in fact rooted in the deep convection zone or below (Schmitt and Rosner 1983; Golub *et al.* 1981); more specifically, theoretical considerations on the stability of magnetic flux make the overshoot (actually "undershoot") region between the convection zone and the radiative interior a favorable place for magnetic flux storage, either in the form of flux tubes (van Ballegooijen 1982) or in the form of a diffuse field (Schmitt and Rosner 1981, 1983; Spiegel and Weiss 1980).

Unfortunately, the character of convective overshoot is one of the major unsolved problems of stellar structure and evolution theory, although a considerable number of important issues hinge on it. Thus, not only may the overshoot region in the Sun hold the key to an understanding of the dynamo process, but it may also provide the explanation of the lithium depletion and neutrino flux problem. Furthermore, gravity waves may be generated in the overshoot region (Hurlburt, Toomre, and Massaguer 1981), which (may) propagate towards the center of the Sun (Press 1981). Our lack of understanding of the phenomenon of convective overshoot is closely linked to our poor understanding of stellar convection phenomena in general; thus, mixing length theory, commonly used in stellar structure computations, does not even introduce the notion of convective overshoot.

Various attempts have been made to estimate the extent of an overshoot region underneath the convection zone proper (Shaviv and Salpeter 1973; Maeder 1975; van Ballegooijen 1982). Press (1980) distinguishes modal and model approaches, depending on whether some Fourier transformed version of the hydrodynamic equations (or equivalent) is solved or, instead, a set of model equations. Our approach will be a model one, taken from meteorological studies of convection; however, we conjecture that the extent of the overshoot region is actually not sensitively dependent on the precise nature of the assumed flows, analogous to the virtually adiabatic temperature stratification in the deep solar convection zone, which is independent of the details of the flow because of the high efficiency of convection (Biermann 1933).

The outline of this paper is as follows: In § II*a*, we discuss the phenomenon of convective overshoot in general. Sections II*b* and II*c* briefly review some of the modal and model approaches, with particular reference to our model approach, whose physical basis is discussed in detail in § II*d*. In § III we give a detailed theoretical description of the motions of plumes in a stably stratified medium, leading (in §§ III*a* and III*b*) to a "derivation" of the plume equations from the hydrodynamic equations. In § III*c* we discuss entrainment and show (in § III*d*) how the plume equations can be used to compute convective overshoot in an astrophysical context, i.e., in the Sun. In § III*e* we discuss the limitations of the plume model as applied to convective overshoot in the Sun, arguing (in the following § III*f*) that a thin (i.e., <500 km) boundary layer must exist which separates convective and radiative regions. The results of numerical integrations of the plume equations, as applied to the region of convective overshoot underneath the solar convection zone, are given in § IV; § V contains a discussion and our conclusions.

II. CONVECTION AND CONVECTIVE OVERSHOOT

In this section, we discuss the mean stratification of the boundary layer separating the convection zone proper and the radiative interior and examine previous work on calculating the mean stratification relevant to our plume model.

a) Mean Stratification

Most calculations of stellar structure employ the familiar Schwarzschild criterion (see Cox and Giuli 1968) to distinguish between convective or radiative equilibrium. If the criterion is satified, mixing length theory is used to determine the actual logarithmic temperature gradient ∇; this gradient depends on the adiabatic (logarithmic) temperature gradient $\nabla_{\rm ad}$, which can be calculated from thermodynamics, and the convective efficiency Γ, defined to be the ratio of excess thermal energy contained in a convecting eddy and the energy lost by radiation during its lifetime (Cox and Giuli 1968). For the lower part of the solar convection zone, mixing length models predict superadiabatic gradients $\nabla - \nabla_{\rm ad}$ of the order 10^{-7} to 10^{-8}, i.e., convection is extremely efficient. This result is a consequence of the assumption that the sizes of the energy-carrying eddies are of the order of the pressure scale height ($\sim 50{,}000$ km in the deep convection zone), so that radiative losses become unimportant.

Consider now the run of the mean logarithmic temperature gradient ∇ versus logarithmic pressure at the bottom of the solar convection zone (Fig. 1). Since the matter is almost

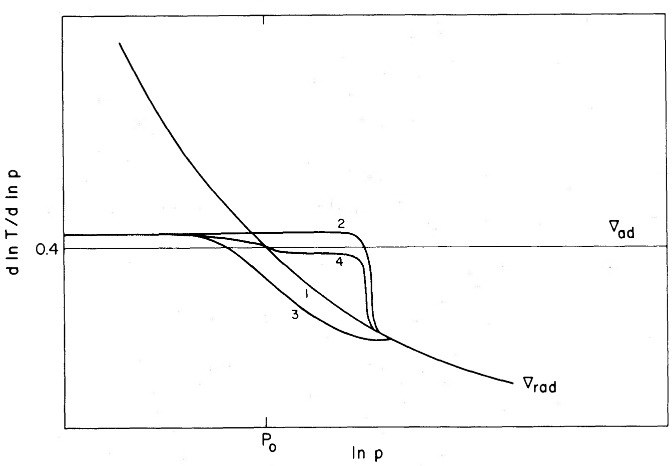

FIG. 1.—Sketch of the possible runs of the (logarithmic) temperature gradient vs. the logarithm of pressure at the bottom of the convection zone. P_0 is the location where the convective flux vanishes (but not the convective velocity); only track 4 is physically acceptable (see discussion in text).

fully ionized, the adiabatic gradient in this region is practically constant; however, the so-called fictitious radiative temperature gradient $\nabla_{\rm rad}$, i.e., this temperature gradient at which the total flux $F_{\rm tot}$ could be solely carried by radiation,

$$\nabla_{\rm rad} = F_{\rm tot} \frac{3\kappa\rho}{4acT^3} \qquad (2.1)$$

(a denotes the radiation density constant, c the speed of light, T the temperature, ρ the density), drops considerably because of the temperature sensitivity of the Rosseland opacity κ. At some point, say P_0, $\nabla_{\rm ad} = \nabla_{\rm rad}$ must hold. At lower pressure, i.e., to the left of P_0 in Figure 1, the actual temperature gradient ∇ will be slightly superadiabatic, and for sufficiently large pressures (depths), all convective motions will have ceased, and the temperature gradient will be purely radiative, $\nabla = \nabla_{\rm rad}$. In Figure 1 we have schematically indicated several possible trajectories in the ∇–log P plane which join these two asymptotic limits across P_0, which we will now discuss in turn.

Track 1 is chosen by mixing length treatments; here the superadiabatic gradient becomes zero exactly at P_0, where $\nabla = \nabla_{\rm ad} = \nabla_{\rm rad}$ holds. In this theory, convective velocities v_c are purely locally determined through the superadiabatic gradient [$v_c \sim (\nabla - \nabla_{\rm ad})^{1/2}$], and thus vanish exactly at P_0. This is of course quite unsatisfactory since convecting eddies sinking from a location above P_0 will not be stopped abruptly there and are hence expected to penetrate beyond P_0 and thus modify the temperature gradient in this region.

On track 2 the Schwarzschild criterion is satisfied in regions spatially below P_0, so that the medium is (locally) unstable. However, in general we expect the value of the actual gradient ∇ to lie between the radiative gradient $\nabla_{\rm rad}$ and the adiabatic gradient $\nabla_{\rm ad}$, which is not the case on this track. This unphysical behavior leads us to reject track 2.

On track 3 the inequalities $\nabla < \nabla_{\rm rad} < \nabla_{\rm ad}$ hold spatially below P_0, which again is unphysical. Furthermore, convecting eddies cannot move adiabatically on track 3, because they would then become hotter than their mean surroundings, and hence contribute a negative (instead of a positive) convective flux. In the language of mixing-length theory, convection would have to turn from being extremely efficient in the convection zone proper to being extremely inefficient in the overshoot zone. This also seems quite implausible, and we therefore reject track 3.

On track 4, $\nabla > \nabla_{\rm rad}$, but $\nabla < \nabla_{\rm ad}$ below P_0, i.e., the temperature gradient is subadiabatic but can be arbitrarily close to the adiabatic trajectory; any change in convective efficiency is therefore small, and the convective flux is negative. Penetrating eddies will become hotter than their surroundings and be subject to buoyancy braking, so that the temperature gradient becomes radiative. We thus regard track 4 as the only physically acceptable trajectory.

Denoting the position (pressure) where the superadiabatic gradient changes sign by P_δ, the position where $\nabla_{\rm ad} = \nabla_{\rm rad}$ by P_ϵ, the position where the convective flux changes sign by P_T and, finally, the position where convection ceases by P_v, we therefore expect the ordering

$$P_\delta < P_\epsilon < P_T < P_v \, , \qquad (2.2)$$

which is in fact precisely what Shaviv and Salpeter (1973) found in their simple extension of mixing length models. It appears that the arguments presented above have general validity as long as largest eddies can be treated adiabatically;

since the thermal microscale in the lower part of the solar convection zone is of the order of a few kilometers, eddies may be considerably smaller than a pressure scale height without losing their adiabaticity.

b) *Modal Approaches*

Two distinct methodologies have emerged for dealing with problems of convection and convective overshoot, the so-called modal and model approaches (Press 1980). In the modal approach, one basically tries to solve the hydrodynamic equations in Fourier space; the mathematical difficulties are enormous, since all scales (i.e., modes) are coupled due to the nonlinearity of the hydrodynamic equations. It is therefore usually necessary to make simplifying assumptions (such as the anelastic approximation, single-mode approximation, etc.). Although much progress in theoretical development has been made over the past few years (Marcus, Press, and Teukolsky 1983), relatively little has been accomplished in terms of application to actual astrophysical systems.

The only exception seems to be the work of Latour, Toomre, and collaborators (Latour *et al.* 1976; Toomre *et al.* 1976; Latour, Toomre, and Zahn 1981), who studied convection in A type dwarf stars. For these stars standard mixing length theory predicts two distinct convection zones, a helium convection zone lying underneath a hydrogen convection zone, separated by a convectively stable region of about one pressure scale height in extent. Latour, Toomre, and Zahn (1981) used the anelastic single-mode approximation, i.e., they considered only one, arbitrarily imposed, horizontal scale length; this is done both because of mathematical convenience *and* because of the argument that "turbulent convection under a broad gamut of circumstances has a cellular appearance." The results of such an approach pertaining to convective overshoot may have to be taken with caution (Marcus, Press, and Teukolsky 1983), but we nevertheless will choose Latour, Toomre, and Zahn's (1981) results as exemplifying this class of calculations in our discussion.

Latour, Toomre, and Zahn (1981) resolve the ambiguity in the choice of the horizontal scale length by choosing the hexagonal planforms that maximize the convective heat transport; they then consider cells with central upward flow and peripheral downward flow ("up-solutions") and vice versa ("down-solutions"). The "down-solutions" most resemble the plume scenario we have in mind and which we will explore in the following sections; therefore, we focus attention on them. These solutions have the following properties, as shown in Figures 2 and 3 of Latour, Toomre, and Zahn (1981): underneath the second (i.e., helium) convection zone, the convective flux, as well as the temperature fluctuations, are negative; the pressure fluctuations are small, and therefore density fluctuations have the opposite sign, but almost the same magnitude, as temperature fluctuations. Consequently, most of the work is not done by pressure fluctuations, but by buoyancy, and the presure and mechanical fluxes are small when compared to the convective flux in this region. As Latour, Toomre, and Zahn (1981) write, "the overall dynamics is largely controlled by buoyancy driving in the second zone and buoyancy braking below it, with work by the pressure and viscous terms having lesser roles."

As we will see in § III*a*, our plume model captures all these essential features that characterize Latour, Toomre, and Zahn's (1981) nonlinear "down-solutions" of the anelastic, single-mode approximation.

c) *Model Approaches*

In the model approach, one does not attempt to solve the full hydrodynamic equations, but is instead content with solving a set of model equations which are thought to capture the essential physics of the underlying, complex hydrodynamic phenomenon. A model that generalizes traditional mixing-length theory to include convective overshoot is that of Shaviv and Salpeter (1973); because, as we shall show, our model, although based on a rather different physical picture, formally reduces to a similar set of equations in the case of no entrainment, we discuss their approach in somewhat more detail.

i) *The Approach of Shaviv and Salpeter* (*1973*)

Shaviv and Salpeter (1973) adopt the mixing-length view of convective transport and thus think in terms of blobs of material driven by buoyancy. A blob, or bubble, originating at some point r_1 will acquire some excess temperature $\delta T(r_2, r_1)$ and velocity $v(r_2, r_1)$ at some level r_2, given by

$$\Delta T(r_2, r_1) = -\int_{r_1}^{r_2} dr \left(\frac{dT}{dr}\bigg|_{\text{actual}} - \frac{dT}{dr}\bigg|_{\text{ad}} \right) \tag{2.3}$$

$$v^2(r_1, r_2) = 2 \int_{r_1}^{r_2} dr \, \frac{g(r)}{T(r)} \Delta T(r, r_1) \, . \tag{2.4}$$

They choose a maximum distance l a blob can travel without losing its identity, i.e., a mixing length, and write the resulting convective flux as

$$F_{\text{conv}}(r) = f c_p \, \rho(r) v(r, r - l) \Delta T(r, r - l) \, , \tag{2.5}$$

with c_p the specific heat at constant pressure and f the (area) filling factor of the bubbles.

Finally, with the constraint of the flux constancy condition

$$F_{\text{tot}}(r) = F_{\text{rad}}(r) + F_{\text{conv}}(r) \, , \tag{2.6}$$

Shaviv and Salpeter (1973) show that equations (2.3)–(2.6) can be solved iteratively once the superadiabatic gradient is known over one mixing length. Interestingly enough, all their solutions were similar to track 4 and obey the ordering of equation (2.2), when one takes into account the fact that they studied a core, rather than a surface, convection zone.

ii) *The Approach of van Ballegooijen* (*1982*)

A different approach to convective overshoot has been pursued by van Ballegooijen (1982). He considers expressions for the velocity amplitudes u_z, u_x of the vertical and horizontal velocity components of the form

$$\boldsymbol{v}(x, z, t) = [-iu_x(z), u_z(z)] \exp(ikx + \gamma t) \, , \tag{2.7}$$

where k (the horizontal wavenumber) and γ (a linear growth rate) are both treated as free parameters. Together with similar expressions for temperature T and entropy S, e.g., $T = T_0 + T_1(z) \exp(ikx + \gamma t)$, and $S = S_0 + S_1(z) \exp(ikx + \gamma t)$, he then derives a single ordinary differential equation for the linear, adiabatic evolution of these modes of the form

$$\frac{d^2}{dz^2}(\rho u_z) + \frac{3}{5}\frac{g\rho}{p}\frac{d}{dz}(\rho u_z) - (\rho u_z)\left[\left(\frac{k}{\gamma}\right)^2 \frac{g}{c_p}\frac{dS}{dz} + k^2\right] = 0 \, . \tag{2.8}$$

Here planar geometry, full ionization, and constant molecular weight have been assumed; also note that in our coordinate system, gravity points in the negative z-direction and energy fluxes are positive when directed in the positive z-direction.

The entropy gradient is then related by van Ballegooijen (1982) to the convective flux through a diffusion-like equation

$$F_{\rm conv} = -\rho T D \frac{dS}{dz}, \quad (2.9)$$

with the diffusion coefficient D given by

$$D = u_z{}^2 \tau, \quad (2.10)$$

with $\tau\,(=\gamma^{-1})$ the characteristic convective turnover time scale. Using the flux constancy condition (2.6) and the diffusion approximation (2.1) for the radiative flux, the entropy gradient can then be expressed as

$$\frac{dS}{dz} = \frac{gK - c_p F_{\rm tot}}{TK + \rho c_p TD}, \quad (2.11)$$

with

$$K = \frac{4}{3}\frac{acT^3}{\kappa\rho}. \quad (2.12)$$

With this substitution for dS/dz, equation (2.8) becomes an ordinary second-order differential equation for the vertical mass flow, once the temperature and density stratification are known. Instead of using depth (scaled by the pressure scale height) as van Ballegooijen (1982), we use the logarithmic pressure as independent variable and obtain

$$y'' + y' \log\left(\frac{g\rho}{p}\right)' - \frac{3}{5}y' + y\left(\frac{kp}{g\rho}\right)^2\left[\left(\frac{5g}{3a\gamma}\right)^2\frac{1}{c_p}S' - 1\right] = 0 \quad (2.13)$$

(a is the local sound speed), where we set

$$y \equiv \rho u_z, \quad (\;)' = \frac{d}{\ln p} = \frac{d}{dx}. \quad (2.14)$$

and note that

$$S' = -\frac{2c_p}{5}\frac{K - (c_p/g)F_{\rm tot}}{K + \rho c_p D}, \quad \frac{S'}{c_p} = \nabla - \nabla_{\rm ad}. \quad (2.15)$$

Apart from boundary values for pressure, density, temperature, gravity, and total flux to be specified, say, at the top of the overshoot layer, there are three independent model parameters: a characteristic velocity u_z, an instability time scale τ, and a wave number k.

For the special case $S' = 0$, i.e., adiabatic stratification, which van Ballegooijen assumes in order to calculate the temperature and density stratification, equation (2.13) reduces to

$$y'' - y' - y\exp\left(\tfrac{4}{5}x\right)\epsilon^2 = 0, \quad (2.16)$$

where

$$\epsilon = \frac{kp}{g\rho} \quad (2.17)$$

is a constant to be evaluated at the top of the overshoot region. The transformation

$$f = y\exp\left(\frac{x}{2}\right), \quad \eta = \frac{5}{2}\epsilon\exp\left(\frac{2x}{5}\right) \quad (2.18)$$

finally results in the equation

$$\frac{d^2f}{d\eta^2}\eta^2 + \frac{df}{d\eta}\eta - f\left[\eta^2 + \left(\frac{5}{4}\right)^2\right] = 0, \quad (2.19)$$

whose solutions are modified Bessel functions of the first and second kind with index $\nu = 5/4$.

Unfortunately, equation (2.19) is not likely to have much physical significance: although the flows in the overshoot zone are supposed to be influenced by buoyancy braking, it is exactly the braking term $\sim S'$ that has been omitted in equation (2.19). Let us therefore focus attention on the expression for the entropy gradient S' in equation (2.15). Since the top of the overshoot layer is defined by the condition $K = F_{\rm tot}\,c_p/g$, we expect $K \sim O(F_{\rm tot}\,c_p/g)$ throughout the overshoot layer. On the other hand, we find for the ratio of the two terms in the denominator of equation (2.15)

$$\frac{\rho c_p D}{K} \sim 1.6 \times 10^7\left(\frac{u}{8 \times 10^3}\right)^2\left(\frac{\tau}{\rm months}\right), \quad (2.20)$$

if typical values at the bottom of the convection zone are used ($p \sim 6.7 \times 10^{13}$ dyn cm^{-2}, $T \sim 2.2 \times 10^6$ K, $\mu = 0.67$, $\rho \sim 0.24$ g cm^{-3}, $K \sim 7.6 \times 10^{14}$ g cm s^{-3} K^{-1}). Therefore, equation (2.15) immediately implies that the subadiabatic gradient remains very small (i.e., $\ll 1$) throughout most of the overshoot layer until vertical velocities become of the order of a few cm s^{-1}. Since the temperature gradient is thus almost adiabatic, the radiative flux exceeds the total flux, and the convective flux must equal the negative of the excess flux; the rapid increase of the absolute value of the convective flux almost through the entire overshoot layer (see van Ballegooijen's Fig. 3*a*) comes therefore as no surprise.

The transition from adiabatic to radiative temperature gradient requires velocities u of a few cm s^{-1}; with these velocities, typical length scales l are

$$l \sim u\tau \sim 50\frac{\tau}{\rm months}\ {\rm km}, \quad (2.21)$$

or roughly 10^{-3} pressure scale heights. Since l is of the order of the thickness of the boundary layer at the bottom of the overshoot zone (as asserted by van Ballegooijen 1982), the model can obviously *not* be employed to compute the thickness of this boundary layer.

Finally, we note that a further consistency condition must be imposed on the free parameters entering equation (2.13): if the stratification, i.e., the superadiabatic gradient, is to affect the vertical motions, we expect, from equation (2.13),

$$\left(\frac{5g}{3a\gamma}\right)^2\frac{1}{c_p}S' > 1. \quad (2.22)$$

Using equations (2.9) and (2.10), this can be rewritten as

$$\tau > \frac{\rho u^2 a^2}{gF_{\rm conv}}. \quad (2.23)$$

Setting $F_{\rm conv} \sim F_{\rm tot}$ and $u \sim 8 \times 10^3$ cm s^{-1}, we obtain $\tau > 10^6$ s ~ 12 days; this is perfectly reasonable, because the turnover times of the largest eddies in the deep convection zone are believed to be of the order of a solar day. On the other hand, evaluating van Ballegooijen's (1982) adopted parameters (say, in his Fig. 3*a*) yields $\tau \sim 3 \times 10^5$ s; therefore he simply integrates equation (2.19) over most of the overshoot region. In fact, from the plot of the superadiabatic gradient (Fig. 3*a* in van Ballgooijen 1982) one can easily convince oneself that, as long as $\delta < 10^{-6}$ (which is the case for the bulk of the overshoot region), the constants dominate over the terms proportional to the superadiabatic gradient in the terms proportional to ρu_z in the equations (2.8) and (2.13).

d) Modeling Convective Overshoot with Plumes

The shortcomings of traditional mixing-length theory are well known and have been widely discussed in the literature. For example, Moore (1981) points out that the predicted spectrum of eddies, ranging in size from ~ 500 km to $\sim 50{,}000$ km (the pressure scale heights at top and bottom of the convection zone), are not observed; only two scale sizes are evident, granulation and supergranulation, and the observational evidence for so-called giant cells is weak at best. On the other hand, numerical experiments tend to produce cells sizes dependent on the computational domain, and it is controversial whether this is due to the imposed idealized initial or boundary conditions, inadequate horizontal grid resolution, the approximation used in deriving the underlying equations of motions, or "real physics."

Before abandoning mixing-length theory entirely, it may be appropriate to first carefully examine what assumptions our beliefs about the stratification of the deep solar convection zone are based upon.

What is the dominant scale of motion?—In the outer part of the convection zone, the predicted mixing-length sizes become small, and convective velocities and departures from adiabaticity large, whereas in the deep convection zone, large scale sizes imply small convective velocities and small superadiabaticity. Since mixing-length theory scales all relevant length scales with the pressure scale height, which becomes large at large depths, the conclusion is inevitable that convection is efficient, and departures from adiabaticity unimportant for deep temperature stratification. Note, however, that the thermal microscale at the bottom of the solar convection zone, i.e., the scale on which cooling and turnover time are the same, is of the order 1 km, roughly six orders of magnitude smaller than the corresponding density scale height; therefore, even if the size of the energy-carrying eddies is considerably smaller than that assumed by mixing length theory, or if instead an entire spectrum of energy-carrying eddies is present, we do not expect this conclusion to change. Of course, questions such as how far a Kolmogorov cascade (inertial range) extends in wavenumber space, what the spectrum of eddies as a function of height is, and so forth, can obviously not be answered with mixing-length theory, but require a detailed solution of the hydrodynamic equations, as discussed by Marcus, Press, and Teukolsky (1983).

Are there coherent motions in large Reynolds number flows?—It is evident that the largest scales are most important for energy transport; unfortunately, we only have theoretical suggestions as to what the nature of the flows on these scales in the deep solar convection zone ought to be. One manifestation of these uncertainties is the fact that in both laboratory and meteorological measurements, highly ordered flows are encountered even though the large computed Reynolds numbers would indicate highly turbulent flows. For example, in icewater convection experiments (Adrian 1975), "intermittent narrow plumes that extended across the convection layer were the prevalent structure," and Adrian (1975) concludes that "it may be very feasible to model many of the properties of convection in water over ice (and other types of convection) by essentially neglecting its turbulent character and modelling only the low wavenumber motions." Similarly, in the meteorological context it has long been known that in the planetary boundary layer under conditions of a large vertical heat flux, i.e., on a sunny day, free convection occurs in form of plumes and thermals, that is, in the form of *coherent* motions (Wallace and Hobbs 1977).

The appearance of coherent motions under high Reynolds number conditions is also found in the numerical calculations of Hurlburt, Toomre, and Massaguer (1981) of fully compressible model convection. These authors consider a convectively unstable region overlying a convectively stable region, as is the case at the bottom of the solar convection zone; the computational domain spans about a factor of 50 in density. Hurlburt, Toomre, and Massaguer (1981) do not find a succession of rolls, but rather find that the flows occur "in prominent downward directed plumes surrounded by broader regions of upflow." Considerable overshoot into the stably stratified region and ensuing mixing is also found.

Why plumes?—Plumes may indeed be a natural consequence of the compressibility of astrophysical convective flows (Massaguer and Zahn 1980; Zahn 1980). Consider the lower boundary of a series of overturning eddies (cf. Fig. 1 in Zahn 1980): In regions of converging flow the flow pressure fluctuations are negative and the flow is predominantly upward, whereas in regions of diverging flow the flow is predominantly downward and the pressure fluctuations are positive. Since the downward flow presumably has negative temperature fluctuations, its positive density fluctuations (and thus antibuoyancy) are *enhanced* over what one would expect in the Boussinesq approximation (which ignores pressure fluctuations). Thus it is hardly surprising that downward moving fluid is antibuoyant and overshoots into an adjacent stably stratified region.

For these various reasons, it seemed worthwhile to us to explore the astrophysical application of a phenomenological model that successfully describes plumelike motions in the laboratory and in the meteorological context, although it cannot be rigorously derived from the hydrodynamic equations. The plume equations have been introduced by Morton, Taylor, and Turner (1956), who referred to them as the conservation equations for volume, momentum, and density deficiency, in an attempt to provide a theoretical framework for laboratory studies of release of buoyant material into incompressible fluids. They also considered thermals and showed that the equations governing the rise of plumes and thermals are very similar.

Since then, Morton, Taylor, and Turner's (1956) theory has been applied, in modified form, to a variety of problems, including studies of pollutant emission into the atmosphere (Fay 1973) and the scatology of wastewater in the ocean (Koh and Brooks 1975); comprehensive reviews of the subject have been given by Turner (1969, 1973) and List (1982). The only application of the original Morton, Taylor, and Turner (1956) theory in the astrophysical context seems to be due to Moore (1967), who used their equations to study convective overshoot (!) into the photosphere and chromosphere.

Because the plume equations are not widely known in the general astrophysical literature, and because there is also some confusion as to what the correct equations of motion ought to be (Schatzmann 1979), we present our version of the plume equations in § III and show how they can be "derived" from the hydrodynamic equations.

III. PLUME THEORY AND ITS APPLICATION TO CONVECTIVE UNDERSHOOT

In this section, we consider the equations of motion for buoyant plumes (§§ III*a* and III*b*), discuss the entrainment problem (§ III*c*), and show how the plume equations may be applied to the convective undershoot problem in the Sun (§ III*d*).

a) *Plume Equations*

For simplicity, we will assume planar geometry, with gravity pointing in the $-x_3$ direction and neglect (for the moment) rotation and magnetic fields. The stationary hydrodynamic equations are then given by

$$\frac{\partial}{\partial x_i}(\rho v_i) = 0\,, \tag{3.1}$$

$$\rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho g \delta_{i3} + \text{viscous terms}\,, \tag{3.2}$$

$$\rho T v_j \frac{\partial S}{\partial x_j} = \text{viscous + heat conduction terms}\,, \tag{3.3}$$

where ρ, T, p, g, and S denote density, temperature, pressure, gravity, and specific entropy, respectively; v_i denotes the ith component of velocity and δ_{i3} is the usual Kronecker symbol.

With no flows present, only the x_3 component of the Navier-Stokes equation is nontrivial:

$$\frac{\partial p}{\partial x_i} = -g\rho_0\,, \tag{3.4}$$

i.e., the equation of hydrostatic equilibrium. (The subscript "0" in eq. [3.4] and thereafter refers to the stratification of the ambient medium.)

The concept of plumes obviously requires us to think in terms of regions with and without flows; let us therefore suppose that in the flow-free regions the runs of temperature, density, and pressure as functions of height x_3 are given as T_0, ρ_0, and p_0. Subtracting equation (3.4) from equation (3.2), we then obtain

$$v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p_r}{\partial x_i} - g\delta_{i3}\frac{\rho - \rho_0}{\rho} + \text{viscous terms}\,, \tag{3.5}$$

where we have introduced the reduced pressure

$$p_r = p - p_0\,. \tag{3.6}$$

If we furthermore assume the flow within the plume to be axisymmetric, the r and z components of the Navier-Stokes equations become

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p_r}{\partial r} + \text{viscous terms}\,, \tag{3.7}$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p_r}{\partial z} - g\delta_{i3}\frac{\rho - \rho_0}{\rho} + \text{viscous terms}\,. \tag{3.8}$$

Similarly, we write the equation of continuity as

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{\partial v_z}{\partial z} = -v_z \frac{\partial \ln \rho_0}{\partial z}\,, \tag{3.9}$$

where we replace $\rho(r, z)$ by $\rho_0(z)$ in the buoyancy term; this replacement basically amounts to a Boussinesq approximation.

The basic idea in deriving the plume equations consists in integrating the continuity equation and the Navier-Stokes equation over a horizontal cross section of a given plume, i.e., from zero out to some radius b, which is assumed to be a function of height. The effect of turbulent diffusion at the plume boundary is only taken into account through an entrainment hypothesis, and the reduced pressure is neglected altogether. The neglect of pressure perturbations may be justified in the light of the discussion in § II*b*, but the entrainment assumption can only be justified *a posteriori*. From the continuity equation we then obtain:

$$2\pi r v_r \bigg|_{r=b(z)} + \frac{d}{dz}\int_0^{b(z)} dr 2\pi r v_z - 2\pi r v_z \frac{db(z)}{dz}\bigg|_{r=b(z)} = -\left\{\frac{d}{dz}\ln\left[\rho_0(z)\right]\right\}\int_0^{b(z)} dr 2\pi r v_z\,. \tag{3.10}$$

The integral appearing on the right-hand side of equation (3.10) is simply the volume flow through the plume at height z; similarly, the first term on the left-hand side is the radial volume flow into the plume at the same height z. Since we assumed the flows to be turbulent, this quantity need not be zero, because the plume may entrain material from outside, i.e., "quiescent," regions.

According to Taylor's hypothesis (Taylor 1945) we can express this radial volume flow with the help of the vertical component of velocity as

$$2\pi r v_r \bigg|_{r=b(z)} = -2\pi w \alpha v_2(r = 0, z)\,, \tag{3.11}$$

where w is the effective plume width [$\sim b(z)$ to be precisely defined later] and α is a dimensionless entrainment function; Morton, Taylor, and Turner (1956) choose α as a constant, but more complicated entrainment functions can be found (cf. § III*c*). The hope underlying equation (3.11) of course is that the plume dynamics be not sensitively dependent on the complicated details of turbulent mixing and entrainment; the experiments suggest that this is the case, at least in zeroth order.

If we furthermore assume that vertical flows vanish at the plume boundary, we can drop the third term on the left-hand side of equation (3.10) and rewrite equation (3.10) as

$$-2\pi r v_r \rho_0 \bigg|_{r=b(z)} = \frac{dM}{dz}\,, \tag{3.12}$$

where M is the mass flow through the plume defined as

$$M = \int_0^{b(z)} dr 2\pi r v_z \rho_0\,. \tag{3.13}$$

Let us now focus attention on the z-component of the Navier-Stokes equation, where the buoyancy terms appear. First we manipulate the equation by multiplying by ρ_0, using the continuity equation (3.9), and neglecting the reduced pressure p_r to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 v_r v_z) + \frac{\partial}{\partial z}(\rho_0 {v_z}^2) = -g(\rho - \rho_0)\,, \tag{3.14}$$

where we put, in the spirit of the Boussinesq approximation, $\rho_0/\rho \equiv 1$, but $\rho_0 - \rho \neq 0$. If we now integrate over a plume cross section as before and again assume that v_z vanishes at the plume boundary, we obtain the simple equation

$$\frac{d}{dz}P_{\text{mom}} = F_B\,, \tag{3.15}$$

where P_{mom} is the momentum flow through the plume at depth z, defined as

$$P_{mom} = \int_0^{b(z)} dr 2\pi r \rho_0 v_z^2 , \quad (3.16)$$

and F_B is the buoyancy force per unit length along the plume, defined as

$$F_B = \int_0^{b(z)} dr 2\pi r g(\rho_0 - \rho) . \quad (3.17)$$

Strictly speaking, we could have written down equation (3.15) immediately, because it expresses nothing else than Newton's Second Law.

b) Application of Plume Equations to Stellar Interiors

In the astrophysical context, there is the complication that one specifies the total energy flux in the plume (in order to guarantee steady state energy flux conservation); in contrast, the total energy flux is initially unknown (and indeed is to be determined) in meteorological applications. As usual in stellar structure calculations, we only seek to determine the stationary mean stratification and can therefore impose the flux constancy condition (2.6) through $F_{tot} \sim$ constant (for planar geometry). Thus both F_{tot} and F_{rad} (in the diffusion approximation; cf., eq. [2.1]) are purely locally determined, and so the mean convective flux must also be locally determined from equation (2.6) to guarantee energy conservation. For the convective flux we will adopt the simple expression

$$F_{conv} = \langle \rho v_z T \Delta S \rangle_{mean} , \quad (3.18)$$

which is very similar to equation (2.5), i.e., we neglect the contributions of pressure and mechanical flux toward the convective flux. The first assumption is consistent with our assumption of pressure equilibrium across the plume; the mechanical flux at the bottom of the solar convection zone is indeed likely to be quite small because of the low convective velocities in this region. Note that we also require

$$\langle \rho v_z \rangle_{mean} = 0 , \quad (3.19)$$

i.e., conservation of mass; this means that the rapid downward mass motions within a plume must be compensated by a slow upward motion in the environment, which is not explicitly taken into account by the plume equations themselves.

The convective flux, written in the form of equation (3.18), can obviously be related to the energy equation, which we use in its stationary form without dissipative processes. Then the (specific) entropy in the plume flow must obey

$$v_j \frac{\partial S_{pl}}{\partial x_j} = 0 , \quad (3.20)$$

which can be rewritten, using the continuity equation (3.9), to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r \rho_0 S_{pl}) + \frac{\partial}{\partial z}(v_z \rho_0 S_{pl}) = 0 . \quad (3.21)$$

The ambient medium will be characterized by some mean entropy stratification S_0, given by

$$S_0 \sim \log (p_0 \rho_0^{-\gamma}) . \quad (3.22)$$

Thus, we obtain from equation (3.21)

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r \rho_0 \Delta S) + \frac{\partial}{\partial z}(v_z \rho_0 \Delta S) = -v_z \rho_0 \frac{\partial S_0}{\partial z} ; \quad (3.23)$$

the quantity $\Delta S = S_{pl} - S_0$ (S_{pl} is the specific entropy inside the plume) is the entropy excess of the plume with respect to the environment that gives rise to convective energy transport or, since $\Delta S \propto \Delta T \propto \Delta \rho$, to a buoyancy flux (in the language of meteorology). The energy equation is now in a suitable form for the integration over horizontal plume cross sections. We find that

$$2\pi \rho_0 v_r r \Delta S \Big|_{r=b(z)} + \frac{d}{dz}\int_0^{b(z)} dr 2\pi r \rho_0 v_z \Delta S - \frac{db(z)}{dz}\rho_0 v_z \Delta S = -\int_0^{b(z)} dr 2\pi r v_z \rho_0 \frac{dS_0}{dz} = -\frac{dS_0}{dz} M . \quad (3.24)$$

We again neglect the third term on the left-hand side (because $v_z \to 0$ on the boundary) and recognize that the second term on the left-hand side is simply the derivative of the convective flux through the plume. The first term on the right-hand side is usually neglected on the grounds that the temperature should be a continuous function across the interface between plume and ambient medium. This is inconsistent with the entrainment assumption. According to Taylor's hypothesis, the quantity $2\pi r v_r \rho_0$ is regarded as a radial mass flux into the plume due to entrainment, and therefore the first term in equation (3.24) represents the entropy flux into the plume. This entropy flux thus does not vanish, first, because entropy is an extensive thermodynamic quantity and, second, because the entropies inside and outside the plume must differ (otherwise there would be no buoyancy). We therefore write

$$2\pi \rho_0 v_r r(S_{pl} - S_0)\Big|_{r=b(z)} = -2\pi\alpha w \rho_0 v_z \Delta S , \quad (3.25)$$

and finally obtain the plume energy equation in the form

$$-2\pi\alpha w \rho_0 v_z \Delta S + \frac{dF_{ent}}{dz} = -\frac{dS_0}{dz} M , \quad (3.26)$$

with the radial (i.e., across the boundary) entropy flux F_{ent} defined as

$$F_{ent} = \int_0^{b(z)} dr 2\pi r v_z \rho_0 \Delta S . \quad (3.27)$$

In order to make further progress, we must adopt some functional form for the radial profiles of velocity, entropy, and density; we shall assume that these are Gaussian in a horizontal plume cross section. This assumption seems to be a good description of laboratory jets and plumes, but is not essential to the following. Thus we posit that

$$v_z = u_z(z) \exp\left[-\left(\frac{r}{r_V}\right)^2\right] \quad (3.28)$$

$$\Delta S = \Delta S(z) \exp\left[-\left(\frac{r}{r_\rho}\right)^2\right] \quad (3.29)$$

$$\Delta \rho = \Delta \rho(z) \exp\left[-\left(\frac{r}{r_\rho}\right)^2\right] , \quad (3.30)$$

where we allow for different dispersions in the velocity and density profiles as is customary in the description of laboratory and atmospheric plumes. Defining a dimensionless parameter $\lambda \equiv r_\rho / r_V$, we obtain the following expressions for the mass, momentum, and buoyancy flux, respectively:

$$\frac{d}{dz}(\pi \rho_0 u_z w^2) = 2\pi\alpha\rho_0 u_z w \quad (3.31)$$

$$\frac{d}{dz}(\pi\rho_0 u_z{}^2 w^2) = -2g\lambda^2\pi w^2\Delta\rho \tag{3.32}$$

$$\frac{\lambda^2}{1+\lambda^2}\frac{d}{dz}(\pi\rho_0 u_z w^2\Delta S) = -\pi\rho_0 u_z w^2 \frac{dS_0}{dz} + 2\pi\alpha\rho_0 u_z w\Delta S\,. \tag{3.33}$$

c) *Entrainment*

Entrainment into turbulent flow is an extremely complex, time-dependent hydrodynamic phenomenon, the details of which are only poorly understood (Townsend 1966). Clearly, for the purposes of a stationary, integral theory of plume rise the prescription of a time-independent mean entrainment rate is sufficient. This can be accomplished ad hoc (Morton, Taylor, and Turner 1956), empirically (as in parameterization of cumulus convection), or, more formally, by considering velocity moments of the Navier-Stokes equations; in any case, an entrainment hypothesis is needed in order to close the system of equations (3.31)–(3.33). It should not come as a surprise that a great many entrainment functions can be generated with ease (see Schatzmann 1978). We therefore only show how Morton, Taylor, and Turner's (1956) entrainment hypothesis can be "derived," and consider somewhat more complicated entrainment functions that do not lead to the singular behavior at the point of maximum penetration (as the Morton, Taylor, and Turner 1956 model).

Because the sound speed at the bottom of the solar convection zone is large ($\sim 10^7$ cm s^{-1}), the flow Mach number M is small, i.e., $M \ll 1$. According to Markovin's hypothesis (Bradshaw and Ferriss 1971) we do not expect compressibility to affect the structure of turbulence; in other words, we will ignore the compressibility of the turbulence, but not of the mean flow. Thus, decomposing the velocity field into mean and fluctuating quantities, we assume (for stationary axisymmetric turbulent flow)

$$\operatorname{div} \boldsymbol{v}' = \frac{1}{r}\frac{\partial v_r'}{\partial r} + \frac{\partial v_z'}{\partial z} = 0\,, \tag{3.34}$$

where the prime denotes fluctuating quantities. Because of Markovin's hypothesis, the mean continuity equation (3.9) remains unchanged, since fluctuating quantities always appear in connection with the fluctuating density ρ'. On the right-hand side of the mean z-component of the Navier-Stokes equation (3.8), however, the additional Reynolds stress terms

$$-\frac{1}{r}\left\langle\frac{\partial}{\partial r}(rv_r'v_z')\right\rangle_{\text{mean}} - \left\langle\frac{\partial}{\partial z}v_z'^2\right\rangle_{\text{mean}}$$

appear, where equation (3.34) has been used. We shall neglect all but the off-diagonal terms in the Reynolds stress tensor, because they are usually most important for momentum transfer in turbulent flows. Note that, upon integration over a plume cross section, equation (3.15) also remains unchanged if we assume

$$\langle rv_r'v_z'\rangle_{\text{mean}}\Big|_{r=b(z)} = 0\,. \tag{3.35}$$

Fox (1970) and Hirst (1971) show how closure of the plume equations can be achieved by taking moments of the Navier-Stokes equations including Reynolds stress terms. Taking therefore the moment of equation (3.8), with respect to v_z, including the stress term (3.34), we obtain after a little manipulation

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 v_r v_z{}^2) + \frac{\partial}{\partial z}(\rho_0 v_z{}^3) = -gv_z(\rho-\rho_0) - \frac{v_z\rho_0}{r}\left\langle\frac{\partial}{\partial r}(rv_z'v_r')\right\rangle_{\text{mean}}\,. \tag{3.36}$$

Integrating over a plume cross section as in § IIIa, dropping boundary terms proportional to v_z or db/dz, assuming Gaussian profiles (3.28)–(3.30), and assuming that the shear stress integral scales like (Hirst 1971)

$$\int_0^{b(z)} dru(r,z)\left\langle\frac{\partial}{\partial r}(rv_z'v_r')\right\rangle = Jv_z{}^3 w \tag{3.37}$$

for some constant J, we obtain

$$\frac{d}{dz}(\rho_0 wu_z{}^3) = -6g\frac{\lambda^2}{1+\lambda^2}u_z w^2\Delta\rho - 12Jw\rho_0 u_z{}^3\,. \tag{3.38}$$

Writing equation (3.31) in the more general form

$$\frac{d}{dz}(\rho_0 w^2 u_z) = E\,, \tag{3.39}$$

we obtain after some manipulation, and upon using equation (3.32),

$$E = 2g\frac{w^2}{v_z}\Delta\rho\left(\frac{3\lambda^2}{1+\lambda^2} - 2\lambda^2\right) + 12Jw\rho_0 u_z\,. \tag{3.40}$$

In terms of the entrainment function α, we have instead

$$\alpha = \frac{\alpha_2}{F} + \alpha_1\,, \tag{3.41}$$

where

$$\alpha_1 = 6J\,,\quad \alpha_2 = \frac{3\lambda^2}{1+\lambda^2} - 2\lambda^2\,,\quad F = \frac{v_z{}^2}{gw}\frac{\rho_0}{\Delta\rho}\,. \tag{3.42}$$

F is the densimetric Froude number (a measure of the relative strengths of inertial and (effective) gravity forces, in analogy to the Reynolds number).

In the limit $F \to \infty$, we have therefore recovered Morton, Taylor, and Turner's (1956) entrainment constant. Experiments at both small and large Froude numbers suggest $\lambda \sim 1.11$–1.16; for our application we find with $v \sim 8 \times 10^3$ cm s^{-1}, $w \sim 5 \times 10^8$ cm, and $\Delta\rho/\rho \sim -10^{-8}$, $F \sim -260$ at the bottom of the convection zone. Therefore, the first term in the entrainment function (3.41) will become important; note also that it changes sign when $\Delta\rho$ changes sign, and thus α may actually become negative. We shall see that this circumvents the singularities intrinsic to the Morton, Taylor, and Turner (1956) model. We also remark in passing that jets with small Froude numbers can be straightforwardly generated in the laboratory; as long as the law of similarity holds, our conclusions should not be affected by scaling up from laboratory to solar convection zone dimensions.

d) *Inclusion of Plume Effects in Stellar Structure Equations*

In the preceding sections we derived the plume equations in planar geometry. In laboratory and meteorological applications the entropy stratification, i.e., dS_0/dz, is known, and thus the system equations (3.31)–(3.33) can be solved as an initial value problem. In astrophysical applications one is of course interested in determining the stratification in order to integrate the stellar structure equations; this can be done by utilizing the flux constancy condition.

In hydrodynamics the stationary energy equation in its general form can be expressed as

$$\operatorname{div}\left(\tfrac{1}{2}\rho v^2 v + \rho v h - v\sigma_V - \rho c_p \sigma \nabla T\right) = 0 , \qquad (3.43)$$

where the four terms represent the mechanical, enthalpy, viscous, and radiative flux, respectively, and σ is the thermometric radiative conductivity, defined as

$$\sigma \equiv \frac{4acT^3}{3\kappa c_p \rho^2} . \qquad (3.44)$$

We shall consider—as explained above—only the convective flux (i.e., the contributions of entropy fluctuations to the enthalpy flux) and the radiative flux. Marcus, Press, and Teukolsky (1983) point out that neglect of the diffusive terms in the entropy flux equation

$$\frac{\partial \rho S}{\partial t} + \nabla F_S = S_v + S_\sigma , \qquad (3.45)$$

where F_S is the (total) entropy flux defined as

$$F_S = \rho v S - c_p \rho \sigma \nabla(\ln T) \qquad (3.46)$$

and where S_σ and S_v are the radiative and viscous entropy source terms, defined as

$$S_\sigma = c_p \rho \sigma \frac{\nabla T}{T}\frac{\nabla T}{T} \qquad (3.47)$$

and

$$S_v = \frac{\eta}{2T}\sigma_{ij}\sigma_{ij} + \frac{\xi}{T}(\nabla v)^2 \qquad (3.48)$$

(σ_{ij} is the stress tensor, η and ξ are the bulk and dynamic viscosities) leads to an incorrect calculation of the entropy flux in mixing-length theory. Is this objection relevant to our discussion? In the radiative interior (i.e., if there are no flows) the entropy flux equation is trivially satisfied given that the energy equation (3.34) is satisfied, i.e., if the radiative flux is divergence free (the usual condition to calculate radiative interiors). Consider now a convective region where $v \neq 0$; in mixing-length theory the advected entropy flux, i.e., $\rho v S$, is linked to the entropy content of the stellar material, a purely thermodynamic quantity, by assuming $F_{S,\text{flow}} \sim \rho v \Delta S$, where ΔS is the entropy difference between a rising (sinking) bubble and the environment. If the temperature gradient were precisely adiabatic in such a zone, then the entropy flux equation (3.45) would obviously be violated (and there would be zero convective flux). However, mixing-length theory assumes the temperature gradient to be slightly superadiabatic, and it is straighforward to show that in a mixing-length theory model of the deep solar convection zone, the advected entropy flux is of the same order as the radiative entropy source term; the latter is in turn of the same order as the viscous entropy source term if $F_{\text{rad}} \sim F_{\text{conv}}$, as shown by Marcus, Press, and Teukolsky (1983). In other words, mixing-length theory does not calculate the entropy flux in detail, but simply allows (or rather assumes) sufficient small-scale dynamics (i.e., breakup of bubbles) so that equation (3.45) is satisfied. Since the plume model presented in the previous section limits convective overshoot by buoyancy braking, and because buoyancy forces arise from entropy excesses, we will now show how the diffusion terms in equation (3.36) can be explicitly included in our model equations; the additional terms turn out to be small over the range of applicability of the plume model.

First we subtract from equation (3.45), the entropy flux equation, the quantity

$$\nabla(\rho v S_0) = -\rho v \frac{dS_0}{dz} \qquad (3.49)$$

as in § III*b*; this is obviously necessary because buoyancy requires a density or temperature *contrast* at some horizontal level. We then integrate the resulting equation over a control volume consisting of a horizontal plume slice of thickness Δz. Using Gauss's theorem, we write the result as

$$\begin{aligned}\rho v(S_{\text{pl}} - S_0)A\Big|_{\text{bot}} - \rho v(S_{\text{pl}} - S_0)A\Big|_{\text{top}} + \rho v(S_{\text{pl}} - S_0)A\Big|_{\text{sides}} \\ = c_p \rho_0 \sigma \frac{\nabla T}{T} A\Big|_{\text{bot}} - c_p \rho_0 \sigma \frac{\nabla T}{T} A\Big|_{\text{top}} + c_p \rho_0 \sigma \frac{\nabla T}{T} A\Big|_{\text{sides}} \\ + \Delta z \rho u \frac{dS_0}{dz} A + \int_{\text{Vol}} dx^3 \left[S_v + c_p \rho_0 \sigma \left(\frac{\nabla T}{T}\right)^2\right] .\end{aligned} \qquad (3.50)$$

The first two terms on the left-hand side represent the net excess entropy flux due to vertical fluid motions through the control volume; since we think of the convective flux, as in mixing-length theory, as being merely due to entropy fluctuations, we can write

$$\rho v(S_{\text{pl}} - S_0)A\Big|_{\text{bot}} - \rho v(S_{\text{pl}} - S_0)A\Big|_{\text{top}} = \Delta z \frac{d}{dz}(\pi \rho_0 u w^2 \Delta S) , \qquad (3.51)$$

as in § III*b*. The third term on the left-hand side represents the flux of entropy into the plume due to horizontal mixing, i.e., entrainment, and in the spirit of Taylor's entrainment hypothesis we write

$$\rho v(S_{\text{pl}} - S_0)A\Big|_{\text{sides}} = -\Delta z 2\pi\alpha w u \Delta S . \qquad (3.52)$$

Similarly, the first two terms on the right-hand side simply represent the net radiative flux through the plume; we assume that the radiative fluxes inside and outside the plume are the same, so that we can ignore the radiative entropy generation term altogether. The third term on the left-hand side of equation (3.50) represents the entropy loss due to horizontal temperature differences and ensuing radiation losses. Since such horizontal temperature differences are quite small compared to vertical temperature differences, and as the dimensions of the plume are assumed to be sufficiently large, we will ignore this term altogether; in the language of mixing-length theory, this simply means that convection is efficient. In the limit $\Delta z \to 0$ we then obtain, returning to the nomenclature of § III*b*.

$$\frac{\lambda^2}{1+\lambda^2}\frac{d}{dz}(\rho_0 u w^2 \Delta S) = 2\alpha w u \Delta S - \rho u w^2 \frac{dS_0}{dz} + w^2 S_v . \qquad (3.53)$$

We thus obtain the same equation as before, but now augmented by a viscous dissipation term. Since $S_v/S_\sigma \sim F_{\text{conv}}/F_{\text{rad}}$ (Marcus, Press, and Teukolsky 1983), and since

$F_{\rm rad} \sim F_{\rm conv}$ in the overshoot zone, the neglected diffusive terms should not seriously affect our results.

For our application to stellar interiors it turns out to be useful to write the plume equations (3.31)–(3.33) in the variables $W \equiv \rho_0 uw^2$, $V \equiv \rho_0{}^2u^4w^4$, and ΔS, with logarithmic pressure as the independent variable, and u positive in direction of increasing pressure. After some manipulation, we find

$$\frac{dW}{d \ln p} = 2\alpha \frac{p}{g\rho_0{}^{0.5}} V^{0.25} , \tag{3.54}$$

$$\frac{dV}{d \ln p} = \frac{16\lambda^2R^2}{N} \frac{R_G Q}{\mu c_p} W(F_{\rm tot} - F_{\rm rad}) , \tag{3.55}$$

$$\frac{d\Delta S}{d \ln p} = -\frac{1+\lambda^2}{\lambda^2} \frac{dS_0}{d \ln p} + \frac{2\alpha}{\lambda^2} \frac{p}{g\rho_0{}^{0.5}} \frac{4R^2}{NT} (F_{\rm tot} - F_{\rm rad}) \frac{V^{0.25}}{W^2} , \tag{3.56}$$

where μ is the molecular weight, R_G is the gas constant, the dimensionless factor Q is defined as in Cox and Giuli (1968, p. 291), and where we used relation (2.6) together with

$$\frac{\Delta\rho}{\rho} = -Q\frac{\Delta T}{T} = -Q\frac{\Delta S}{c_p} = \frac{Q}{c_p}\frac{4R^2}{TN}\frac{F_{\rm conv}}{\rho uw^2} . \tag{3.57}$$

The system (3.54–(3.57) has to be solved simultaneously with the equations of hydrostatic equilibrium

$$\frac{dr}{d \ln p} = \frac{p}{g\rho_0} , \tag{3.58}$$

to establish the height scale and temperature stratification

$$\frac{d \ln T}{d \ln p} = \nabla_{\rm ad} + \frac{1}{c_p}\frac{dS_0}{d \ln p} , \tag{3.59}$$

and with an equation (similar to eq. [2.1]) relating the radiative flux to the logarithmic temperature gradient.

e) Discussion and Limitations of the Plume Model

At this point, it seems worthwhile to pause, and to see what has actually been achieved when compared to the Shaviv and Salpeter (1973) model. Upon differentiating equations (2.3) and (2.4) with respect to r_2, keeping r_1 fixed, we obtain

$$\frac{dv^2}{dz} = 2g\frac{\Delta T}{T} = -2g\frac{\Delta\rho}{Q\rho_0} , \tag{3.60}$$

$$\frac{d\Delta T}{dz} = \left.\frac{dT}{dz}\right|_{\rm ad} - \left.\frac{dT}{dz}\right|_{St} = -\frac{T}{c_p}\frac{dS_0}{dz} , \tag{3.61}$$

or

$$\frac{d\Delta S}{dz} = -\left.\frac{dS}{dz}\right|_{St} , \tag{3.62}$$

if we neglect variations of T and c_p with height. It is immediately clear that equations (3.60) and (3.61) are the same (up to factors of order unity) as the plume equations (3.32) and (3.33), *if we ignore entrainment* and assume the quantity $\rho_0 w^2$ constant. Therefore, plume theory adds a mass flux equation (3.31) and the effects of entrainment on the mass and entropy content of the buoyant material to the physics embodied in the Shaviv-Salpeter model. Note also that we could have written equations (3.31)–(3.33) simply on dimensional grounds, up to factors of order unity; this, however, would leave the entrainment function α unspecified.

Conceptually, the plume approach offers two advantages over the mixing-length-based Shaviv and Salpeter (1973) model. First, no mixing length enters the theory and, second, the area filling factor of convecting material is a function of height. If one thinks of blobs of material in the mixing-length picture, detached from their environment and traveling adiabatically over one pressure scale height (down), then a typical blob dimension R scales like $R \sim p^{1/3\gamma}$ and thus its area would shrink roughly by factor of 0.7 (using the parameters given above). Mixing-length theory ignores this effect by assuming perfect symmetry between up- and downflows; in an overshoot layer, however, it is not obvious why up- and downflows ought to be symmetric, and in fact we justified our assumption of an extremely *asymmetric* flow pattern in § II*d*.

A drawback of the plume model appears to be its failure to describe the transition from the overshoot region to the quiescent, i.e. radiative region, in contrast to van Ballegooijen (1982; cf., however, our discussion in § II*c*[ii]). This failure, however, is intrinsic to the original Morton, Taylor, and Turner (1956) equations; for example, in their similarity solution, the buoyancy flux remains constant at maximum penetration depth, and the plume cross section goes to infinity because the velocity $u \to 0$ and the volume flux uw^2 (in our application the mass flux $\rho_0 uw^2$) can only increase; (cf. Fig. 1 in Morton, Taylor, and Turner 1956). Obviously, the plume model must break down at large penetration depths, where a singularity (in the model) is expected to occur: mass, momentum, and buoyancy are continuously provided at the plume base, but dissipation is not (Fox 1970). This conclusion could be avoided, if, for example, the plume could *detrain* beyond some point, say, where it looses its buoyancy with respect to the environment; buoyant material could then be expelled and the convective flux reduced. Note that the more general entrainment function (3.40) does *not* lead to detrainment either, since $\Delta\rho < 0$ in an overshoot region.

A thin boundary layer separating the overshoot region from the radiative interior of thickness $\sim 10^{-3}$ to 10^{-2} pressure scale heights, such as claimed by van Ballegooijen (1982), could provide a *stable* interface between convective and nonconvective regions (Press 1980), and would not significantly change the total penetration depth. In any event, for magnetic flux storage (and its release within a time scale of less than a solar cycle) only those regions with small absolute value of superadiabatic gradient are important (Schmitt and Rosner 1981, 1983; van Ballegooijen 1982), and we argue that the plume model (using the Morton, Taylor, and Turner 1956 entrainment function) actually *overestimates* the size of this region because of the following reason: plumes can transport an arbitrary convective flux simply by spreading out, whereas in the Shaviv and Salpeter (1973) and van Ballegooijen (1982) model a decrease in velocity must always be accompanied by an increase value of superadiabatic gradient because the filling factor is fixed. If the overshoot region were deeper, it could not be quasi-adiabatically stratified; this would then affect only the problem of chemical mixing, but not the magnetic flux storage problem.

f) The Boundary Layer between Convective and Radiative Regions in the Sun

In the previous sections we pointed out that the plume equations become singular at the point of maximum penetration, i.e., at the interface between regions with and without flows. In this section we derive an upper bound for the thickness of the boundary layer separating the solar convection zone from the radiative interior and argue that the physics of this layer is largely determined by radiative diffusion.

Since photon mean free paths in the solar interior are of the order of a few centimeters, energy transport by radiation is very accurately given by the diffusion approximation (2.1); with planar geometry, i.e., $F_{\rm tot}$ = constant and g = constant, and Kramer's opacities (i.e., $\kappa \sim \rho T^{-3.5}$), and constant molecular weight (i.e., $\rho \sim p/T$), then, given some logarithmic temperature gradient ∇, the radiative flux will be

$$F_{\rm rad} \sim \frac{T^{8.5}}{p^2} \nabla . \tag{3.63}$$

In the deep solar convection zone, we have, to a very good approximation, $\nabla = \nabla_{\rm ad} = \frac{2}{5}$ = constant, $T \sim p^{2/5}$, and thus

$$F_{\rm rad,CZ} \sim p^{1.4} \nabla_{\rm ad} , \tag{3.64}$$

whereas in the radiative zone

$$F_{\rm rad,RZ} = F_{\rm tot} = \text{constant} , \quad \nabla = \tfrac{4}{17} . \tag{3.65}$$

We note the strong increase of the radiative flux in the convection zone with pressure; if, for example, the logarithmic temperature gradient remains constant over one pressure scale height, the radiative flux increases by about a factor of 4. If convection continues beyond the point where $\nabla_{\rm rad} = \nabla_{\rm ad}$ such that the logarithmic temperature gradient stays almost adiabatic, then obviously $F_{\rm rad} > F_{\rm tot}$ and $F_{\rm conv} < 0$ in this region, if all other forms of energy flux are negligible. Clearly, there must exist a transition region where the radiative flux drops to its value in the radiative core, i.e., $F_{\rm rad} = F_{\rm tot}$; in this layer, the inequality

$$\frac{d \ln F_{\rm rad}}{d \ln p} = 8.5\nabla - 2 + \frac{d \ln \nabla}{d \ln p} < 0 \tag{3.66}$$

must be satisfied. Note that if $\nabla > 4/17$, the inequality (3.66) can only be satisfied if the temperature gradient decreases sufficiently rapidly with depth.

We now calculate the run of the logarithmic temperature gradient, assuming $F_{\rm rad}$ = constant; this will *overestimate* the size of the transition region, since we *overestimate* the second derivative of temperature. The condition $F_{\rm rad}$ = constant is equivalent to the logarithmic temperature gradient ∇ satisfying the nonlinear first order differential equation (the prime denotes derivative with respect to logarithmic pressure as in § II*c*)

$$\nabla' + 8.5\nabla^2 - 2\nabla = 0 , \tag{3.67}$$

which is a Riccati equation. Its general solution is given by

$$\nabla_{(p/p_0)} = \frac{4}{17}\left[1 + \frac{1}{z_0(p/p_0)^2 - 1}\right] , \tag{3.68}$$

where p_0 is the pressure at the point, where the radiative flux stops increasing with depth, and the constant z_0 is related to the logarithmic temperature gradient at this point through

$$\nabla\Big|_{p=p_0} = \nabla_0 = \frac{4}{17}\frac{z_0}{z_0 - 1} . \tag{3.69}$$

At large depths, i.e., for $p \to \infty$, the solution (3.68) relaxes to the radiative equilibrium (3.66), as expected; the Taylor expansion of the solution (3.68) about $p = p_0$ is given by

$$\nabla_{(\Delta p/p_0)} = \nabla(p_0) - \frac{8}{17}\frac{z_0}{(z_0 - 1)^2}\frac{\Delta p_0}{p_0} . \tag{3.70}$$

If we assume that the logarithmic temperature gradient in the overshoot region is *exactly* adiabatic, then $\nabla(p_0) = 2/5$ and $z_0 = 17/7$, and we obtain for the run of the superadiabatic gradient with pressure

$$\nabla - \nabla_{\rm ad} = -\frac{14}{25}\frac{\Delta p}{p_0} . \tag{3.71}$$

We want to reemphasize that the run of temperature (3.71) ensures only $F_{\rm rad}$ = constant, but that the actual temperature gradient must drop even more steeply; note that so far only radiative energy transport in the diffusion approximation has been considered, and that *no* assumptions on convective energy transport have been made.

The crucial point now is that a temperature stratification of the form (3.71) represents an almost insurmountable barrier for *adiabatic* convective motions; since the thermal microscale at the bottom of the solar convection zone is of the order of a few kilometers, close to the viscous cutoff, our conclusions should hold for all but the very smallest eddies. For adiabatic motions, equations (3.60)–(3.62) are expected to be quite accurate approximations to the equations of motion, because the buoyancy force terms should dominate as a consequence of the strong temperature stratification (3.71). If the logarithmic pressure is used as the independent variable, the temperature stratification (3.71) is adopted, and if the initial condition $\Delta S(p_0) = 0$ is used, we obtain upon integrating eq. (3.61) (treating T as constant)

$$\frac{\Delta T}{T} = \frac{\Delta S}{c_p} = \frac{7}{25}\left(\frac{\Delta p}{p_0}\right)^2 . \tag{3.72}$$

Using this result to integrate equation (3.60) with the initial condition $v(p_0) = v_0$, and upon assuming $p/\rho = p_0/\rho_0$ = constant, we find

$$\left(\frac{v}{v_0}\right)^2 = 1 - \frac{14}{75}\frac{p_0}{\rho_0 v_0^2}\left(\frac{\Delta p}{p_0}\right)^3 . \tag{3.73}$$

Thus the velocity becomes zero if

$$\frac{\Delta p}{p_0} = \left(\frac{75}{14}\frac{\rho_0 v_0^2}{p_0}\right)^{1/3} \sim 1 \times 10^{-2}\left(\frac{v_0}{8 \times 10^3}\right)^{2/3} , \tag{3.74}$$

where we have inserted typical densities ($\rho \sim 0.25$ g cm^{-3}) and pressures ($p \sim 7 \times 10^{13}$ dyn cm^{-2}) at the bottom of the solar convection zone. We note especially that this result does *not* sensitively depend on the chosen velocities and point out again that equation (3.74) is an *upper* limit on the boundary layer thickness. Objections may be raised against the use of the simple equations of motions (3.60)–(3.62), but the fundamental point remains that the scale of the change of the temperature stratification (eqs. [3.70] and [3.71]) is solely set by the properties of radiative transport.

To summarize, at the bottom of the overshoot region in the Sun we expect a thin boundary layer (thickness < 500 km), in which the radiative flux drops from approximately $F_{\rm rad} = F_{\rm tot}(p_0/p_\epsilon)^{1.4}$ (p_0 and p_ϵ as defined previously) to its radiative equilibrium value $F_{\rm rad} = F_{\rm tot}$. Thus, the temperature gradient changes almost discontinuously; this discontinuity provides the stability of the interface between convective and radiative regions against turbulent erosion as suggested by Press (1981). We conjecture that the scale and the detailed thermodynamic properties of this interface are predominantly set by radiative energy transport.

IV. SOLUTIONS OF THE PLUME EQUATIONS

In this section we present the results of numerical integrations of equations (3.54)–(3.59) and (2.1).

a) Initial Conditions and Method of Solution

i) *Choice of Initial Conditions*

The original plume equations were derived (and applied) by Morton, Taylor, and Turner (1956) to describe the rise of buoyant material into a *stably* stratified medium; care has to be exercised when applying them to a neutrally or unstably stratified medium. We do not propose to use the plume equations for the convection zone proper (where flows are driven, presumably by superadiabatic temperature gradients). Instead, we use as a starting point the location where $\nabla_{ad} = \nabla_{rad}$, which is a well defined point in the solar interior. At lower pressure, i.e., for larger radii, $\nabla_{rad} > \nabla_{ad}$, so that the medium is (locally) unstable to adiabatic perturbations, we expect convective flows to cause the temperature stratification to be almost adiabatic; at higher pressure, i.e., for smaller radii, radiation may carry all the required flux (to satisfy eq. [2.6]) at a *subadiabatic* temperature gradient; in Shaviv and Salpeter's (1973) nomenclature, we use the point r_ϵ as starting point.

We choose an initial (area) filling factor f, a characteristic velocity v_0, and the number of plumes N present at any given time; it is then straightforward to calculate the quantities V, W, and ΔS at the initial level. We have of course no clues as to how large the filling factor f ought to be; mixing-length theory predicts convective velocities of about 10^4 cm s^{-1} in the lower part of the solar convection zone; however, these values may well be an "artifact" of this "theory"; N can be estimated by assuming that each convective cell gives rise to one plume; if the cell sizes are of the order of a pressure scale height, as assumed in mixing-length theory, we find $N \sim 400$.

ii) *Method of Solution*

As pointed out in the previous section, it is possible to choose initial conditions for the variables W, V, ΔS, r, and T at some level, say r_ϵ; if the superadiabatic gradient were known, we could then integrate the system of equations (3.54)–(3.59) and (2.1) numerically. However, the superadiabatic gradient is not known *a priori*, albeit it is presumably small throughout the overshoot layer. For each integration step we choose a (constant) trial value for $\nabla - \nabla_{ad}$, integrate equations (3.54)–(3.59) and (2.1), calculate the convective flux carried in the plumes through $F_{conv} = (NT/4R^2)W\Delta S$, and iteratively adjust $\nabla - \nabla_{ad}$ until $F_{conv} = F_{tot} - F_{rad}$ to some specified accuracy, where F_{rad} is calculated in diffusion approximation (2.1); in other words, $\nabla - \nabla_{ad}$ is taken to be piecewise continuous over each integration interval; since $\nabla - \nabla_{ad}$ turns out to be only slightly varying throughout the overshoot layer, we feel that this relatively crude integration scheme is quite appropriate. The actual numerical integration was performed using an extrapolation scheme based on rational functions as described by Bulirsch and Stoer (1966).

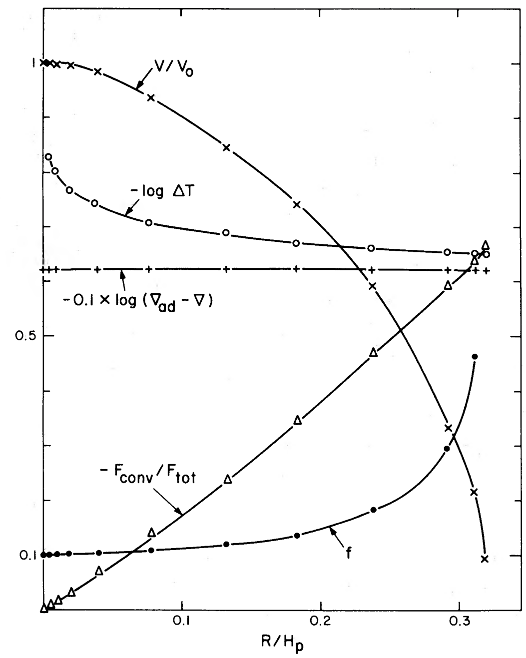

FIG. 2.—Plot of the scaled velocity v/v_0 (X), the scaled convective flux $-F_{conv}/F_{tot}$ (△), the filling factor f (●), the negative logarithm of the temperature difference ΔT between plume interior and exterior (○), and the negative logarithm of the subadiabatic gradient (+) for a typical plume model with $v_0 = 10^4$ cm s^{-1}, $N = 200$, $f_0 = 0.1$, $\alpha = 0.14$ (see text) vs. the overshoot length scaled by the pressure scale height.

b) Results

First, consider a typical result of the numerical integration of the plume equations (3.54)–(3.59) and (2.1). In Figure 2 we plot the runs of velocity (v/v_0), filling factor (f), convective flux ($-F_{conv}/F_{tot}$), temperature difference (ΔT) between plume interior and exterior, and subadiabatic gradient ($\nabla_{ad} - \nabla$) as a function of height (z/H_p) for the case $N = 200$, $v_0 = 10^4$ cm s^{-1}, $f_0 = 0.1$, and a constant entrainment function ($\alpha = 0.14$); the height and velocity variables are scaled with respect to their respective initial values, the convective flux is scaled to the negative total flux, and we plot the negative logarithms of ΔT and $\nabla_{ad} - \nabla$, respectively. The above choice of initial conditions results in an overshoot of $\sim$0.31 pressure scale heights. The following characteristics of this particular solution have been found in all the solutions we have obtaind so far: (1) The filling factor stays almost constant over most of the overshoot region before the plume spreads out. (2) The spreading out is accompanied by a rapid decrease in velocity; note that Figure 2 nicely illustrates the singular character of the plume equations at the point of maximum penetration as discussed in § III. (3) The convective flux steadily increases and does not approach zero; the reason for this is that both the temperature difference ΔT between plume interior and exterior and the filling factor steadily increase, and therefore any desired convective flux may be transported. The subadiabatic gradient, on the other hand, hardly changes over the entire overshoot region (only by 10% in Fig. 2), and does not approach the subadiabatic gradient required in a purely radiative core; again this is not surprising in the light of our discussion of the plume equations in § III.

Next, we demonstrate that the amount of overshoot predicted by the plume model largely depends on v_0 and f_0

(i.e., the initial velocity and filling factor), to a lesser extent on a specific entrainment model, and hardly at all on the number of plumes N (within physically reasonable limits); in other words, only the plume density matters in the parameter regime we studied. In Figure 3 we plot the extent of the overshoot region, again scaled to the initial pressure scale height (which turns out to be $\sim$66,000 km for our choice of parameters) as a function of initial velocity v_0 for several choices of the initial filling factor ($f_0 = 0.1, 0.0316, 0.01$) and the mean number of plumes ($N = 200$ denoted by crosses, $N = 800$ denoted by dots). It is evident that the results, at least for smaller initial velocity, are almost independent of N, whereas f has a profound influence on the resulting overshoot. Indeed, our results may be represented parametrically by the three curves also shown in Figure 3, whose functional dependence upon v and f is given by the relation

$$\frac{R}{H_p} = 0.42 \frac{v^{3/2}}{10^4} f^{1/2} . \tag{4.1}$$

This analytic expression seems to provide a reasonable description of all the numerically calculated points; for larger velocities equation, equation (4.1) slightly overestimates the actual amount of overshoot.

We explore the influence of a specific choice of an entrainment function on the resulting amount of convective overshoot in Figure 4. For simplicity, we restrict ourselves to the case of a constant entrainment function (parameterized by $\alpha = 0$, 0.057, 0.14, and 0.28) and plot overshoot versus initial velocity, keeping f and N fixed (at $f = 0.1$, $N = 200$, respectively). Different entrainment constants become important only at rather large initial velocities, and for velocities less than $\sim 10^4$ cm s^{-1}, different entrainment constants lead to quite similar results.

In summary, we can draw the following conclusions from our numerical solutions of the plume equations (3.54)–(3.59) and (2.1): (1) The predicted overshoot *neither* depends sensitively on our admittedly relatively crude parameterization of the complicated entrainment process, *nor* does it depend sensitively on assumptions concerning the mean number of plumes. (2) The important parameters are *initial velocity* and *filling factor*; using an entrainment parameterization that describes a wide variety of laboratory and atmospheric measurements best, we obtain to good approximation

$$R^2 \approx v^3 f . \tag{4.2}$$

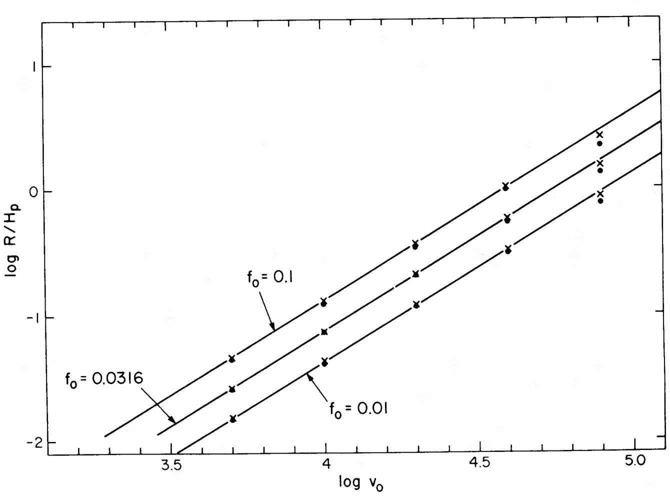

FIG. 3.—Extent of the overshoot region (scaled by pressure scale height) vs. (initial) velocity v_0 for $f_0 = 0.1$, 0.0316, 0.01, and $N = 200$ (X) and $N = 800$ (●). The straight line represents the analytical approximation given by equation (4.1).

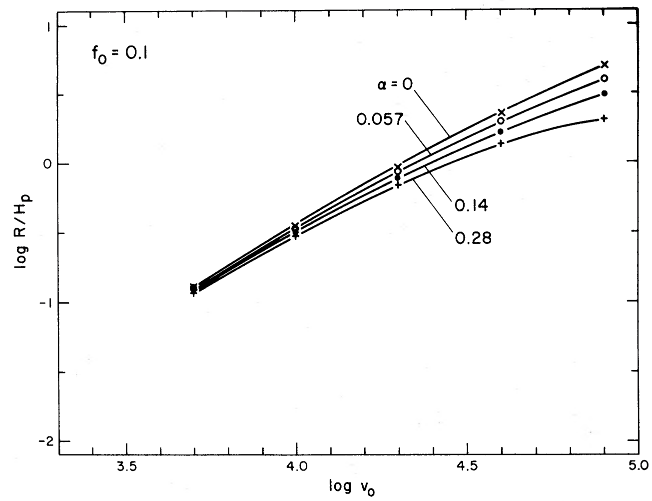

FIG. 4.—Extent of the overshoot region (scaled by pressure scale height) vs. (initial) velocity v_0 for several values of the entrainment constant α (for the case $N = 200$, $f_0 = 0.1$); the case $\alpha = 0$ is denoted by X, $\alpha = 0.057$ by ○, $\alpha = 0.14$ by ●, and $\alpha = 0.28$ by +.

V. DISCUSSION AND CONCLUSIONS

a) Plume Dynamics as a Model of Convective Overshoot in the Solar Interior

Our principal results may be summarized as follows: We have derived the phenomenological plume equations appropriate to the solar interior and have shown (by explicit numerical integration) that the resulting penetration of downward-moving plumes into the stably stratified layers below the solar convection zone is relatively shallow; typically, i.e., for initial conditions as predicted by mixing-length models, the ensuing penetration depth is substantially less than a local pressure scale height. In the limit of zero entrainment, this model leads to a formalism essentially identical to that of Shaviv and Salpeter (1973). We have furthermore shown that one can bound the thickness of the boundary layer which separates the overshoot region proper from the radiative zone; we find that this layer must be less than about 500 km in depth, so that the transition from adiabatic to radiative temperature gradients occurs virtually discontinuously.

b) Implications for Storage of Magnetic Fields within the Overshoot Layer

Our interest in the properties of the region of convective overshoot beneath the solar convection zone developed from the possibility of magnetic flux storage in this overshoot region during the course of the solar cycle, rather than from the possible effects of convective overshoot on stellar structure and evolution: whereas in the latter case, we would have to consider time scales of the order of the nuclear burning time scale $\sim 10^9$ yr, the time scale of interest to us is, at the most, $\sim$10 yr. In a previous paper (Schmitt and Rosner 1983) we explored doubly diffusive instabilities in such an overshoot region on time scales considerably less than those associated with the solar cycle; assuming kinetic equipartition magnetic field strengths of $\sim 10^4$ gauss and a layer thickness of $\sim$20,000 km, we showed that doubly diffusive instabilities with inverse growth rates of a few months can occur as long as the subadiabatic gradient is less than $\sim 10^{-5}$ to 10^{-4}. In order to obtain an overshoot region of $\sim$20,000 km extent, our present plume model requires

$$\frac{v}{10^4} f \approx 0.5 , \tag{5.1}$$

which leads to initial velocities in excess of 10^4 cm s^{-1}, for filling factors of less than 0.1 (larger filling factors lead to geometries where the plume model would not no longer be physically reasonable). Mixing length models of the solar convection zone (Spruit 1976; Gough 1982) predict convective velocities of only $(5–7) \times 10^3$ cm s^{-1} in the lower half of the solar convection zone, but this discrepancy is not worrisome because velocities derived from mixing-length theory may in any case be artifactual; furthermore, even if the mixing-length values are appropriate so that the overshoot region has a thickness of, say, only one-tenth or so of a pressure scale height H_p, doubly diffusive instabilities could still operate: For example, for $B \sim 10{,}000$ gauss and a layer thickness $R \sim 5{,}000$ km, we would find stable diffusionless modes (i.e., stable modes on large scales), since the inequality (cf. eq. [3.15] in Schmitt and Rosner 1983)

$$\frac{dS}{dz} + \frac{d}{dz} \ln \left(\frac{B}{\rho}\right)\left(\frac{V}{a}\right)^2 > 0 \tag{5.2}$$

(S denotes entropy; B, magnetic field; ρ, density; V, Alfvén speed; a, sound speed; z, height) is still amply satisfied; because the ratio of Alfvén and sound speeds is quite small, even large (negative) magnetic field gradients can be stabilized by a small subadiabatic temperature gradient, but only on large scales. On intermediate scales (cf. Schmitt and Rosner 1983) doubly diffusive instability will set in with growth rates even larger than in the numerical example of Figure 6 in Schmitt and Rosner (1983), since the instability-generating magnetic field gradient becomes larger (if we still assume equipartition magnetic field strengths).

The thickness of the overshoot layer therefore enters in two ways: On the one hand it determines the total amount of magnetic flux that can be stored, and on the other it sets the scale for the magnetic field gradient that drives the doubly diffusive instability. The value of the subadiabatic gradient within the overshoot region then determines the spatial length scale of the instability by suppressing instability on large scales ("diffusionless modes"), but not on intermediate scales; in our plume model, the subadiabatic gradient is chosen—as in conventional mixing-length theory—in order to conserve the total energy flux. Now, even small subadiabatic gradients lead to efficient buoyancy braking of large-scale motions, which are, for all practical purposes, adiabatic; only nonadiabatic motions on sufficiently small scales (i.e., on thermal and viscous microscales) can avoid buoyancy braking, and hence produce as large an overshoot as suggested by Press (1980). The question then arises how much energy can be cascaded down to these small scales; the answer must await calculations of the kind pursued by, for example, Marcus, Press, and Teukolsky (1983).

For the moment, we therefore draw the following (tentative) conclusions: the region of convective overshoot underneath the solar convection zone is thin, i.e., less than about 10,000 km, and hence considerably thinner than a pressure scale height; in this overshoot region the subadiabatic gradient is small, i.e., 10^{-7} to 10^{-6}. In our scenario, doubly diffusive instabilities driven by an unstable magnetic field gradient can operate with (linear) inverse growth rates of a few months. These growth rates are a very sensitive function of (meridional) scale (cf. Schmitt and Rosner 1983); therefore, doubly diffusive instabilities are a natural mechanism for producing magnetic flux *ropes* over a narrow band of spatial scales from a diffuse magnetic field with no intrinsic length scale (other than its gradient scale).

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