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LOCAL STABILITY OF THICK ACCRETION DISKS. I. BASIC EQUATIONS AND PARALLEL PERTURBATIONS IN THE NEGLIGIBLE VISCOSITY CASE

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ABSTRACT

We describe a rather general approach to the stability of axisymmetric rotating non–self-gravitating bodies to local perturbations, and then specialize to systems such as thick accretion disks, where viscosity is negligible. A detailed analysis of the case where velocity perturbations are parallel to the non-azimuthal flow is carried out by examining the dominant terms in the fifth order dispersion relation. We find that the stability criteria applicable in different regimes are close analogs of the Hoiland condition, and that oscillations similar to the *g*- and *p*-modes in stars exist in thick disks.
Subject headings: stars: accretion

I. INTRODUCTION

Recently, thick accretion disks have been extensively studied by many authors (see Appendix A for a review). Such disks may exist in active galactic nuclei or in exotic objects like SS 433, Cyg X-1, β Lyr, etc. It has also been suggested that the (unstable) inner parts of thin accretion disks may become thick.

The properties of thick disks differ, even locally, from those of thin disks and rotating stars. Therefore, existing results concerning the stability of the latter (see Appendix B for a review) cannot be applied to thick disks without a careful examination.

The present paper initiates a series devoted to the study of the pulsations and stability of thick accretion disks. We examine here only local dynamical stability, i.e., we neglect the effects of boundary conditions. Although there is no formal justification for this procedure, it is well known that a local analysis usually yields a fairly good description of the pulsation modes and the criteria for stability. The full behavior, propagation, etc., can of course only be studied in a global framework. The object of this research is to eventually treat a flow with viscosity and thermal effects fully included. This first paper presents the general method of analysis that will be expanded upon in subsequent papers.

We consider local, axisymmetric, dynamical perturbations of stationary rotating fluid bodies. First, a rather general fifth order dispersion relation is derived. Since it is an impossible task to solve the general dispersion relation, we introduce a method in which the various terms in this dispersion relation are classified according to their dependence on five dimensionless parameters (Appendix C). Our analysis proceeds by examining the order of magnitude of all terms rather than directly neglecting terms in the coefficients for the low viscosity case, thus making clear the general method to be employed. The dominant terms found for all points in the five-dimensional phase space compose an approximate dispersion relation whose stability properties should closely approach those of the complete system. As an example, we study the stability of a two-dimensional subspace of the five-dimensional parameter space. Although our attention is directed to thick accretion disks, our approach is valid for any rotating configuration which obeys our basic assumptions.

In § II of this article we describe our basic assumptions and present a general discussion of the order of magnitude of the variables determining the pulsation modes of rotating objects. We begin § III with the general equations governing rotating non-self-gravitating fluids. Aiming at thick accretion disks, we specialize to the negligible viscosity case for which we write down the dispersion relation and devise a "magnitude" method for analyzing it. In formulating this method we again focus on thick accretion disks (which allows us to restrict the relevant parameters to five). We examine in § IV the stability conditions for a two-dimensional subspace of the parameter space. Section V includes a physical discussion of the results and conclusions. Appendices A and B review the theory of thick disks and of the stability of rotating configurations. Appendix C includes a catalog of the terms appearing in the dispersion relation for a three-dimensional subspace of the five-dimensional parameter space.

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TABLE 1

BASIC ASSUMPTIONS OF THIS APPROACH

A. EQUILIBRIUM

a	$X_0 = X_0(r, z)$, $\partial X/\partial\varphi = 0 = \partial X_0/\partial t$
b	The configuration is not self-gravitating
c	No electromagnetic field present
	B. PERTURBATIONS
d	$X(t, r, z) - X_0(r, z) = \delta X(t, r, z) = \langle \delta X \rangle \exp\left[i(\omega t + \kappa_r r + \kappa_z z)\right]$
e	Angular momentum and entropy are conserved
f	$\dfrac{\text{Wavelength of perturbation}}{\text{scale-height}} \ll 1$

II. BASIC ASSUMPTIONS, DEFINITIONS, AND ORDER-OF-MAGNITUDE ESTIMATES

We consider stationary and axisymmetric configurations. Unperturbed quantities (which are denoted with a subscript zero) depend on r, the cylindrical radius, and z. The fluid is rotating in the external gravitational field of a central object. We ignore the self-gravitation of the fluid, and we use the Newtonian gravitational force. Long-range electromagnetic forces are not present. Only axisymmetric, local, and dynamical (adiabatic) perturbations will be discussed. These basic assumptions are summarized in Table 1.

Five forces should balance for a dynamical equilibrium. These are: F_i, F_g, F_c, F_p, and F_v, the inertial (other than centrifugal), gravitational, centrifugal, pressure gradient, and viscous forces, respectively. We can estimate the order of magnitude of these forces as:

$$F_i \approx \frac{(\text{velocity of non-azimuthal flow})^2}{\text{scale height}} = \frac{v^2}{H}, \tag{2.1}$$

$$F_g \approx \frac{(\text{Keplerian velocity})^2}{\text{radius}} = \frac{v_K{}^2}{r}, \tag{2.2}$$

$$F_c \approx \frac{(\text{velocity of rotation})^2}{\text{radius}} = \frac{v_\varphi{}^2}{r}, \tag{2.3}$$

$$F_p \approx \frac{(\text{velocity of sound})^2}{\text{scale height}} = \frac{v_s{}^2}{H}, \tag{2.4}$$

$$F_v \approx \alpha \frac{(\text{velocity of sound})^2}{\text{scale height}} = \alpha \frac{v_s{}^2}{H}. \tag{2.5}$$

The scale height H in these formulae is either one of H_r or H_z:

$$H_r \equiv \left(\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r}\right)^{-1}, \qquad H_z \equiv \left(\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}\right)^{-1}, \tag{2.6}$$

where ρ_0 denotes the unperturbed density distribution. The dimensionless viscosity parameter, α, is defined as

$$\alpha = -\eta p^{-1} r(\partial\Omega/\partial r), \tag{2.7}$$

where η, p, and Ω are the viscosity, pressure, and angular velocity, respectively.

The seven parameters appearing in (2.1)–(2.7) determine the local dynamical equilibrium of rotating fluid masses. These are four velocities, two characteristic scale heights and the viscosity parameter α. The relative importance of these parameters is shown in Table 2 for rotating stellar atmospheres, thin disks, and thick disks.

Pulsations with a wave vector κ_r, κ_z and a wavelength λ, have typical frequencies, ω, which can be constructed from these variables:

$$\omega = f^{1/2} V/L, \tag{2.8}$$

where $V = (v, v_K, v_\varphi, v_s)$ is one of the characteristic velocities, $L = (r, H_r, H_z, \kappa_r{}^{-1}, \kappa_z{}^{-1})$ is one of the characteristic lengths, and f is a dimensionless, model-dependent factor.

The equilibrium parameters are not completely independent since the following order-of-magnitude relations generally hold for rotating objects:

$$\frac{h}{r} \approx \frac{H_z}{H_r} \approx \frac{v_z}{v_r} \approx \frac{v_s}{v_\varphi}. \tag{2.9}$$

TABLE 2

EQUILIBRIUM PARAMETERS FOR ROTATING FLUIDS

Parameter	Rotating Stellar Atmosphere	Thin Disk	Thick Disk
Shape	Spheroidal	Disklike	Toroidal
Rotation	Slow: $v_\varphi \ll v_K$, $v_\varphi \ll v_s$	Fast: $v_\varphi = v_K$, $v_\varphi \gg v_s$	Fast: $v_\varphi \approx v_K$, $v_\varphi \approx v_s$
Pressure	Important: $v_s \approx v_K$	Unimportant: $v_s \ll v_K$, $v_s \ll v_\varphi$	Important: $v_s \approx v_\varphi$, $v_s \approx v_K$
Viscosity	Unimportant: $\alpha v_\varphi \ll v_s$	Large: $\alpha \approx 1$	Small: $\alpha \ll 1$
Non-azimuthal flow	Unimportant: $v \ll v_s$, $v \ll v_\varphi$, $v \ll v_K$	Unimportant in central and outer parts: $v \ll v_\varphi$ Important in innermost parts: $v \approx v_s$	Unimportant in central and outer parts: $v \ll v_\varphi$ Important in innermost parts: $v \approx v_s$
Horizontal and vertical structure	Similar: $H_r \approx H_z$	Very different: $H_r \gg H_z$	Similar: $H_r \approx H_z$

Here $h = h(r)$ describes the location of the surface $(p = 0)$ of the object; the equation for the surface is $z = h$. Therefore, any pulsation frequency in the disk will be a combination of eight standard frequencies, constructed according to (2.8). These frequencies are listed in Table 3. In this table and elsewhere, we use the terms "infinitesimal" (or "finite") for perturbations which are independent (or dependent) on the scale heights H or on r.

III. BASIC EQUATIONS AND METHOD OF SOLUTION

The basic equations describing both the unperturbed and perturbed states are:
Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial z}(\rho v_z) = 0 \; ; \tag{3.1'}$$

conservation of momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r}\right) + \rho \frac{\partial \Phi}{\partial r} + \frac{\partial p}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\partial \tau_{rz}}{\partial z} = 0 \, , \tag{3.2}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\right) + \rho \frac{\partial \Phi}{\partial z} + \frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr}) - \frac{\partial \tau_{zz}}{\partial z} = 0 \, , \tag{3.3}$$

$$\rho\left(\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + v_z \frac{\partial v_\varphi}{\partial z} + \frac{v_\varphi v_r}{r}\right) - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{\varphi r}) + \frac{\tau_{\varphi r}}{r} + \frac{\partial \tau_{\varphi z}}{\partial z}\right] = 0 \; ; \tag{3.4}$$

isentropy:

$$\rho T\left(\frac{\partial S}{\partial t} + v_r \frac{\partial S}{\partial r} + v_z \frac{\partial S}{\partial z}\right) - \left(\frac{\tau_{r\varphi}^{\ 2}}{\eta} + \frac{\tau_{z\varphi}^{\ 2}}{\eta} + \frac{\tau_{rr}^{\ 2}}{\eta} + \frac{\tau_{zz}^{\ 2}}{\eta} + \frac{\tau_{zr}^{\ 2}}{\eta}\right) - \nabla \cdot \boldsymbol{F} = 0 \; ; \tag{3.5}$$

TABLE 3

EIGHT STANDARD TYPES OF PULSATIONS

Type	"Infinitesimal"	"Finite"
Sound	$\omega_{(1)} = v_s \kappa$ [a]	$\omega_{(2)} = v_s/H$ [b]
Inertial	$\omega_{(3)} = v\kappa$ [c]	$\omega_{(4)} = v/H$
Rotational	$\omega_{(5)} = v_\varphi \kappa$	$\omega_{(6)} = v_\varphi/r$
Gravitational	$\omega_{(7)} = v_k \kappa$	$\omega_{(8)} = v_k/r$

[a] Sound wave: stable.
[b] Brunt-Väisälä frequency.
[c] Perturbation of the velocity field: stable.

where τ_{ij} is the viscous stress tensor, $\boldsymbol{F}$ the radiative flux, and Φ the gravitational potential. But as we are currently interested in the stability of *thick* disks, we can neglect all of the terms which include viscosity and thermal effects. Then (3.1)–(3.5) assume the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial z}(\rho v_z) = 0 , \tag{3.1'}$$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r}\right) + \rho\frac{\partial \Phi}{\partial r} + \frac{\partial p}{\partial r} = 0 , \tag{3.2'}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) + \rho\frac{\partial \Phi}{\partial z} + \frac{\partial p}{\partial z} = 0 , \tag{3.3'}$$

$$r\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial}{\partial r}(v_\varphi r) + rv_z\frac{\partial v_\varphi}{\partial z} = 0 , \tag{3.4'}$$

$$\frac{\partial p}{\partial t} + v_r\frac{\partial p}{\partial r} + v_z\frac{\partial p}{\partial z} - \frac{p}{\rho}\Gamma_1\left(\frac{\partial \rho}{\partial t} + v_r\frac{\partial \rho}{\partial r} + v_z\frac{\partial \rho}{\partial z}\right) = 0 , \tag{3.5'}$$

where $\Gamma_1 \equiv (\partial \ln p/\partial \ln \rho)_s$.

We are now concerned with the local stability of perturbations about stationary solutions to equations (3.1′)–(3.5′). Although specific solutions to these equations have yet to be derived, as they demand a three-dimensional numerical treatment, such solutions certaintly exist (if time-averaged values in convective zones are used; Paczyński and Abramowicz 1982). For such an equilibrium, (3.4′) and (3.5′) can be stated as: $\boldsymbol{v} \cdot \nabla l = \boldsymbol{v} \cdot \nabla S = 0$, with l the specific angular momentum and S the specific entropy. If we have an axisymmetric stationary fluid with no dissipation and nonzero v_r and/or v_z, then the level surfaces of S and l coincide; but if $v_r = v_z = 0$, then the surfaces of S and l need not do so. As many thick disk models have regions where $(\partial l/\partial r) = 0$ (e.g., Jaroszyński, Abramowicz, and Paczyński 1980), then (3.4′) means only that $v_z = 0$ or that l is fixed on cylinders and $\nabla S \neq 0$ is still possible for finite $\boldsymbol{v}$. When S- and l-surfaces do coincide, reasonable thick disk models have been produced (Paczyński and Abramowicz 1982; Rózyczka and Muchotrzeb 1982). This might seem to be overly restrictive, but it must be stressed that thick disks are intermediate between thin disks and spherical accretion, and can have nonzero v_r and v_z with extremely small values of α. Surfaces of S and l do not necessarily coincide to the extent that the non-azimuthal velocities are small with respect to v_φ, and this is a key condition for thick disks (cf. Abramowicz and Zurek 1981). The immediate application of the above restrictions would enable us to achieve some simplification of the dispersion relation derived below (3.26), but we wish to demonstrate the general procedure by retaining all terms at this stage. Simpler ways of obtaining some of our results are discussed in § V.

Using the standard linear perturbation technique, we rewrite the equations (3.1′)–(3.5′) as equations for the linear parts of the perturbations $\delta\rho$, δp, δv_φ, δv_r, δv_z in the form

$$\begin{bmatrix} (i\omega + \rho_1) & 0 & 0 & r_1 & z_1 \\ \rho_2 & p_2 & \varphi_2 & (i\omega + r_2)\rho & z_2 \\ \rho_3 & p_3 & 0 & r_3 & (i\omega + z_3)\rho \\ -v_s^2(i\omega + \rho_4) & (i\omega + p_4) & 0 & r_4 & z_4 \\ 0 & 0 & (i\omega + \varphi_5) & r_5 & z_5 \end{bmatrix} \begin{bmatrix} \delta\rho \\ \delta p \\ \delta v_\varphi \\ \delta v_r \\ \delta v_z \end{bmatrix} = 0 . \tag{3.6}$$

The quantities $\rho_1, \ldots, z_1, \ldots, z_5$ are defined by

$$\rho_1 = i\chi + \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} , \tag{3.7}$$

$$r_1 = \frac{\partial \rho}{\partial r} + \frac{\rho}{r} + i\rho\kappa_r , \tag{3.8}$$

$$z_1 = \frac{\partial \rho}{\partial z} + i\rho\kappa_z , \tag{3.9}$$

$$\rho_2 = v_r\left(\frac{\partial v_r}{\partial r}\right) + v_z\left(\frac{\partial v_r}{\partial z}\right) - \frac{v_\varphi^2}{r} + \frac{\partial \Phi}{\partial r} , \tag{3.10}$$

$$p_2 = i\kappa_r , \tag{3.11}$$

$$\varphi_2 = -2v_\varphi \frac{\rho}{r}, \tag{3.12}$$

$$r_2 = \frac{\partial v_r}{\partial r} + i\chi, \tag{3.13}$$

$$z_2 = \left(\frac{\partial v_r}{\partial z}\right)\rho, \tag{3.14}$$

$$\rho_3 = v_r\left(\frac{\partial v_z}{\partial r}\right) + v_z\left(\frac{\partial v_z}{\partial z}\right) + \frac{\partial \Phi}{\partial z}, \tag{3.15}$$

$$p_3 = i\kappa_z, \tag{3.16}$$

$$r_3 = \left(\frac{\partial v_z}{\partial r}\right)\rho, \tag{3.17}$$

$$z_3 = i\chi + \frac{\partial v_z}{\partial z}, \tag{3.18}$$

$$\rho_4 = i\chi - \left[\frac{v_r}{\rho}\left(\frac{\partial \rho}{\partial r}\right) + \frac{v_z}{\rho}\left(\frac{\partial \rho}{\partial z}\right)\right], \tag{3.19}$$

$$p_4 = i\chi - \Gamma_1\left[\frac{v_r}{\rho}\left(\frac{\partial \rho}{\partial r}\right) + \frac{v_z}{\rho}\left(\frac{\partial \rho}{\partial z}\right)\right], \tag{3.20}$$

$$r_4 = \frac{\partial p}{\partial r} - v_s^2\left(\frac{\partial \rho}{\partial r}\right), \tag{3.21}$$

$$z_4 = \frac{\partial p}{\partial z} - v_s^2\left(\frac{\partial \rho}{\partial z}\right), \tag{3.22}$$

$$\varphi_5 = \frac{v_r}{r} + i\chi, \tag{3.23}$$

$$r_5 = \frac{v_\varphi}{r} + \frac{\partial v_\varphi}{\partial r}, \tag{3.24}$$

$$z_5 = \frac{\partial v_\varphi}{\partial z}, \tag{3.25}$$

where $\chi = v_r \kappa_r + v_z \kappa_z$.

The condition for a nontrivial solution of (3.6) is the vanishing of the determinant of the matrix which appears there. This can be written as the fifth order dispersion relation:

$$ia_5\left(\frac{\omega}{\Omega}\right)^5 + a_4\left(\frac{\omega}{\Omega}\right)^4 - ia_3\left(\frac{\omega}{\Omega}\right)^3 - a_2\left(\frac{\omega}{\Omega}\right)^2 + ia_1\left(\frac{\omega}{\Omega}\right) + a_0 = 0, \tag{3.26}$$

where the dimensionless coefficients a_0–a_5 are:

$$a_5 = 1, \tag{3.27}$$

$$a_4 = \Omega^{-1}(p_4 + r_2 + z_3 + \varphi_5 + \rho_1), \tag{3.28}$$

$$\begin{aligned} a_3 = -(\rho\Omega)^{-2}[&\rho\varphi_2 r_5 - \varphi_5 \rho_1 \rho^2 - (\varphi_5 + \rho_1)(p_4 + r_2 + z_3)\rho^2 - \rho^2(p_4 r_2 + p_4 z_3 + r_2 z_3) \\ &+ \rho(r_4 p_2 + z_4 p_3) + z_2 r_3 + r_1(\rho_2 \rho + v_s^2 p_2 \rho) + v_s^2 z_1 p_3 \rho + \rho\rho_3 z_1], \end{aligned} \tag{3.29}$$

$$\begin{aligned} a_2 = -(\rho^{-2}\Omega^{-3})\{&\varphi_2 r_5 \rho(\rho_1 + p_4 + z_3) - \varphi_2 z_5 r_3 - (p_2 r_3 z_4 + p_3 r_4 z_2 + \rho^2 p_4 r_2 z_3 - z_2 r_3 p_4 - \rho z_3 r_4 p_2 - \rho z_4 p_3 r_2) \\ &- (\varphi_5 + \rho_1)[\rho^2(p_4 r_2 + p_4 z_3 + r_2 z_3) - z_2 r_3 - \rho r_4 p_2 - \rho z_4 p_3] - \varphi_5 \rho_1 \rho^2(p_4 + r_2 + z_3) + r_1 \varphi_5 \rho(\rho_2 + v_s^2 p_2) \\ &- r_1[\rho_3 z_2 - v_s^2 p_2 \rho(\rho_4 + z_3) + v_s^2 p_3 z_2 - \rho_2 \rho(z_3 + p_4)] + z_1 \varphi_5 \rho(\rho_3 + v_s^2 p_3) \\ &+ z_1[(p_4 + r_2)\rho_3 \rho - v_s^2 p_2 r_3 + v_s^2 p_3 \rho(\rho_4 + r_2) - r_3 \rho_2]\}, \end{aligned} \tag{3.30}$$

$$
\begin{aligned}
a_1 = (\rho\Omega^2)^{-2}\{&\varphi_2 r_5[p_3 z_4 + \rho_3 z_1 + v_s^2 z_1 p_3 - \rho(\rho_1 p_4 + p_4 z_3 + \rho_1 z_3)] - \varphi_2 z_5[p_3 r_4 + \rho_3 r_1 + v_s^2 p_3 r_1 - r_3(p_4 + \rho_1)] \\
&+ (\varphi_5 + \rho_1)[p_2 r_3 z_4 + p_3 r_4 z_2 - z_2 r_3 p_4 + \rho^2 p_4 r_2 z_3 - \rho(z_3 r_4 p_2 + z_4 p_3 r_2)] \\
&+ \varphi_5 \rho_1[\rho^2(p_4 r_2 + p_4 z_3 + r_2 z_3) - z_2 r_3 - \rho(r_4 p_2 + z_4 p_3)] \\
&+ r_1(\rho_2 p_3 z_4 + \rho_3 z_2 p_4 - v_s^2 \rho_4 z_3 p_2 \rho + v_s^2 \rho_4 p_3 z_2 - \rho_3 p_2 z_4 - z_3 p_4 \rho_2 \rho) \\
&+ r_1 \varphi_5[\rho_3 z_2 - v_s^2 p_2 \rho(\rho_4 + z_3) + v_s^2 p_3 z_2 - \rho_2 \rho(z_3 + p_4)] \\
&- z_1(\rho_2 p_3 r_4 + \rho_3 \rho p_4 r_2 - v_s^2 p_2 r_3 \rho_4 + v_s^2 \rho_4 r_2 \rho p_3 - r_3 \rho_2 p_4 - r_4 p_2 \rho_3) \\
&- z_1 \varphi_5[\rho_3 \rho(p_4 + r_2) - v_s^2 p_2 r_3 + v_s^2 \rho p_3(\rho_4 + r_2) - r_3 \rho_2]\}\,,
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
a_0 = (\rho^2\Omega^5)^{-1}\{&\varphi_2 r_5[p_3 z_4 \rho_1 + p_4 \rho_3 z_1 - \rho p_4 \rho_1 z_3 + v_s^2 z_1 p_3 \rho_4] - \varphi_2 z_5[\rho_1 p_3 r_4 + \rho_3 r_1 p_4 - r_3 p_4 \rho_1 + v_s^2 \rho_4 p_3 r_1] \\
&+ \varphi_5 \rho_1[p_2 r_3 z_4 + p_3 r_4 z_2 - z_2 r_3 p_4 + \rho^2 p_4 r_2 z_3 - z_3 r_4 p_2 \rho - z_4 p_3 r_2 \rho] \\
&+ r_1 \varphi_5[\rho_2 p_3 z_4 + \rho_3 z_2 p_4 - v_s^2(\rho_4 z_3 p_2 \rho - \rho_4 p_3 z_2) - z_3 p_4 \rho_2 \rho - \rho_3 p_2 z_4] \\
&- z_1 \varphi_5[\rho_2 p_3 r_4 + \rho_3 p_4 r_2 \rho - r_3 \rho_2 p_4 - v_s^2(p_2 r_3 \rho_4 - \rho_4 r_2 p_3 \rho) - r_4 p_2 \rho_3]\}\,.
\end{aligned}
\tag{3.32}
$$

In practice it is impossible to examine the stability conditions for the full dispersion relation (3.26). Such a general discussion is only of academic interest anyway. Since we are interested in local stability, it is sufficient to consider the stability of given small regions of the whole configuration. In a small region particular conditions hold, and these can be used to simplify the dispersion relation. In the following section we develop a method for specializing the dispersion relation to particular regions (in a specified parameter space) by picking up only its dominant terms. The resulting dispersion relation is tractable.

We have already shown that the variables v_r, v_z, v_φ, v_K, v_s, H_r, H_z, κ_r, κ_z, r, and α determine the local conditions in any non-self-gravitating rotating object. In the following discussion we are interested only in the order of magnitude of these variables; therefore, for disks (in fact, for any rapidly rotating object) we can eliminate v_K since $v_K = O(v_\varphi)$. For thick accretion disks we can further eliminate a few other variables, since using (2.9) $v \approx v_r \approx v_z$ and $H \approx H_r \approx H_z$. For these disks the scale height H is also of the same order of magnitude as the radius r.

We now convert all quantities to dimensionless variables. The basic approximation (f) provides us with a standard dimensionless variable:

$$\epsilon = (\lambda/H) \ll 1\,. \tag{3.33}$$

We next define five other dimensionless variables: α, (v/v_φ), (v_s/v_φ), $\mu \equiv |\boldsymbol{\kappa} \cdot \boldsymbol{v}|/|\boldsymbol{\kappa}||\boldsymbol{v}|$, and (ω/Ω). Without loss of generality we can express these five variables as:

$$\mu \sim \epsilon^j\,, \tag{3.34}$$

$$(v_s/v_\varphi) \sim \epsilon^k\,, \tag{3.35}$$

$$\alpha \sim \epsilon^l\,, \tag{3.36}$$

$$(\omega/\Omega) \sim \epsilon^m\,, \tag{3.37}$$

$$(v/v_\varphi) \sim \epsilon^n\,. \tag{3.38}$$

Each term in the dispersion relation can be expressed in the form

$$\text{term} \sim \mu^{N_1}(v_s/v_\varphi)^{N_2}\alpha^{N_3}(\omega/\Omega)^{N_4}(v/v_\varphi)^{N_5}(\lambda/H)^{N_6}\,. \tag{3.40}$$

This defines the "magnitude," M, of each term:

$$M = N_1 j + N_2 k + N_3 l + N_4 m + N_5 n + N_6\,. \tag{3.41}$$

We classify different regions in the disks of interest according to their locations in the five-dimensional phase space $[j, k, l, m, n]$, and compute the value of M for each term in the dispersion relation. We retain only the terms of lowest magnitude and solve the resulting approximate dispersion relations. Note that sometimes higher order terms ought to be kept in order to discover slower instabilities for which $|\mathrm{Im}\,\omega| \ll |\mathrm{Re}\,\omega|$.

Systematic application of this procedure to the entire parameter space $[j, k, l, m, n]$ can provide us with detailed information about the local stability of all the possible configurations satisfying our basic assumptions, in particular thick and thin accretion disks and rotating stellar atmospheres. The stability conditions vary from one point to another in the parameter space, but usually a given condition holds in some finite subspace. The purpose of this series of papers is to scan the parameter space and to derive the relevant stability conditions.

We now examine the allowed range for the parameters j, k, l, m, and n. The parameter m can range from $-\infty$ to $+\infty$. Both μ and α are less than (or of order) unity, and therefore j and l vary between 0 and $+\infty$. In principle n could vary between $-\infty$ and $+\infty$, however negative n-values mean that $v > v_\varphi$, i.e., a configuration in which the non-azimuthal (radial infall) velocity is larger than the rotational velocity; clearly the current approach is not intended to analyze such situations.

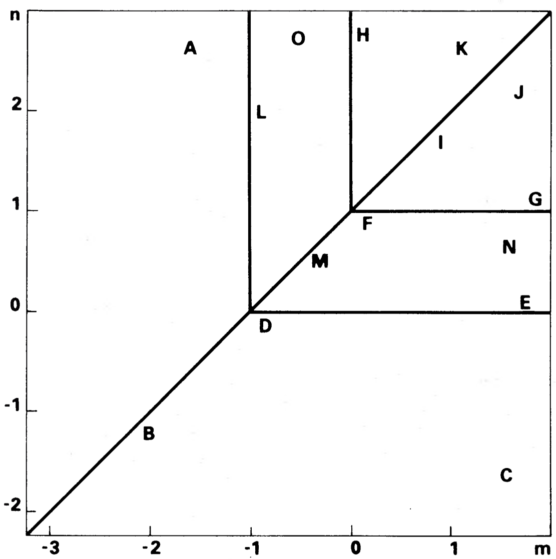

FIG. 1

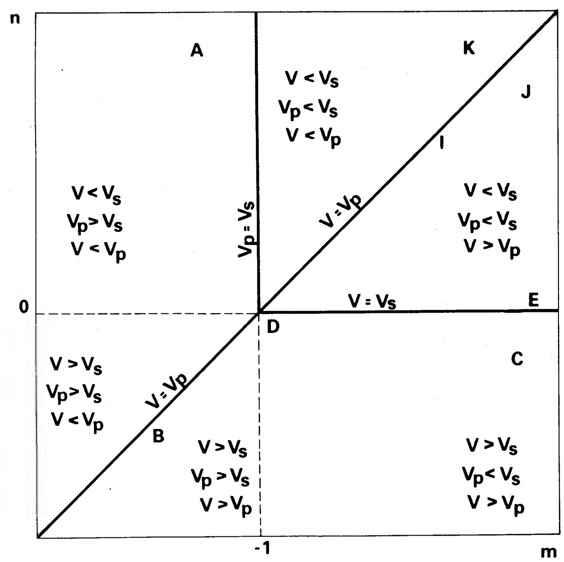

FIG. 2

FIG. 1.—The division of the *m-n* plane into the 15 regions corresponding to the different first order dispersion relations. The areas are labelled A, C, J, K, N, and O, the lines dividing them are labeled B, E, G, H, I, L, and M, and the points of intersection are D and F.

FIG. 2.—The physical properties corresponding to the different regions in the *m-n* plane. Here v = the accretion velocity, v_p = the velocity of the perturbation, and v_s = the sound speed in the disk. For our thick disks, the Keplerian (v_K) and azimuthal (v_φ) velocities are approximately equal to v_s.

The two parameters l and k are independent, but both are related to the disk's thickness. We have already implicitly set $l = \infty$ by neglecting viscous terms in equation (3.1′–3.5′). The dispersion relation (3.26) is valid for every k; still for thick disks we expect $k = 0$ and we restrict the rest of the discussion, and in particular the catalog given in Appendix C, to this special case. In addition to these restrictions, note that some part of the parameter phase space is not physically accessible since certain combinations of j, k, l, m, and n lead to contradictory physical assumptions.

IV. THE NEGLIGIBLE VISCOSITY, PARALLEL PERTURBATION, THICK DISK CASE

As a physically relevant example we consider in this section the question of dynamical stability of thick disks ($k = 0$) of negligible viscosity ($l = \infty$) and we examine perturbations which are parallel to the non-azimuthal flow ($j = 0$). In addition, we shall assume that $\kappa_r \approx \kappa_z$, i.e., that the perturbation is isotropic.

In light of the specializations discussed above, the magnitudes of the terms in the dispersion relation now only involve N_4, N_5, and N_6, so that

$$M = N_4 m + N_5 n + N_6 . \tag{4.1}$$

For each individual term in the dispersion relation the values for N_4, N_5, and N_6 are given in the Catalog (Appendix C).

We compute the magnitude of all terms for

$$-\infty \leq m \leq +\infty , \qquad -\infty \leq n \leq +\infty . \tag{4.2}$$

The $[m, n]$-plane is divided into 15 regions in which there are different lowest order dispersion relations. These regions comprise six different areas, A, C, O, N, K, and J, seven different lines, L, H, I, G, M, E, and B, and two points, D and F, as shown in Figure 1.

The diagonal line across the plane, $n = m + 1$, corresponds to the condition that the velocity of the perturbation is approximately equal to the non-azimuthal velocity. Indeed, the velocity of the perturbation can be estimated by

$$v_p \approx \lambda\omega , \tag{4.3}$$

and therefore,

$$v_p/v_\varphi \approx (\lambda/H)(\omega/\Omega) \sim \epsilon^{m+1} . \tag{4.4}$$

On the other hand, $(v/v_\varphi) \sim \epsilon^n$, and the condition $(v/v_\varphi) \approx (v_p/v_\varphi)$ is equivalent to $n = m + 1$. Other lines also have clear physical meanings. The condition that the velocity of the perturbation is about equal to the speed of sound corresponds to the equation $m = -1$, i.e., to the line L. Also, the condition that the velocity of sound is approximately equal to the non-azimuthal velocity is equivalent to the equation $n = 0$, or the line E.

The physical meanings of the different parts of the $[m, n]$-plane are shown in Figure 2. Note the similarity between Figures 1 and 2, which, of course, stems from the fact that different stability properties are connected with different physical regimes.

We shall now explain how the stability of the 15 different cases has been examined by taking the line H as an example. On that line we have

$$v_r \approx v_z \approx 0\ , \tag{4.5}$$

$$|\omega| \approx \Omega\ . \tag{4.6}$$

There are seven relevant terms of dominating order ϵ^{-2} in the dispersion relation, viz., 137, 153, 188, 213, 238, 263, and 373 in the Catalog. The dispersion relation for this case reads

$$\left(\frac{\omega}{\Omega}\right)^2 B_0 = \kappa_z^2\left[r^{-3}\left(\frac{\partial l^2}{\partial r}\right) + g_r Q_r\right] + \kappa_r^2[g_z Q_z] + \kappa_r \kappa_z\left[-r^{-3}\left(\frac{\partial l^2}{\partial z}\right) - g_r Q_z\right] + \kappa_z \kappa_r[-g_z Q_r] \tag{4.7}$$

$$\equiv \kappa_z^2 B_{zz} + \kappa_r \kappa_z B_{rz} + \kappa_z \kappa_r B_{zr} + \kappa_r^2 B_{rr} \equiv H(\kappa_r, \kappa_z)\ .$$

Here $l = \Omega r^2$ is the specific angular momentum and

$$Q_r = \left[\frac{\partial p}{r} - v_s^2\left(\frac{\partial \rho}{\partial r}\right)\right]\frac{1}{\Gamma_1 p}\ , \tag{4.8a}$$

$$Q_z = \left[\frac{\partial p}{\partial z} - v_s^2\left(\frac{\partial \rho}{\partial z}\right)\right]\frac{1}{\Gamma_1 p}\ , \tag{4.8b}$$

$$g_r = -\frac{v_\varphi^2}{r} + \frac{\partial \Phi}{\partial r}\ , \qquad g_z = \frac{\partial \Phi}{\partial z}\ , \qquad B_0 > 0\ . \tag{4.9}$$

Note that from the mechanical equilibrium conditions (3.2) and (3.3) with $v_r = v_z = 0$ it follows that

$$(\nabla \rho^{-1}) \times (\nabla p) = (r^{-3}\nabla l^2) \times e_r\ , \tag{4.10}$$

which is equivalent to

$$Q_r g_z - Q_z g_r = r^{-3}(\partial l^2/\partial z)\ . \tag{4.11}$$

Thus the matrix B_{ij} is symmetric.

The necessary and sufficient condition for stability is that the form $\kappa_i \kappa_j B_{ij}$ should be positive definite, i.e.,

$$\text{Trace}\ (B_{ij}) > 0\ , \qquad \text{Det}\ (B_{ij}) > 0\ . \tag{4.12}$$

These two conditions are equivalent to

$$r^{-3}\left(\frac{\partial l^2}{\partial r}\right) - \left(\frac{\partial T}{\partial p}\right)_s (\nabla p \cdot \nabla S) > 0\ , \tag{4.13}$$

$$-\left(\frac{\partial p}{\partial z}\right)\left[\left(\frac{\partial l^2}{\partial r}\right)\left(\frac{\partial S}{\partial z}\right) - \left(\frac{\partial l^2}{\partial z}\right)\left(\frac{\partial S}{\partial r}\right)\right] > 0\ . \tag{4.14}$$

This is exactly the well known Hoiland criterion for stability (e.g., Tassoul 1978). When no rotation is involved ($l = 0$), it yields

$$(-g)\nabla S > 0\ , \tag{4.15}$$

which is just the Schwarzschild criterion for convective stability. It also ensures that all g-modes are stable, as will be discussed later (see also Cox 1980). In the case of a homentropic configuration ($\nabla S = 0$) the Hoiland criterion becomes

$$\frac{d(l^2)}{dr} > 0\ . \tag{4.16}$$

Therefore, for stability the specific angular momentum must increase outward. Condition (4.16) is the Solberg criterion which generalizes to homentropic configurations the Rayleigh criterion for an inviscous incompressible fluid.

In a similar way, the other 14 cases have been studied. The results are summarized in Table 4.

V. DISCUSSION

Let us now discuss the results presented in Table 4 in those regions where definite solutions have been obtained. In order to do this, it is useful, for example, to move from high frequencies to low ones (from left to right in the [m, n]-plane). Area A (Fig. 1) gives inconsistent results. This fact is not surprising since it is the consequence of the requirement that ω is so large so that only the term ω^5 survives in the dispersion relations. This, however, leads to the solution, $\omega = 0$, which contradicts the assumption that ω is large.

Line L, which corresponds to a high-frequency, low-velocity limit, gives a stable frequency

$$\omega = \pm v_s(\kappa_r^2 + \kappa_z^2)^{1/2}\ . \tag{5.1}$$

TABLE 4

STABILITY PROPERTIES ON THE $j = 0$, $k = 0$, $l = +\infty$ SURFACE

Region	Typical Point	Terms Included	Dispersion Relation, Its Solution	Result[a]
A (area)	$n = 0$, $m = -2$	380	$\omega^5 = 0$, No solution	impossible
B (line)	$n = -1$, $m = -2$	33, 163, 177, 294 304, 312, 365, 368 378, 380	$(\omega + \chi)^5 = 0$, $\omega = -\chi$	stable
C (area)	$n = -1$, $m = -1$	33	$\chi^5 = 0$, No solution	impossible
D (point)	$n = 0$, $m = -1$	33, 68, 113, 163, 177, 198, 223, 253, 279, 294, 304, 312, 322, 329, 340, 357, 365, 368, 373, 378, 380	$(\omega + \chi)^5 - v_s^2(\kappa_r^2 + \kappa_z^2) \times (\omega + \chi)^3 = 0$, $\omega = -\chi \pm v_s(\kappa_r^2 + \kappa_z^2)^{1/2}$	stable
E (line)	$n = 0$, $m = 0$	33, 68, 113	$\chi^2 - v_s^2(\kappa_r^2 + \kappa_z^2) = 0$, ?	second order terms
F (point)	$n = 1, m = 0$	9, 24, 54, 68, 87, 93, 113, 126, 137, 153, 188, 198, 213, 223, 238, 253, 263, 279, 322, 329 340, 357, 373	$v_s^2(\kappa_r^2 + \kappa_z^2)(\omega + \chi)^3 - H(\kappa_r, \kappa_z)(\omega + \chi) = 0$, $\omega = -\chi$; $\omega = -\chi \pm [H(\kappa_r, \kappa_z)/(\kappa_r^2 + \kappa_z^2)]^{1/2}$	Two modes: Hoiland and stable
G (line)	$n = 1$, $m = 0$	9, 24, 54, 68, 87, 93, 113, 126	$\chi^2 - \dfrac{H(\kappa_r, \kappa_z)}{(\kappa_r^2 + \kappa_z^2)} = 0$, ?	second order terms
H (Line)	$n = 2$, $m = 0$	137, 153, 188, 213, 238, 263, 373	$\omega^2 B_0 - \Omega^2 B_{ij} \kappa_i \kappa_j = 0$, $\omega = \left(\dfrac{H(\kappa_r, \kappa_z)}{(\kappa_r^2 + \kappa_z^2)}\right)^{1/2}$	Hoiland
I (line)	$n = 2$, $m = 1$	9, 24, 54, 87, 93, 126, 137, 153, 188, 213, 238, 263	$H(\kappa_r, \kappa_z)(\chi + \omega) = 0$, $\omega = -\chi$	stable
J (area)	$n = 2$, $m = 2$	9, 24, 54, 87, 93, 126	$H(\kappa_r, \kappa_z)\chi = 0$, No solution	impossible
K (area)	$n = 3$, $m = 1$	137, 153, 188, 213, 238, 263	$H(\kappa_r, \kappa_z)\omega = 0$, $\omega = 0$	neutral
L (Line)	$n = 1$, $m = -1$	373, 380	$\omega^2 - v_s^2(\kappa_r^2 + \kappa_z^2) = 0$, $\omega = v_s(\kappa_r^2 + \kappa_z^2)^{1/2}$	stable
M (line)	$n = 0.1$, $m = -0.9$	68, 113, 198, 223, 253, 279, 322, 329, 340, 357, 373	$v_z^2(\kappa_r^2 + \kappa_z^2)(\omega + \chi)^3 = 0$, $\omega = -\chi$	stable
N (area)	$n = 0.2$, $m = 1.0$	68, 113	$v_s^2(\kappa_z^2 + \kappa_r^2)\chi^3 = 0$, No solution	impossible
0 (area)	$n = 0.9$, $m = -0.1$	373	$v_z^2(\kappa_r^2 + \kappa_z^2)\omega^3 = 0$, No solution	impossible

[a] The Result column indicates if the region under consideration is always stable, if it is neutral to perturbations, if the Hoiland criterion determines the stability, or if the region corresponds to a physically contradictory situation ("impossible"), where there is no consistent solution of the dispersion relations for the assumed values of m and n. A question mark in the Solution column indicates that the first order terms identically vanish so that second order terms ought to be included in determining the stability criterion in that region.

This is the equivalent in thick disks of high order p-modes in stars (e.g., Cox 1980). It is also a known frequency limit in adiabatic stability analyses of stars.

At point D (Fig. 1) the velocity field becomes important, and the frequency becomes that of a sound wave in a moving medium; this represents essentially the Doppler effect (see, e.g. Landau and Lifshitz 1959),

$$\omega = -\boldsymbol{\kappa} \cdot \boldsymbol{v} \pm v_s(\kappa_r^2 + \kappa_z^2)^{1/2} . \tag{5.2}$$

The same result can be obtained by a different procedure. For large ω one writes $\omega = \omega_0 + \Delta\omega$, where ω_0 is given by (5.1), and $\Delta\omega \ll \omega_0$. Neglecting second and higher powers of $\Delta\omega$ and keeping only leading terms in the coefficients, one can show (after some tedious but straightforward algebra) that $\Delta\omega = -\boldsymbol{\kappa} \cdot \boldsymbol{v}$.

The results for line H have been discussed thoroughly in § IV. Here we would like to note that the same results can be obtained formally by setting $v_r = v_z = 0$ in the initial equations and dropping terms in the coefficients of the resulting dispersion relation according to basic approximation (f).

In the limit of negligible rotation ($l \to 0$) we obtain on line H:

$$\omega = \pm[(g_r Q_r)^{1/2}\kappa_z - (g_z Q_z)^{1/2}\kappa_r)](\kappa_r^2 + \kappa_z^2)^{-1/2} . \tag{5.3}$$

It is instructive to evaluate (5.3) in the equatorial plane (where $g_z = Q_z = 0$); there we obtain

$$\omega = \pm \frac{\kappa_z}{(\kappa_r^2 + \kappa_z^2)^{1/2}} \left[\frac{g_r}{\Gamma_1 p}\left(\frac{\partial p}{\partial r} - v_s^2 \frac{\partial \rho}{\partial r}\right)\right]^{1/2} = \frac{\kappa_z}{(\kappa_r^2 + \kappa_z^2)^{1/2}} N , \tag{5.4}$$

where N is the Brunt-Väisaäla frequency. The frequency (5.4) (and in fact [5.3]) represents the high order g-modes of stars (e.g., Cox 1980). Stability or instability is obtained (in the limit of negligible rotation) according to whether

$$A \equiv -Q_r = \rho^{-1}\left(\frac{d\rho}{dr}\right) - (\Gamma_1 p)^{-1}\left(\frac{dp}{dr}\right) \tag{5.5}$$

is negative (stable) or positive (unstable) (e.g., Ledoux and Walraven 1958).

The region O (Fig. 1), while giving inconsistent results when only the leading term is considered, gives the Hoiland stability criterion when second order terms are included and terms are dropped in the coefficients according to the basic assumption (f). The dispersion relation assumes the form

$$\omega^4 - \Gamma_1(p/\rho)(\kappa_r^2 + \kappa_z^2)\omega^2 + \Gamma_1(p/\rho)H(\kappa_r, \kappa_z) = 0 , \tag{5.6}$$

so for stability we need $H(\kappa_r, \kappa_z) > 0$, which leads to the Hoiland criterion, as shown in § IV.

As a final case, we would like to mention that the destabilizing effect of the velocity field can be demonstrated when second order terms are considered in the low frequency domian (such as $m = 1$, $n = 1$ on line G). In the equatorial plane we obtain in this case

$$\omega = -\mathrm{D}/\mathrm{C} + i\mathrm{B}/\mathrm{C} , \tag{5.7}$$

where

$$\mathrm{C} = [r^{-3}(\partial l^2/\partial r) + g_r Q_r]\kappa_r^2 - 3l^2(\kappa_r^2 + \kappa_z^2) , \tag{5.8}$$

$$\mathrm{D} = \chi\{\kappa_z^2[r^{-3}(\partial l^2/\partial r) + g_r Q_r] - \chi^2(\kappa_r^2 + \kappa_z^2)\} , \tag{5.9}$$

$$\begin{aligned} \mathrm{B} = v_r\Bigg\{&\left(\frac{\kappa_z^2}{r}\right)\left[r^{-3}\left(\frac{\partial l^2}{\partial r}\right) + g_r Q_r\right] - \left(\frac{\kappa_z^2}{r^3}\right)\left(\frac{\partial l^2}{\partial r}\right)\left[\rho^{-1}\left(\frac{\partial \rho}{\partial r}\right) + \frac{1}{r}\right] + (\kappa_z\chi)^2\left[\rho^{-1}\left(\frac{\partial \rho}{\partial r}\right) - \frac{v_r}{r} - \frac{\partial v_r}{\partial r}\right] + (\kappa_r\chi)^2\left[\rho^{-1}\left(\frac{\partial \rho}{\partial r}\right) - \frac{v_r}{r}\right]\Bigg\} \\ &+ \left(\frac{\partial v_z}{\partial r}\right)\chi^2\kappa_r\kappa_z - \chi^3\frac{\kappa_r}{r} . \end{aligned} \tag{5.10}$$

This clearly shows that the velocity field introduces the destabilizing part B. In particular, had we taken the limit $\chi \to 0$ (this is not inconsistent with the way the leading terms were kept since $\chi \sim \epsilon^0$), then

$$\omega \sim iv_r\left\{-\left[r^{-3}\rho^{-1}\left(\frac{\partial l^2}{\partial r}\right)\left(\frac{\partial \rho}{\partial r}\right)\right] + g_r\frac{Q_r}{r}\right\}\left[r^{-3}\left(\frac{\partial l^2}{\partial r}\right) + g_r Q_r\right]^{-1} , \tag{5.11}$$

which is unstable ($v_r < 0$).

To summarize, various regimes in the [m, n]-plane, corresponding to different physical situations, give us a variety of pulsations and stability criteria. We have been able to demonstrate that most of the dynamical pulsations obtained in thick disks are very similar to those of ordinary stars; in particular, there exist pulsations that are equivalent to p and g modes. In the limit of low non-azimuthal velocities, the stability criterion for thick disks is the known Hoiland criterion for rotating stars. In the low frequency limit the velocity field introduces an overstability with a frequency of order $\omega \approx v\kappa$.

As explained in the introduction, much work is still needed to sort out cases not discussed in the present paper. For example, thin disks have to be dealt with separately, and "orthogonal" perturbations need to be treated in a similar fashion. Once the latter work is completed, actual disk models will be tested for stability. Finally, we would like to again remark that while caution must be exercised when interpreting the results of a local analysis, the identification of many of the modes obtained with familiar results is very encouraging, and we feel confident that much valuable information can be gained by continuing this approach.

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APPENDIX A

THICK ACCRETION DISK MODELS

The standard models of accretion disks are geometrically thin and assume that the material in the disk follows Keplerian orbits (e.g., Shakura and Sunyaev 1973). For central black holes, such disks must terminate with their inner edge at the last stable circular orbit, $r_{\rm ms}$ ($= 6GM/c^2 = 3r_{\rm S}$ for a Schwarzschild black hole). But analyses of the equilibrium distributions of perfect fluids around black holes (e.g., Fishbone and Moncrief 1976; Abramowicz, Jaroszyński, and Sikora 1978) showed that a non-Keplerian angular momentum distribution could exist if there were sufficient pressure in the fluid, and that material could approach the marginally bound orbit, $r_{\rm ms}$ ($= 2r_{\rm S}$ in the Schwarzschild case). This material would then flow in through a cusp, analogous to flow through the inner Lagrange point in close binary systems (Kozlowski, Jaroszyński, and Abramowicz 1978). A detailed analysis of this flow (Abramowicz and Zurek 1981) has shown that the infall is subsonic at the cusp, thus justifying the common assumption that the radial motion is slower than azimuthal motion in thick disks. In discussing quasars, Lynden-Bell (1978) showed that the accreting material could form a pair of very steep vortices or whirlpools along the accretion axes and that any relativistic plasma generated by the central object could be shot out as a pair of jets.

The first attempt at the full-fledged thick disk models was that of Paczyński and Wiita (1980). They employed several simplifying assumptions including the following: a global energy balance was assumed, i.e., the total amount of energy generated by viscosity within the disk was radiated from its bloated surfaces; because of radiation pressure domination each element of the surface was taken to radiate critically, with the flux $= cg_{\rm eff}/\kappa$; a pseudo-Newtonian potential $[\Phi = GM/(R - r_{\rm S})]$ was used in lieu of full general relativity; a barytropic equation of state was used; and the self-gravity of the disk material was ignored. Now, it turns out that, if one specifies the distribution of angular momentum per unit mass, $l(r)$, on the surface of the disk and picks an inner edge for the disk, $(r_{\rm mb} < r_i \leq \tau_{\rm ms})$, then the balance of gravitational, rotational, and pressure forces immediately determine the shape of the disk. The most basic of stability requirements, that $dl/dr \geq 0$ (see, e.g., eq. [4.16]), and the demand that the flow be inward $(d\Omega/dr \leq 0)$ significantly restrict the allowed forms of $l(r)$. When the flux is integrated over the surface area, the total luminosity is found, as long as the outer boundary, r_0, of the thick portion of the disk is known. Once it is matched to a standard thin solution at large enough radius, then the accretion rate can be determined. Surprisingly, all of these results are independent of any knowledge of the disk's interior. These rotationally supported disks can remain in mechanical equilibrium even for significantly supercritical accretion rates, when $\dot{M} > \dot{M}_{\rm cr} = 8\pi c r_i/\kappa$, and can radiate $\approx 10L_{\rm Edd}$, where $L_{\rm Edd} = 4\pi cGM/\kappa$. The bulk of the energy is emitted deep within the funnels so that these thick disks resemble stars that have had a pair of holes bored into them, exposing their hot cores to the rest of the universe.

Shortly thereafter, Jaroszyński, Abramowicz, and Paczyński (1980) extended these models to a fully general relativistic framework in both the Schwarzschild and Kerr metrics. They used an angular momentum distribution, which, while only marginally stable, was later shown (Abramowicz, Calvani, and Nobili 1980) to maximize the emitted luminosity for given values of r_i and r_0; (they were also able to drop the barytropic assumption). Luminosities from disks around rotating holes can exceed those from disks around nonrotating ones by $\sim 30\%$. The validity of the pseudo-Newtonian potential was basically confirmed. They also showed that the equivalent of the thin-disk α viscosity parameter had to be much less than unity in thick disks. Even though these supercritical disks do depend on general relativistic effects $(r_i < r_{\rm ms})$ Abramowicz, Calvani, and Nobili (1980) showed that good approximation can be obtained using a purely Newtonian potential. They found that $(L/L_{\rm Edd})_{\rm max} \approx -2 \ln (r_i/r_0) - 2.44$ as long as $r_i/r_0 \lesssim 10^{-2}$, and they also realized that an important consistency requirement was that the mass of the disk must be less than that of the central object. This implies that the more massive the black hole, the lower the ratio of $L/L_{\rm Edd}$; however, a $10^8\ M_\odot$ black hole could conceivably entertain a disk giving off $100L_{\rm Edd}$ or 10^{48} ergs s^{-1}, adequate for any active galactic nucleus.

Quite a wide variety of approaches have since been followed in trying to discuss the interior properties and stability of thick disks. The first detailed specific model was proposed by Paczyński (1980), who analyzed the case in which the accretion flow is confined to a thin surface layer; then the assumptions of hydrostatic equilibrium and local heat balance suffice to describe the shape, luminosity, and angular momentum distribution once the accretion rate is specified. A complementary model, where the accretion is confined to a thin region along the equatorial plane, has been considered by Paczyński and Abramowicz (1982). This is a more likely approximation in that the viscous processes of heat generation and accretion flow should be more efficient in the central plane and convection can develop; they argue that this leads to the specific entropy being a function of the specific angular momentum alone. A few midly supercritical disks around 10 $M_\odot$ black holes have been computed and are stable to the Hoiland criterion (4.13), while Goldreich-Schubert-Fricke instabilities only operate over long thermal time scales. A set of assumed self-similar models has been analyzed by Begelman and Meier (1982), who found that the interior structure is sensitive to the viscosity law and the degree of equatorial pressure support. Their models appear to be unstable to local axisymmetric perturbations at high latitudes and to convection in most regions. However, these disks differ significantly from the others discussed here in that a very large fraction of the radiation generated is assumed to be trapped and swallowed by the hole, and they require the unphysical assumption that $\dot{M} > 10^3 \dot{M}_{\rm cr}$. Two different approaches have been used by Wiita (1982*a*) to estimate the central temperatures and densities of standard thick disks around black holes ranging from 1 to 10^8 $M_\odot$, for accretion rates from 1.3 to 89 $\dot{M}_{\rm cr}$, and for polytropic indices of 0, 3/2, and 3. Important limits are found for fully self-consistent models: for low-mass black holes, nuclear fusion can dominate over viscosity for energy generation; if the radiation pressure is too high, mechanical equilibrium can be violated; and, as also noted by Abramowicz, Calvani, and Nobili (1980), if the mass of the hole is too high, then the disk's mass must exceed that of the hole (but Wiita argues that the upper limit set by this constraint is $\sim 10^7$ $M_\odot$). Of course, rotation and higher accretion rates could loosen this limit, but self-gravitating disks may have to be considered.

Although the super-Eddington luminosities exhibited by these disks are very interesting, probably the main reason for studying them involves the possibilities for the production of collimated relativistic jets. All of the work completed to date assumes that a relatively small amount of material is lost from the surface of the funnels and is available for acceleration as an optically thin plasma. Although Jaroszyński, Abramowicz, and Paczyński (1980) had derived an approximate equation for the acceleration of particles in funnels, Abramowicz and Piran (1980) made the first estimates of the terminal velocity of ejected material, claiming $\sim 0.75c$ to $\sim 0.91c$ for ordinary plasma. Models of the reprocessed radiative field involving most general relativistic effects were developed by Sikora (1981); deep within the funnel the field is nearly isotropic and exerts a drag, while at larger heights the anisotropic flux begins the acceleration. Detailed calculations by Sikora and Wilson (1981) and general arguments by Piran (1982) yield lower terminal velocities, although fairly narrow ($\lesssim 10°$ opening angle) jets are produced by all these investigators. The unbalanced tangential force exerted by the radiation on the material in the funnel walls was considered by the earlier authors to drive unimportant mixing motions, but Nityananda and Narayan (1982) argued that it would inevitably lead to the ejection of large chunks of the walls and implied a gross inconsistency. A more careful analysis of this issue has shown that this usually is not the case, and that the ejected particle luminosity depends on both the sound speed in the disk material and the exact strength of the shear induced turbulence in the outer layers (Narayan, Nityanada, and Wiita 1983). A discussion of these jet phenomena in the context of observations of active galactic nuclei has been given by Wiita, Kapahi, and Saikia (1982), while applications to SS 433 have been made by Calvani and Nobili (1981) and Calvani, Sharp, and Turolla (1982).

The only semiclassical stability analysis of thick disks other than the present paper is due to Hacyan (1982), who employs a global stability technique (see Appendix B) to conclude that the restricted set of $n = 3$ models should be considered unstable. While this is an interesting approach, the choice of basis eigenvectors he uses to analyze the stability is not very suitable and the conclusions are very uncertain. More detailed reviews of thick accretion disk models have recently been published by Paczyński (1982) and Wiita (1982*b*).

APPENDIX B

STABILITY OF ROTATING STARS AND THIN DISKS

The analysis of the stability of rotating stars has a long history, much of it summarized in the books by Chandrasekhar (1969) and Tassoul (1978). In this appendix we shall merely mention some of the important approaches that have been used to investigate various aspects of the stability of both rotating stars and thin disks. In the case of stars we will touch on both local and global approaches to dynamical stability. The literature on disks is less extensive and is more concentrated on thermal instabilities, and this discussion will reflect that fact.

Because the von Zeipel (1924) paradox shows that pseudo-barytropic models in a state of permanent rotation cannot correctly describe rotating stars in strict radiative equilibrium, any plausible model of a star must either have $\Omega = \Omega(r, z)$ or a permanent meridional circulation. Assuming the former, the key question which local stability analyses try to answer is: With a prescribed $\Omega(r, z)$, what are the conditions for the star to be stable with respect to small isentropic perturbations? A general answer for axisymmetric perturbations was provided by Hoiland (1939, 1941) and can be phrased in the following way (Tassoul 1978): A baroclinic star [one with $p = p(\rho, T, \lambda_1, \lambda_2, \ldots)$, with the λ's variables depending on r and z] in permanent rotation is dynamically stable with respect to axisymmetric motions if and only if the two following conditions are satisfied: (i) the entropy per unit mass S never decreases outward, and (ii) on each surface $S = \text{const.}$, the angular momentum per unit mass, Ωr^2, increases as we move from the poles to the equator. A simpler version for isentropic stars had been found by Solberg (1936), and more detailed discussions were given by Fjortoft (1946) and Holmboe (1948). In the present paper, studying thick disks, we have specialized to local perturbations, so that $\lambda \ll H$, and this type of result then becomes a necessary, but not a sufficient, criterion for stability. The question of nonaxisymmetric motions is much more complicated, but several authors have produced partial generalizations (Spiegel and Zahn 1970; Zahn 1974; Sung 1974*a*, 1975). Even if a star satisfies the Hoiland criterion and is thus dynamically stable, it was shown by Goldreich and Schubert (1967) and by Fricke (1968) that such a chemically homogeneous, inviscid star still exhibits slower thermal instabilities with respect to axisymmetric disturbances if Ω depends on the coordinate z. More refined analyses were later produced by Sung (1974*b*) and Smith and Fricke (1975).

Global methods of stability analysis basically depend on considerations of integrated energy functionals. Here one computes the variation of the total energy of the system (W) and a moment of inertia (I) including perturbations; appropriate values of the normal frequencies come from $\sigma^2 = -W/I$, where trial displacements are used. This variational technique was applied to uniformly rotating stars by Clement (1964) and to general differentially rotating systems by Lynden-Bell and Ostriker (1967). Subsequent workers have expanded on this technique and applied it to specific problems (Unno 1968; Lebovitz 1970; Schutz 1972). A similar method involves using the second order virial theorem to obtain more easily understandable representations of small departures from permanent rotation. Pioneered by Ledoux (1945), it was very successfully applied to a variety of problems by Chandrasekhar and Lebovitz (Lebovitz 1961; Chandrasekhar and Lebovitz 1962*a*, *b*, 1973; Chandrasekhar 1969). The recent attempt by Hacyan (1982) to treat the stability of thick disks follows this approach. A very general way of analying the vibrational stability of differentially rotating bodies is due to Aizenman and Cox (1975). They specialize to linear, non-adiabatic arbitrary perturbations on steady, inviscid flows in the unperturbed state and compute formal solutions for the perturbations; however, these results depend on unknown nonadiabatic eigenfunctions and are very difficult to apply to actual models.

Because of their basically Keplerian structure, little concern has been expressed over the dynamical stability of thin accretion disks. However, Lightman and Eardley (1974; Lightman 1974) showed that the inner regions of standard thin disks around black holes were subject to secular clumping instabilities as long as the usual assumptions about viscosity were made. Shakura and Sunyaev (1976) produced a more general analysis; in addition to the relatively slow, long-wavelength, Lightman-Eardley instabilities, they showed that a thermal instability, whose growth rate was nearly independent of wavelength, should exist. This problem was further generalized by Pringle (1976), who basically agreed with the previous authors but also showed that radiation pressure–dominated regimes could be thermally stable if electron scattering did not dominate the opacity. An extension to non-standard thin disk models was made by Piran (1978), who provided a useful test for stability in terms of parameterized heating (viscosity) and cooling rates. He also showed how mass loss via a wind from the disk could aid in stabilizing it. Abramowicz (1981), expanding upon this point, noted that the relativistic potential near a black hole forces an overflow at the cusp at the inner edge of the disk which is probably adequate to stabilize the innermost region of disks around black holes.

Several other approaches to the study of the stability of thin disks have been investigated. Shakura, Sunyaev, and Zilitinkevich (1978) claimed that the thermal instabilities remained even if convective energy transport was included. Tayler (1980) argued that convection would always occur in standard disks if $P_{\text{gas}}/P_{\text{rad}} \lesssim 1$, and that convective energy transport would be important in any thermally unstable disks. This analysis has been extended (Robertson and Tayler 1981) to show that convection does set in before thermal instabilities, that it can slow, but not eliminate, the growth of such instabilities, and that it could never carry the bulk of the energy. The case of pulsational perturbations which are of long wavelength with respect to the disk thickness, but are otherwise local ($H_z \ll \lambda \ll H_r$) for both optically thin and optically thick disks, has been analyzed by Kato (1978, 1979). He showed that if the coefficient of viscosity increases by more than a critical amount during the compressional phase of the oscillation, then such quasi-radial pulsations can grow by extracting thermal and dynamical energy from the shear. Perhaps the approach closest to the one that we have used in this paper is due to Livio and Shaviv (1977, 1981). They analyzed linearized perturbations where the vertical dimension is not integrated out, in distinction to almost all of the other work. Gas pressure dominated regions were first considered, and the stability conditions included the Schwarzschild criterion in the vertical direction, the Hoiland criterion in the radial direction and an upper limit on the value of α (Livio and Shaviv 1977). Later, short-wavelength perturbations in the z direction were treated, and they found that such modes would always grow for high enough viscosity; in radiation pressure–dominated regions a sonic group velocity is achieved and the instability grows rapidly (Livio and Shaviv 1981).

APPENDIX C

A CATALOG OF TERMS IN THE DISPERSION RELATION

A total of 96 expressions combine to form the coefficients a_0–a_5, but most of these produce several different terms when considered in light of the magnitudes as defined by (4.1). In Table 5 all 380 such possible contributions are given, grouped first by N_4, the power of (ω/Ω) that multiplies that coefficient. Two basic groupings are determined by whether the perturbation is "orthogonal" to the nonazimuthal flow ($\mu = 0$, or $j = 0$) or whether it is "parallel" to the velocity ($\mu = 1$, or $j = +\infty$); the latter case is analyzed in this paper, and the former is even more complicated. The columns headed i indicate whether the term is real ($i = 0$) or imaginary ($i = 1$). The columns headed N_6 give the power to which $\epsilon = (\lambda/H)$ contributes to the terms, while N_5 indicates the power to which (v/v_φ) enters. This catalog is restricted to the thick disk case in that $k = 0$, while $l = +\infty$.

TABLE 5

ORDER-OF-MAGNITUDE CATALOG

		$\mu \equiv 1$				$\mu \equiv 0$			
No.	Term	i	N_4	N_6	N_5	i	N_4	N_6	N_5
1	$\varphi_2 r_5 p_3 z_4 \rho_1$	0	0	−2	1				
2		1	0	−1	1	1	0	−1	1
3	$\varphi_2 r_5 p_4 \rho_3 z_1$	0	0	−2	1				
4		1	0	−1	1	1	0	−1	1
5		0	0	0	1	0	0	0	1
6		0	0	−2	3				
7		1	0	−1	3	1	0	−1	3
8		0	0	0	3	0	0	0	3
9	$\varphi_2 r_5 \Gamma_1(p/\rho) z_1 p_3 \rho_4$	1	0	−3	1	0	0	−2	1
10		0	0	−2	1	1	0	−1	1
11		1	0	−1	1				
12	$-\varphi_2 r_5 \rho \rho_1 p_4 z_3$	1	0	−3	3				
13		0	0	−2	3				
14		1	0	−1	3				
15		0	0	0	3	0	0	0	3
16	$-\varphi_2 z_5 \rho_1 p_3 r_4$	0	0	−2	1				
17		1	0	−1	1	1	0	−1	1
18	$-\varphi_2 z_5 \rho_3 r_1 p_4$	0	0	−2	1	1	0	−1	1
19		1	0	−1	1	0	0	0	1
20		0	0	0	1	1	0	−1	3
21		0	0	−2	3	0	0	0	3
22		1	0	−1	3				
23		0	0	0	3				
24	$-\varphi_2 z_5 \Gamma_1(p/\rho) \rho_4 p_3 r_1$	1	0	−3	1	0	0	−2	1
25		0	0	−2	1	1	0	−1	1
26		1	0	−1	1				
27	$\varphi_2 z_5 r_3 p_4 \rho_1$	0	0	−2	3	0	0	0	3
28		1	0	−1	3				
29		0	0	0	3				
30	$\varphi_5 \rho_1 p_2 r_3 z_4$	1	0	−3	3				
31		0	0	−2	3				
32		1	0	−1	3	1	0	−1	3
33	$\varphi_5 \rho_1 \rho^2 p_4 r_2 z_3$	1	0	−5	5				
34		0	0	−4	5				
35		1	0	−3	5				
36		0	0	−2	5				
37		1	0	−1	5				
38		0	0	0	5	0	0	0	5
39	$\varphi_5 \rho_1 p_3 r_4 z_2$	1	0	−3	3				
40		0	0	−2	3				
41		1	0	−1	3	1	0	−1	3
42	$-\varphi_5 \rho_1 z_2 r_3 p_4$	1	0	−3	5				
43		0	0	−2	5				
44		1	0	−1	5				
45		0	0	0	5	0	0	0	5
46	$-\varphi_5 \rho_1 z_3 r_4 p_2 \rho$	0	0	−4	3				
47		1	0	−3	3				
48		0	0	−2	3				
49		1	0	−1	3	1	0	−1	3
50	$-\varphi_5 \rho_1 z_4 p_3 r_2 \rho$	0	0	−4	3				
51		1	0	−3	3				
52		0	0	−2	3				
53		1	0	−1	3	1	0	−1	3
54	$r_1 \varphi_5 \rho_2 p_3 z_4$	1	0	−3	1				
55		0	0	−2	1	0	0	−2	1
56		1	0	−1	1	1	0	−1	1
57		1	0	−3	3				
58		0	0	−2	3	0	0	−2	3
59		1	0	−1	3	1	0	−1	3
60	$r_1 \varphi_5 \rho_3 z_2 p_4$	1	0	−3	3	1	0	−1	3
61		0	0	−2	3	1	0	−1	5
62		1	0	−1	3				
63		1	0	−3	5				
64		0	0	−2	5				
65		1	0	−1	5				
66		0	0	0	3	0	0	0	3
67		0	0	0	5	0	0	0	5
68	$-r_1 \varphi_5 \Gamma_1 p \rho_4 z_3 p_2$	1	0	−5	3	0	0	−2	3
69		0	0	−4	3	1	0	−1	3
70		1	0	−3	3				
71		0	0	−2	3				
72		1	0	−1	3				
73	$r_1 \varphi_5 \Gamma_1(p/\rho) \rho_4 p_3 z_2$	0	0	−4	3				
74		1	0	−3	3				
75		0	0	−2	3	0	0	−2	3
76		1	0	−1	3	1	0	−1	3
77	$-r_1 \varphi_5 z_3 p_4 \rho_2 \rho$	0	0	−4	3				
78		1	0	−3	3				
79		0	0	−2	3				
80		1	0	−1	3	1	0	−1	3
81		0	0	0	3	0	0	0	3
82		0	0	−4	5				
83		1	0	−3	5				
84		0	0	−2	5				
85		1	0	−1	5	1	0	−1	5
86		0	0	0	5	0	0	0	5
87	$-r_1 \varphi_5 \rho_5 p_2 z_4$	1	0	−3	1				
88		0	0	−2	1	0	0	−2	1
89		1	0	−1	1	1	0	−1	1
90		1	0	−3	3				
91		0	0	−2	3	0	0	−2	3
92		1	0	−1	3	1	0	−1	3
93	$-z_1 \varphi_5 \rho_2 p_3 r_4$	1	0	−3	1				
94		0	0	−2	1	0	0	−2	1
95		1	0	−1	1	1	0	−1	1
96		1	0	−3	3				
97		0	0	−2	3	0	0	−2	3
98		1	0	−1	3	1	0	−1	3
99	$-z_1 \varphi_5 \rho_3 \rho p_4 r_2$	0	0	−4	3				
100		1	0	−3	3				
101		0	0	−2	3				
102		1	0	−1	3	1	0	−1	3
103		0	0	0	3	0	0	0	3
104		0	0	−4	5				
105		1	0	−3	5				
106		0	0	−2	5				
107		1	0	−1	5	1	0	−1	5
108		0	0	0	5	0	0	0	5
109	$z_1 \varphi_5 \Gamma_1(p/\rho) p_2 r_3 \rho_4$	0	0	−4	3				
110		1	0	−3	3				
111		0	0	−2	3	0	0	−2	3
112		1	0	−1	3	1	0	−1	3
113	$-z_1 \varphi_5 \Gamma_1 p \rho_4 r_2 p_3$	1	0	−5	3				
114		0	0	−4	3				
115		1	0	−3	3				
116		0	0	−2	3	0	0	−2	3
117		1	0	−1	3	1	0	−1	3
118	$z_1 \varphi_5 r_3 \rho_2 p_4$	1	0	−3	3				
119		0	0	−2	3				
120		1	0	−1	3	1	0	−1	3
121		0	0	0	3	0	0	0	3
122		1	0	−3	5				
123		0	0	−2	5				
124		1	0	−1	5	1	0	−1	5
125		0	0	0	5	0	0	0	5

TABLE 5—*Continued*

No.	Term	$\mu \equiv 1$				$\mu \equiv 0$			
		i	N_4	N_6	N_5	i	N_4	N_6	N_5
126	$z_1\varphi_5 r_4 \rho_3 p_2$	1	0	−3	1				
127		0	0	−2	1	0	0	−2	1
128		1	0	−1	1	1	0	−1	1
129		1	0	−3	3				
130		0	0	−2	3	0	0	−2	3
131		1	0	−1	3	1	0	−1	3
132	$\varphi_2 r_5 p_3 z_4$	1	1	−1	0	1	1	−1	0
133	$\varphi_2 r_5 \rho_3 z_1$	1	1	−1	0	1	1	−1	0
134		1	1	−1	2	1	1	−1	2
135		0	1	0	0	0	1	0	0
136		0	1	0	2	0	1	0	2
137	$\varphi_2 r_5 \Gamma_1(p/\rho) z_1 p_3$	0	1	−2	0	0	1	−2	0
138		1	1	−1	0	1	1	−1	0
139	$-\varphi_2 r_5 \rho\rho_1 p_4$	0	1	−2	2				
140		1	1	−1	2				
141		0	1	0	2	0	1	0	2
142	$-\varphi_2 r_5 \rho p_4 z_3$	0	1	−2	2				
143		1	1	−1	2				
144		0	1	0	2	0	1	0	2
145	$-\varphi_2 r_5 \rho\rho_1 z_3$	0	1	−2	2				
146		1	1	−1	2				
147		0	1	0	2	0	1	0	2
148	$-\varphi_2 z_5 p_3 r_4$	1	1	−1	0	1	1	−1	0
149	$-\varphi_2 z_5 \rho_3 r_1$	1	1	−1	0	1	1	−1	0
150		1	1	−1	2	1	1	−1	2
151		0	1	0	0	0	1	0	0
152		0	1	0	2	0	1	0	2
153	$-\varphi_2 z_5 \Gamma_1(p/\rho) p_3 r_1$	0	1	−2	0	0	1	−2	0
154		1	1	−1	0	1	1	−1	0
155	$\varphi_2 z_5 r_3 p_4$	1	1	−1	2				
156		0	1	0	2	0	1	0	2
157	$\varphi_2 z_5 r_3 \rho_1$	1	1	−1	2				
158		0	1	0	2	0	1	0	2
159	$(\rho_1 + \varphi_5) p_2 r_3 z_4$	0	1	−2	2				
160		1	1	−1	2	1	1	−1	2
161	$(\rho_1 + \varphi_5) p_3 r_4 z_2$	0	1	−2	2				
162		1	1	−1	2	1	1	−1	2
163	$(\rho_1 + \varphi_5)\rho^2 p_4 r_2 z_3$	0	1	−4	4				
164		1	1	−3	4				
165		0	1	−2	4				
166		1	1	−1	4				
167		0	1	0	4	0	1	0	4
168	$-(\rho_1 + \varphi_5) z_2 r_3 p_4$	0	1	−2	4				
169		1	1	−1	4				
170		0	1	0	4	0	1	0	4
171	$-(\varphi_5 + \rho_1) z_3 r_4 p_2 \rho$	1	1	−3	2				
172		0	1	−2	2				
173		1	1	−1	2	1	1	−1	2
174	$-(\varphi_5 + \rho_1) z_4 p_3 r_2 \rho$	1	1	−3	2				
175		0	1	−2	2				
176		1	1	−1	2	1	1	−1	2
177	$\varphi_5 \rho_1 \rho^2 (p_4 r_2 + p_4 z_3 + z_3 r_2)$	0	1	−4	4				
178		1	1	−3	4				
179		0	1	−2	4				
180		1	1	−1	4				
181		0	1	0	4	0	1	0	4
182	$-\varphi_5 \rho_1 z_2 r_3$	0	1	−2	4				
183		1	1	−1	4				
184		0	1	0	4	0	1	0	4
185	$\varphi_5 \rho_1(-\rho r_4 p_2 + z_4 p_3 \rho)$	1	1	−3	2				
186		0	1	−2	2				
187		1	1	−1	2	1	1	−1	2
188	$r_1 \rho_2 p_3 z_4$	0	1	−2	0	0	1	−2	0
189		0	1	−2	2	0	1	−2	2
190		1	1	−1	0	1	1	−1	0
191		1	1	−1	2	1	1	−1	2
192	$r_1 \rho_3 z_2 p_4$	0	1	−2	2	0	1	0	2
193		1	1	−1	2	1	1	−1	2
194		0	1	0	2	1	1	−1	4
195		0	1	−2	4	0	1	0	4
196		1	1	−1	4				
197		0	1	0	4				
198	$-r_1 \Gamma_1 p \rho_4 p_2 z_3$	0	1	−4	2	0	1	−2	2
199		1	1	−3	2	1	1	−1	2
200		0	1	−2	2				
201		1	1	−1	2				
202	$r_1 \Gamma_1(p/\rho) \rho_4 p_3 z_2$	0	1	−2	2	0	1	−2	2
203		1	1	−1	2	1	1	−1	2
204		1	1	−3	2				
205	$-r_1 z_3 p_4 \rho_2 \rho$	1	1	−3	2				
206		0	1	−2	2				
207		1	1	−1	2	1	1	−1	2
208		0	1	0	2	0	1	0	2
209		1	1	−3	4				
210		0	1	−2	4				
211		1	1	−1	4	1	1	−1	4
212		0	1	0	4	0	1	0	4
213	$-r_1 \rho_3 p_2 z_4$	0	1	−2	0	0	1	−2	0
214		0	1	−2	2	0	1	−2	2
215		1	1	−1	0	1	1	−1	0
216		1	1	−1	2	1	1	−1	2
217	$r_1 \varphi_2 \rho_3 z_2$	0	1	−2	2				
218		1	1	−1	2	1	1	−1	2
219		0	1	0	2	0	1	0	2
220		0	1	−2	4				
221		1	1	−1	4	1	1	−1	4
222		0	1	0	4	0	1	0	4
223	$-r_1 \varphi_5 \Gamma_1 p p_2 (\rho_4 + z_3)$	0	1	−4	2				
224		1	1	−3	2				
225		0	1	−2	2	0	1	−2	2
226		1	1	−1	2	1	1	−1	2
227	$r_1 \varphi_5 \Gamma_1(p/\rho) p_3 z_2$	0	1	−2	2	0	1	−2	2
228		1	1	−1	2	1	1	−1	2
229		1	1	−3	2				
230	$-r_1 \varphi_5 \rho_2 (z_3 + p_4)\rho$	1	1	−3	2				
231		0	1	−2	2				
232		1	1	−1	2	1	1	−1	2
233		0	1	0	2	0	1	0	2
234		1	1	−3	4				
235		0	1	−2	4				
236		1	1	−1	4	1	1	−1	4
237		0	1	0	4	0	1	0	4
238	$-z_1 \rho_2 p_3 r_4$	0	1	−2	0	0	1	−2	0
239		0	1	−2	2	0	1	−2	2
240		1	1	−1	0	1	1	−1	0
241		1	1	−1	2	1	1	−1	2
242	$-z_1 \rho_3 \rho p_4 r_2$	1	1	−3	2				
243		0	1	−2	2				
244		1	1	−1	2	1	1	−1	2
245		0	1	0	2	0	1	0	2
246		1	1	−3	4				
247		0	1	−2	4				
248		1	1	−1	4	1	1	−1	4
249		0	1	0	4	0	1	0	4

TABLE 5—*Continued*

No.	Term	$\mu \equiv 1$: i	N_4	N_6	N_5	$\mu \equiv 0$: i	N_4	N_6	N_5
250	$z_1\Gamma_1 p\rho_4 p_3 r_2$	0	1	−2	2	0	1	−2	2
251		1	1	−1	2	1	1	−1	2
252		1	1	−3	2				
253	$-z_1\Gamma_1 p\rho_4 p_3 r_2$	0	1	−4	2				
254		1	1	−3	2				
255		0	1	−2	2	0	1	−2	2
256		1	1	−1	2	1	1	−1	2
257	$z_1 r_3 \rho_2 p_4$	0	1	−2	2				
258		1	1	−1	2	1	1	−1	2
259		0	1	0	2	0	1	0	2
260		0	1	−2	4				
261		1	1	−1	4	1	1	−1	4
262		0	1	0	4	0	1	0	4
263	$z_1 r_4 p_2 \rho_3$	0	1	−2	0	0	1	−2	0
264		1	1	−1	0	1	1	−1	0
265		0	1	−2	2	0	1	−2	2
266		1	1	−1	2	1	1	−1	2
267	$-z_1\varphi_5 \rho_3 \rho(p_4 + r_2)$	1	1	−3	2				
268		0	1	−2	2				
269		1	1	−1	2	1	1	−1	2
270		0	1	0	2	0	1	0	2
271		1	1	−3	4				
272		0	1	−2	4				
273		1	1	−1	4	1	1	−1	4
274		0	1	0	4	0	1	0	4
275	$z_1\varphi_5\Gamma_1(p/\rho)p_2 r_3$	1	1	−3	2				
276		0	1	−2	2				
277		1	1	−1	2	1	1	−1	2
278		0	1	0	2	0	1	0	2
279	$-z_1\varphi_5\Gamma_1 p p_3(\rho_4 + r_2)$	0	1	−4	2				
280		1	1	−3	2				
281		0	1	−2	2	0	1	−2	2
282		1	1	−1	2	1	1	−1	2
283	$z_1\varphi_5 r_3 \rho_2$	0	1	−2	2				
284		1	1	−1	2	1	1	−1	2
285		0	1	0	2	0	1	0	2
286		0	1	−2	4				
287		1	1	−1	4	1	1	−1	4
288		0	1	0	4	0	1	0	4
289	$-\varphi_2 r_5 \rho(\rho_1 + p_4 + z_3)$	1	2	−1	1				
290		0	2	0	1	0	2	0	1
291	$\varphi_2 z_5 r_3$	0	2	0	1	0	2	0	1
292	$p_2 r_3 z_4$	1	2	−1	1	1	2	−1	1
293	$p_3 r_4 z_2$	1	2	−1	1	1	2	−1	1
294	$\rho^2 p_4 r_2 z_3$	1	2	−3	3				
295		0	2	−2	3				
296		1	2	−1	3				
297		0	2	0	3	0	2	0	3
298	$-z_2 r_3 p_4$	1	2	−1	3				
299		0	2	0	3	0	2	0	3
300	$-z_3 r_4 p_2 \rho$	0	2	−2	1				
301		1	2	−1	1	1	2	−1	1
302	$-z_4 p_3 r_2 \rho$	0	2	−2	1				
303		1	2	−1	1	1	2	−1	1
304	$(\varphi_5 + \rho_1)\rho^2(p_4 r_2 + p_4 z_3 + r_2 z_3)$	1	2	−3	3				
305		0	2	−2	3				
306		1	2	−1	3				
307		0	2	0	3	0	2	0	3
308	$-(\varphi_5 + \rho_1)z_2 r_3$	0	2	0	3	0	2	0	3
309		1	2	−1	3				
310	$-(\varphi_5 + \rho_1)\rho(r_4 p_2 + z_4 p_3)$	0	2	−2	1				
311		1	2	−1	1	1	2	−1	1
312	$\varphi_5 \rho_1 \rho^2(p_4 + r_2 + z_3)$	1	2	−3	3				
313		0	2	−2	3				
314		1	2	−1	3				
315		0	2	0	3	0	2	0	3
316	$-r_1\varphi_5 \rho_2 \rho$	0	2	−2	1				
317		1	2	−1	1	1	2	−1	1
318		0	2	0	1	0	2	0	1
319		0	2	−2	3				
320		1	2	−1	3	1	2	−1	3
321		0	2	0	3	0	2	0	3
322	$-r_1\varphi_5\Gamma_1 p p_2$	1	2	−3	1				
323		0	2	−2	1	0	2	−2	1
324		1	2	−1	1	1	2	−1	1
325	$-r_1 \rho_3 z_2$	1	2	−1	1	1	2	−1	1
326		0	2	0	1	0	2	0	1
327		1	2	−1	3	1	2	−1	3
328		0	2	0	3	0	2	0	3
329	$-r_1\Gamma_1(p/\rho)p_2(\rho_4 + z_3)$	1	2	−3	1				
330		0	2	−2	1	0	2	−2	1
331		1	2	−1	1	1	2	−1	1
332	$r_1\Gamma_1(p/\rho)p_3 z_2$	0	2	−2	1	0	2	−2	1
333		1	2	−1	1	1	2	−1	1
334	$-r_1 \rho_2 \rho(z_3 + p_4)$	0	2	−2	1				
335		1	2	−1	1	1	2	−1	1
336		0	2	0	1	0	2	0	1
337		0	2	−2	3				
338		1	2	−1	3	1	2	−1	3
339		0	2	0	3	0	2	0	3
340	$-z_1\varphi_5\Gamma_1 p p_3$	1	2	−3	1				
341		0	2	−2	1	0	2	−2	1
342		1	2	−1	1	1	2	−1	1
343	$-z_1\varphi_5 \rho_3 \rho$	0	2	−2	1				
344		1	2	−1	1	1	2	−1	1
345		0	2	0	1	0	2	0	1
346		0	2	−2	3				
347		1	2	−1	3	1	2	−1	3
348		0	2	0	3	0	2	0	3
349	$-z_1(p_4 + r_2)\rho_3 \rho$	0	2	−2	1				
350		1	2	−1	1	1	2	−1	1
351		0	2	0	1	0	2	0	1
352		0	2	−2	3				
353		1	2	−1	3	1	2	−1	3
354		0	2	0	3	0	2	0	3
355	$z_1\Gamma_1(p/\rho)p_2 r_3$	0	2	−2	1	0	2	−2	1
356		1	2	−1	1	1	2	−1	1
357	$-z_1\Gamma_1 p p_3(\rho_4 + r_2)$	1	2	−3	1	1	2	−3	1
358		1	2	−1	1	1	2	−1	1
359		0	2	−2	1	0	2	−2	1
360	$z_1 r_3 \rho_2$	1	2	−1	1	1	2	−1	1
361		0	2	0	1	0	2	0	1
362		1	2	−1	3	1	2	−1	3
363		0	2	0	3	0	2	0	3
364	$-\varphi_2 r_5$	0	3	0	0	0	3	0	0
365	$\varphi_5 \rho_1 \rho^2$	0	3	−2	2				
366		1	3	−1	2				
367		0	3	0	2	0	3	0	2
368	$(\varphi_5 + \rho_1)(p_4 + r_2$	0	3	−2	2				
369	$+ z_3)\rho^2 + \rho^2(p_4 r_2$	1	3	−1	2				
370	$+ p_4 z_3 + r_2 z_3)$	0	3	0	2	0	3	0	2
371	$-\rho(r_4 p_2 + z_4 p_3)$	1	3	−1	0	1	3	−1	0
372	$-z_2 r_3$	0	3	0	2	0	3	0	2
373	$-[r_1(\rho_2 \rho + \Gamma_1 p p_2)$	0	3	−2	0	0	3	−2	0

TABLE 5—*Continued*

No.	Term	$\mu \equiv 1$ i	N_4	N_6	N_5	$\mu \equiv 0$ i	N_4	N_6	N_5
374	$+ z_1(\rho_3 \rho + \Gamma_1 p p_3)]$	1	3	−1	0	1	3	−1	0
375		1	3	−1	2	1	3	−1	2
376		0	3	0	0	0	3	0	0
377		0	3	0	2	0	3	0	2
378	$\rho^2(p_4 + r_2 + z_3$	1	4	−1	1	1	4	−1	1
379	$+ \varphi_5 + \rho_1)$	0	4	0	1	0	4	0	1
380	1	0	5	0	0	0	5	0	0

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