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ON THE EMISSION OF GRAVITATIONAL RADIATION FROM INHOMOGENEOUS JACOBI CONFIGURATIONS

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ABSTRACT

The effects of inhomogeneity on the emission of gravitational radiation from Jacobi configurations (uniformly rotating nonaxisymmetric configurations) are studied. If the equation of state is approximated by a polytropic relation, a neutron-star core with adiabatic index $\gamma \leq 3$ can radiate up to a fraction $\sim 10^{-3}$ to 10^{-4} of its mass-energy while passing through a Jacobi phase during its evolution. This maximum fraction drops to zero at $\gamma \approx 9/4$, where Jacobi configurations cease to exist in the polytropic approximation. While some of our results suggest that the fast pulsar PSR 1937+214 is hovering near the Jacobi bifurcation point, it appears more likely that the pulsar is not rotating sufficiently rapidly for this to be the case.

Subject headings: gravitation — pulsars — stars: neutron — stars: rotation

I. INTRODUCTION

The expansion of effort to design and bring on line increasingly sensitive gravitational-radiation detectors underscores the need for accurate description of the emission properties of plausible astrophysical sources of gravitational waves. Supernovae, in our own and in neighboring galaxies, have been thought to be particularly good candidates for sources capable of being detected by future, if not current, detectors. (See, e.g., the review by Thorne 1980.)

Most analyses of gravitational radiation from supernovae have concentrated on the waves emitted during the core collapse and bounce phase. With regard to inferred emission strength, Müller's (1982) analyses of this phase are perhaps somewhat discouraging. Müller's calculations are based on what are possibly the most accurate and realistic rotating collapse models available, the axisymmetric models of Hillebrandt and Müller (1981). Those models incorporate in detail the full effects of asphericity, inhomogeneity, microphysics, and shock propagation. Müller finds that the models emit gravitational-wave energies in the range $\sim 10^{-7}\ M_\odot c^2$, to $4 \times 10^{-6}\ M_\odot c^2$, corresponding to efficiencies that are more than a factor of 10 smaller than those suggested by calculations based on cruder, typically homogeneous (i.e., constant density), axisymmetric models.

It has been suggested that nonaxisymmetric collapse and postcollapse phases of supernovae might yield significantly higher efficiencies. Indeed, this stance is supported by calculations of emission from nonaxisymmetric neutron-star cores, using as models the classical homogeneous ellipsoids (Chandrasekhar 1970; Miller 1974; Detweiler and Lindblom 1981; Saenz and Shapiro 1981). However, in view of Müller's demonstration that the effect of inhomogeneity is often to reduce significantly the gravitational-radiation efficiency in axisymmetric collapse, it appears important to assess the effect of inhomogeneity in the nonaxisymmetric phases alluded to earlier.

The purpose of this paper is to investigate some of the effects of inhomogeneity upon the emission of gravitational radiation from one possible postcollapse phase of supernovae; specifically, from a phase in which a rapidly rotating neutron-star core finds itself deformed into a (at least approximately) uniformly rotating, steady-state nonaxisymmetric configuration, i.e., a "Jacobi" configuration. The gravitational radiation emitted by a Newtonian *homogeneous* stellar model as it evolves along the classic homogeneous Jacobi sequence has been calculated by Chandrasekhar (1970) and by Miller (1974). (The latter author has noted that the assumption that evolution proceeds exactly along the Jacobi sequence is necessarily an approximation; see below for further discussion.) Now that fully nonaxisymmetric, inhomogeneous Jacobi configurations have been constructed successfully in the Newtonian approximation by numerical methods (Ipser and Managan 1980; henceforth Paper I), the effects of inhomogeneity in this case can be assessed. We shall see that when allowances are made for the degrees of inhomogeneity appropriate to neutron-star matter, rather strong, though not necessarily discouraging, constraints are obtained for the strengths of emission from Jacobi configurations.

II. EMISSION PROPERTIES OF INHOMOGENEOUS JACOBI CONFIGURATIONS

We imagine that a sufficiently rapidly rotating neutron-star core has been deformed into a Jacobi configuration, perhaps by a nonaxisymmetric instability or by the growth of perturbations during collapse; and, further, that the core remains a Jacobi configuration (as long as it can; see below) as it evolves via the emission of gravitational radiation. We seek to estimate (1) the instantaneous rates of emission of gravitational radiation and the maximum total energy that can be emitted during the Jacobi phase; (2) the characteristic frequencies of emission; and (3) the way in which these quantities vary with the allowed degrees of inhomogeneity.

In the present rough treatment we shall model neutron-star equations of state in terms of polytropic relations of the form

$$p = K\rho^\gamma \,. \qquad (1)$$

Here p and ρ are the pressure and mass-energy density, K is a constant (i.e., independent of position), and γ is the constant adiabatic index. Larger values of γ correspond to stiffer equations of state and smaller degrees of inhomogeneity. A key question concerns what values of γ correspond to semirealistic neutron-star models. From the review conducted by Arnett

TABLE 1

PROPERTIES OF THE JACOBI SEQUENCE WITH $\gamma = 3$

Ω'	R'_y	R'_x/R'_y	E'_T	$10^2 J'$	$10^3 I'_{xx}$	$10^3 I'_{yy}$	$\tilde{L}_{GW}$
6.254	0.1128	1	−0.41837	2.4500	1.9589	1.9589	0
6.243	0.1196	0.89	−0.41871	2.4548	1.7845	2.1475	0.8
6.227	0.1249	0.83	−0.41849	2.4631	1.6790	2.2764	2.3
6.223	0.1261	0.81	−0.41834	2.4671	1.6589	2.3058	2.8
6.213	0.1296	0.78	−0.41806	2.4761	1.6177	2.3677	3.7
6.211	0.1305	0.77	−0.41794	2.4793	1.6102	2.3812	3.8
6.209	0.1312	0.76	−0.41799	2.4797	1.6045	2.3889	4.0
6.207	0.1322	0.76	−0.41817	2.4803	1.5921	2.4042	4.3
6.201	0.1330	0.75	−0.41870	2.4750	1.5790	2.4125	4.4
6.196	0.1343	0.74	−0.41893	2.4730	1.5643	2.4271	4.7
6.191	0.1362	0.72	−0.41890	2.4742	1.5561	2.4400	4.9
6.185	0.1385	0.71	−0.41898	2.4691	1.5444	2.4479	5.2
6.183	0.1390	0.70	−0.41918	2.4689	1.5405	2.4529	5.2

TABLE 2

PROPERTIES OF THE JACOBI SEQUENCE WUTH $\gamma = 8/3$

Ω'	R'_y	R'_x/R'_y	E'_T	$10^2 J'$	$10^3 I'_{xx}$	$10^3 I'_{yy}$	$\tilde{L}_{GW}$
6.353	0.1149	1	−0.41290	2.4046	1.8924	1.8924	0
6.352	0.1212	0.91	−0.41248	2.4168	1.7694	2.0357	0.5
6.351	0.1215	0.91	−0.41243	2.4177	1.7648	2.0418	0.6
6.350	0.1220	0.90	−0.41233	2.4188	1.7577	2.0513	0.7
6.347	0.1253	0.86	−0.41207	2.4265	1.7158	2.1073	1.1
6.345	0.1263	0.85	−0.41184	2.4295	1.7042	2.1245	1.3
6.343	0.1274	0.84	−0.41182	2.4299	1.6923	2.1387	1.4
6.337	0.1296	0.82	−0.41213	2.4275	1.6675	2.1632	1.8
6.330	0.1329	0.79	−0.41252	2.4240	1.6431	2.1864	2.1
6.326	0.1358	0.77	−0.41278	2.4205	1.6326	2.1939	2.3

and Bowers (1977) it would appear that almost all of the popular equations of state thus far proposed for neutron-star matter correspond to values of $\gamma \leq 3$. Faced with this limit, which we shall adopt, one quickly realizes that the amount of gravitational radiation that can in principle be emitted by a Jacobi-like neutron-star core must be much less than what might be inferred from study of the homogeneous configurations. The reason is that, unlike the sequence of homogeneous Jacobi ellipsoids, inhomogeneous Jacobi sequences of polytropes with $\gamma \leq 3$ terminate, due to equatorial mass shedding, at relatively short distances from their points of bifurcation away from axisymmetric configrations (cf. Paper I). In fact, Jacobi configurations do not even exist for $\gamma \lesssim 9/4$.

Given that the relevant range of γ is $9/4 \lesssim \gamma \lesssim 3$, we shall calculate the radiation emitted during evolution along the Jacobi sequences with $\gamma = 3$ and 8/3, corresponding to polytropic indices $n = 0.5$ and 0.6. Some relevant properties of models along the sequences (constructed in Paper I) are exhibited in Tables 1 and 2, in a form presently more convenient than that used in Paper I. Let $\mathscr{M}$ and Φ_p denote the mass and the magnitude of the polar gravitational potential, respectively, of a configuration. The various columns of the tables list in turn the values of the dimensionless quantities

$$\Omega' \equiv 4\pi G \Phi_p^{-3/2} \mathscr{M} \Omega\,, \quad R'_y \equiv (4\pi G)^{-1} \Phi_p \mathscr{M}^{-1} \mathscr{R}_y\,,$$
$$R'_x/R'_y \equiv \mathscr{R}_x/\mathscr{R}_y\,, \quad E'_T \equiv \Phi_p^{-1} \mathscr{M}^{-1} \mathscr{E}_T\,,$$
$$J' \equiv (4\pi G)^{-1} \Phi_p^{1/2} \mathscr{M}^{-2} \mathscr{J}\,,$$
$$I'_{xx} \equiv (4\pi G)^{-2} \Phi_p^2 \mathscr{M}^{-3} \mathscr{I}_{xx}\,, \quad I'_{yy} \equiv (4\pi G)^{-2} \Phi_p^2 \mathscr{M}^{-3} \mathscr{I}_{yy}\,,$$
$$\tilde{L}_{GW} \equiv \mathscr{L}_{GW} \Bigg/ \left[10^{52} \left(\frac{\Phi_p}{0.15c^2} \right)^5 \text{ergs s}^{-1} \right]. \tag{2}$$

In these definitions G is the gravitation constant, Ω is the angular velocity of a configuration, $\mathscr{R}_x$ and $\mathscr{R}_y$ are the principal equatorial radii, $\mathscr{E}_T$ is the total energy (i.e., the sum of kinetic, internal, and gravitational potential energies) excluding rest-mass energy, $\mathscr{J}$ is the angular momentum, $\mathscr{I}_{xx}$ and $\mathscr{I}_{yy}$ are components of the moment-of-inertia tensor, and $\mathscr{L}_{GW}$, the instantaneous rate of emission of gravitational-radiation, is given by

$$\mathscr{L}_{GW} = \frac{32}{5} \frac{G}{c^5} \Omega^6 (\mathscr{I}_{yy} - \mathscr{I}_{xx})^2\,, \tag{3}$$

where c is the speed of light.

It appears appropriate at this stage to comment on our approximation that a configuration remains on the Jacobi sequence as it emits gravitational radiation and evolves. The actual evolution cannot be exactly along the Jacobi sequence for the following reason. As Miller (1974) has pointed out for the homogeneous case, gravitational radiation reaction preserves the circulation, while in fact the circulation varies along the Jacobi sequence. This fact notwithstanding, it is difficult to see how our approximation that evolution proceeds along the Jacobi sequence could introduce overwhelming errors in the present context: Our calculations below and those of Miller demonstrate that Jacobi-like configurations, whether homogeneous or not, evolve toward the axisymmetric (Maclaurin) sequence, from which they bifurcate, in response to the emission of radiation. Since our inhomogeneous Jacobi configurations can never be very far from the axisymmetric sequence (because of sequence termination via mass shedding), they should end up at an axisymmetric configuration not far from the one at the Jacobi bifurcation point. Consequently, we would not expect the error involved in neglecting evolution off the Jacobi sequence to be fatal, especially given the other approximations adopted in the present rough calculation.

The demand that a configuration remain on a Jacobi sequence of fixed polytropic index n, while evolving in response to the emission of gravitational radiation, requires that the physical quantities describing a model vary in a prescribed way along the sequence. Specifically, the familiar fact that

$$\frac{d\mathscr{M}}{dt} = \frac{d\mathscr{E}_T}{dt} - \Omega \frac{d\mathscr{J}}{dt} = 0 \tag{4}$$

during gravitational-radiation emission requires that

$$\Delta\mathscr{M} = \Delta\mathscr{E}_T - \Omega \Delta\mathscr{J} = 0 \tag{5}$$

along the sequence. Here $\Delta\mathscr{E}_T$, for example, is the variation of energy $\mathscr{E}_T$ from one configuration to a nearby one along the sequence. Equations (2) and (5) imply that

$$\frac{\Delta\Phi_p}{\Phi_p} = -\left(1 + \frac{1}{2} \frac{\Omega' J'}{E'_T}\right)^{-1} \left(\frac{\Delta E'_T}{E'_T} - \frac{\Omega' J'}{E'_T} \frac{\Delta J'}{J'} \right) \tag{6}$$

and hence that

$$\frac{\Delta\mathscr{J}}{\mathscr{J}} = \frac{E'_T}{\Omega' J'} \frac{\Delta\mathscr{E}_T}{\mathscr{E}_T} = \left(1 + \frac{1}{2} \frac{\Omega' J'}{E'_T}\right)^{-1} \left(\frac{\Delta J'}{J'} + \frac{1}{2} \frac{\Delta E'_T}{E'_T} \right) \tag{7}$$

along the sequence.

From the tables and equation (7) one finds that $\mathscr{J}$ (and hence $\mathscr{E}_T$) increases in the direction away from the bifurcation point along a sequence. Actually, the numbers indicate that $\mathscr{J}$

peaks and then oscillates near the end of the sequence and thereafter decreases slightly as the shedding point is approached. We think this is a numerical artifact associated with loss of accuracy near the shedding point. Leaving aside this slightly nagging issue, the following picture of evolution emerges (cf. Chandrasekhar 1970 for the corresponding picture in the homogeneous case):

A neutron-star core that reaches the Jacobi sequence evolves toward the bifurcation point (unless instabilities of some sort intercede) via the emission of gravitational radiation, increasing its angular velocity Ω as it does so. Along the sequence with $\gamma = 3$, the maximum total fractions of the angular momentum and of the energy that are lost to gravitational radiation during evolution back to the bifurcation point are

$$\frac{(\Delta \mathscr{E}_T)_{\max}}{|\mathscr{E}_T|} \approx 0.36 \frac{(\Delta \mathscr{J})_{\max}}{\mathscr{J}} \approx 4.7 \times 10^{-3} . \tag{8}$$

Hence the maximum total amount of energy radiated is

$$(\Delta \mathscr{E}_T)_{\max} \approx 4.7 \times 10^{-3} | E_T' | \Phi_p \mathscr{M} \approx 3 \times 10^{-4} \tilde{\Phi}_p \mathscr{M} c^2 \approx 7.9 \times 10^{50} \tilde{\Phi}_p \tilde{\mathscr{M}} \text{ ergs} , \tag{9}$$

where

$$\tilde{\Phi}_p \equiv \frac{\Phi_p}{0.15 c^2} , \quad \tilde{\mathscr{M}} \equiv \frac{\mathscr{M}}{1.5\, M_\odot} . \tag{10}$$

The typical luminosity is

$$\langle \mathscr{L}_{\rm GW} \rangle \approx 3 \times 10^{52} \tilde{\Phi}_p^5 \text{ ergs s}^{-1} . \tag{11}$$

The emission is nearly monochromatic, with the frequencies $\nu_{\rm GW}$ of the emitted gravitational waves lying in a narrow range

$$\frac{(\Delta \nu_{\rm GW})_{\max}}{\nu_{\rm GW}} \approx \frac{(\Delta \Omega')_{\max}}{\Omega'} \approx 6.5 \times 10^{-3} \tag{12}$$

near the value

$$\nu_{\rm GW} = \Omega/\pi \approx 1230 \tilde{\Phi}_p^{3/2} \tilde{\mathscr{M}}^{-1} \text{ Hz} . \tag{13}$$

It follows that most of the radiation is emitted over a time interval

$$\Delta t_{\rm em} = \Delta \mathscr{E}_T / \langle \mathscr{L}_{\rm GW} \rangle \approx 0.022 \tilde{\Phi}_p^{-4} \tilde{\mathscr{M}} \text{ s} \approx 27 \tilde{\Phi}_p^{-5/2} \nu_{\rm GW}^{-1} , \tag{14}$$

corresponding to a wavetrain of approximately $30\tilde{\Phi}_p^{-5/2}$ wavelengths.

These results and those for the Jacobi sequence with $\gamma = 8/3$ are summarized in Table 3.

III. DISCUSSION AND APPLICATION

a) General Remarks

According to the above results, an effect of inhomogeneity is to limit rather strongly the fraction of the mass-energy of a neutron-star core that can be radiated as gravitational waves during a Jacobi phase. In the polytropic approximation the maximum fraction is of order 3×10^{-4} for adiabatic index $\gamma \approx 3$. While this value is significantly larger than what typically comes out of realistic calculations of inhomogeneous axisymmetric collapse, one must remember that in the polytropic approximation the fraction drops to zero at $\gamma \approx 9/4$, where Jacobi configurations cease to exist. This fact adds to the picture a significant measure of doubt whether something resembling a Jacobi configuration can ever be expected to form during gravitational collapse. Perhaps a more likely result for rapidly rotating configurations is significant mass shedding, fission into two or more coherent pieces, or differential rotation.

It seems clear, however, that if Jacobi configurations are permitted, such a source, once formed in a nearby galaxy cluster, can radiate sufficiently strongly to be picked up by envisaged gravitational-wave detectors. Indeed, at Earth the waves from such a source will have dimensionless amplitude

$$h \sim 2 \times 10^{-22} \tilde{\Phi}_p \tilde{\mathscr{M}} \mathscr{D}_{10}^{-1} , \tag{15}$$

where $\mathscr{D}_{10}$ is the distance of the source from Earth in units of 10 megaparsecs. Also, the bandwidth of the radiation will be quite narrow (cf. Table 3), and this feature might be turned into an advantage.

b) The Fast Pulsar

The frequencies listed in the Tables remind one of the recently discovered millisecond pulsar, PSR 1937+214 (Backer *et al.* 1982). Since $\nu_{\rm GW}$ in Table 3 is twice the rotation frequency of the configuration, the appropriate value of $\nu_{\rm GW}$ for PSR 1937+214 is 1282 Hz and is reproduced for $\gamma = 3$ if $\tilde{\mathscr{M}} \tilde{\Phi}_p^{-3/2} \approx 0.96$. At first glance this suggests that PSR 1937+214, assuming its mass is near the maximum neutron-star mass, is hovering near the bifurcation point at which the Jacobi sequence branches off from the axisymmetric sequence. This would correspond to a circumstance in which the pulsar, initially more rapidly rotating than presently, slowed down via gravitational-radiation emission due to instability of modes with spherical azimuthal index $m = 2$ (cf. Friedman 1983 and Wagoner 1983 for a discussion of this and related issues); and in which the pulsar, having reached the Jacobi bifurcation point, found itself stable to $m = 2$ modes, because instability to those naturally ceases there, and also stable to modes with $m > 2$, because instability to these (in this circumstance) is suppressed by, say, viscosity. In this circumstance the pulsar, with a weak magnetic field, would have no way to spin down further and would remain near the Jacobi bifurcation point.

It is perhaps unfortunate that other indicators suggest that the above circumstance may not be the actual one. Our neglect of relativistic effects probably leads to an underestimate of $\nu_{\rm GW}$ at bifurcation. The error may be as much as 10%–20%, in which case the actual value of $\nu_{\rm GW}$ for a relativistic neutron star near maximum mass ($\tilde{\mathscr{M}} \sim 1$) would exceed even 1282 Hz. In this connection, we note that the dimensionless polar radius $R_z' \approx 0.064$ for our Newtonian models, and hence (cf. eqs. [2])

$$\tilde{\Phi}_p \equiv \frac{\Phi_p}{0.15 c^2} = \frac{4\pi G \mathscr{M} R_z'}{0.15 c^2 \mathscr{R}_z} \approx 2.5 \left(\frac{2 G \mathscr{M}}{\mathscr{R}_z c^2} \right) . \tag{16}$$

TABLE 3

CHARACTERISTICS OF EMISSION ALONG JACOBI SEQUENCES

γ	$(\Delta \mathscr{E}_T)_{\max}/\tilde{\Phi}_p \mathscr{M} c^2$	$\langle \mathscr{L}_{\rm GW}(\text{ergs s}^{-1}) \rangle / \tilde{\Phi}_p^5$	$(\Delta \nu_{\rm GW})_{\max}/\nu_{\rm GW}$	$\nu_{\rm GW}(\text{Hz}) \tilde{\mathscr{M}} \tilde{\Phi}_p^{-3/2}$	$\Delta t_{\rm em}\, \nu_{\rm GW}\, \tilde{\Phi}_p^{5/2}$
$\gamma = 3$	3×10^{-4}	3×10^{52}	7×10^{-3}	1230	27
$\gamma = 8/3$	2×10^{-4}	10^{52}	2×10^{-3}	1260	51

This implies that for $\tilde{\Phi}_p \approx 1$ the polar radius $\mathscr{R}_z \approx 2.5\mathscr{R}_S$, where $\mathscr{R}_S$ is the Schwarzschild radius, and hence that a configuration with $\tilde{\Phi}_p \approx 1$ is quite relativistic. In such a case our numbers cannot be trusted to within 10%–20%. We note further that our unpublished studies of the effects of rotation on the properties of neutron-star models lead us to favor values of $\nu_{GW} > 1282$ Hz at bifurcation. Another indicator involves the difficulty one has in understanding how viscosity could be strong enough to suppress instability to all modes with $m > 2$ (Friedman 1983). Instability to modes with $m > 2$ would spin the pulsar down below the Jacobi bifurcation point. This probably has occurred, although we cannot rule out the possibility that the pulsar is in fact hovering at the bifurcation point.

In any case, it would appear worthwhile to keep in mind the possibility that newly born neutron stars might occasionally pass through something resembling a Jacobi phase. We hope that future work will decide this issue more clearly through inclusion of relativity and more realistic equations of state.

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