THE ASTROPHYSICAL JOURNAL, 253:634–654, 1982 February 15

THE PARKER INSTABILITY IN A SELF-GRAVITATING GAS LAYER

BRUCE G. ELMEGREEN
Department of Astronomy, Columbia University
Received 1980 December 11; accepted 1981 August 31

ABSTRACT

The dispersion relation for the Parker instability in a self-gravitating, exponential gas layer is derived and solved explicitly to give the growth time of a perturbation as a function of its dimensions and initial density. Our solution is a general result, expressed in dimensionless form, and valid for arbitrary gas density, magnetic field strength and cosmic ray pressure.

Self-gravity is important as an additional driving force for the Parker instability when a dimensionless density, defined here, becomes comparable to the ratios of the magnetic and cosmic ray pressures to the thermal gas pressure. For the observed scale height and sound speed in the interstellar medium, the dimensionless density equals 1.8 times the ambient density in cm^{-3}. Self-gravity is marginally important for instabilities in the ambient medium. Self-gravity becomes more important than magnetic fields or cosmic rays in regions of higher gas density. For example, in a spiral density wave shock, where the gas density may be 5 cm^{-3}, the initial growth time of the combined instability is only 12 million years. The instability is dominated by self-gravitational forces at these densities, so this growth time is relatively independent of magnetic field strength and cosmic ray pressure. Cloud formation by this Parker-Jeans instability can be so fast that star formation may occur in a moderately compressed interstellar medium within only 20 million years.

Subject headings: hydrodynamics — instabilities — interstellar: matter — stars: formation

I. INTRODUCTION

The interstellar medium is subject to the magnetic Rayleigh-Taylor instability and to the self-gravitational instability. The pure Rayleigh-Taylor instability operates with a characteristic wavelength of about 1 kpc and a growth time comparable to the Alfvén propagation time over one scale height (Parker 1966). Since the pressures from magnetic fields, cosmic rays and rms cloud motions are all comparable, the Parker instability operates with a time scale comparable to the free fall time perpendicular to the galactic plane. The pure self-gravitational instability in the ambient medium would operate with a similar length and time scale except that differential rotation can prevent it. According to Safronov (1960) and Toomre (1964), gravitational instability occurs only when $\pi G\sigma > u\omega_{ep}$ for gravitational constant G, gas column density σ, one-dimensional sound speed u, and epicyclic frequency ω_{ep}. The local interstellar medium barely falls short of this instability criterion, by a factor of 1.5 to 2, so a small decrease in u after a period of cooling (Goldreich and Lynden-Bell 1965), or a small increase in σ (Elmegreen 1979), as may occur in a spiral density wave (Fujimoto 1968; Roberts 1969), could possibly trigger the self-gravitational instability.

The stability of the interstellar medium should be analyzed with all possible forces included at the same time. Here we consider a partial solution to this problem by combining the self-gravitational instability with the Parker instability. The effects of rotation are not included yet. Rotational forces will be minor for the interesting cases of moderate-to-high densities, as in mildly compressed media. For example, rotation changes the pure self-gravitational instability by subtracting the square of the epicyclic frequency from the square of the growth rate derived without rotation (Safronov 1960). The inverse of the epicyclic frequency is about 30 million years locally. Thus rotation affects the pure Parker instability (Lerche 1967*a*; Shu 1974; Zweibel and Kulsrud 1975) and the pure self-gravitational instability, which have growth times comparable to the epicyclic time. Rotation should not affect the combined instability as much as it does each separate instability. The growth times for the combined instability (Table 1) are between 15 and 8 million years for midplane gas densities between 5 and 10 cm^{-3}. If rotation influences the combined instability in the same way as it does the pure self-gravitational instability, then the local rotational forces will decrease the growth times calculated here by only 12% to 3% for these two densities, respectively.

The goal of this work is to understand the formation of giant cloud complexes. These clouds have virial theorem or free fall line widths, so they are strongly self-gravitating at the present time. They cannot be in the *process* of forming by the pure Parker instability, because that instability is driven only by ambient pressure, which is much less than the internal pressure inside the clouds. The pure Parker instability produces density enhancements that are only a factor of 3 to 5 above the ambient density (Lerche 1967*a*; Mouschovias 1974). If giant molecular clouds are formed by large scale instabilities, then self-gravitational forces must be important at some stage. Our results suggest that self-gravitational forces could have been important from the beginning of the instability if the interstellar medium was slightly compressed when these clouds formed.

Section II describes our procedure for determining the growth rate of the combined instability. Section III derives the dispersion relation for arbitrary self-gravitational accelerations, and § IV determines the self-gravitational forces for our model. Then in §§ V and VI, we consider each separate instability for comparison and the final results for the combined instability. A summary of our results is in § VII.

II. MODEL: AN EXPONENTIAL, SELF-GRAVITATING GAS LAYER WITHOUT ROTATION

a) *Modeling the Forces in the Parker Instability*

The Parker instability may be analyzed most easily for a gas field layer that has a constant gravitational acceleration perpendicular to the layer, g_0. A more realistic layer has an acceleration that increases with height, similar to the tanh z/H dependence for an isothermal layer with scale height H. Parker (1966) gives a solution for the growth rate when $g \propto z$. Most previous studies use constant g because the layer is unstable whether or not g is constant, and the mathematics is much simpler when g is constant. We use a constant g for this analysis. An appropriate choice for g_0 is the observed value of the galactic acceleration at one scale height, which is 3.5×10^{-9} cm s^{-2} in the solar neighborhood (Oort 1965) for $H=160$ pc (Falgarone and Lequeux 1973).

The equilibrium layer with constant g has a density and pressure distribution that is exponential:

$$\frac{\rho}{\rho_0}=\frac{P}{P_0}=\frac{P_{\rm CR}}{P_{\rm CR0}}=\frac{B^2}{B_0^2}=e^{-z/H}, \qquad (1)$$

where the scale height is

$$H=\frac{P_0+B_0^2/8\pi+P_{\rm CR0}}{\rho_0 g_0}=\frac{u^2(1+\alpha+\beta)}{g_0}, \qquad (2)$$

and

$$P=\rho u^2, \qquad (3)$$

$$\alpha=B_0^2/(8\pi P_0), \qquad (4)$$

$$\beta=P_{\rm CR}/P_0; \qquad (5)$$

u is the one-dimensional rms velocity in the interstellar medium, which we take to be $u=7$ km s^{-1} (Falgarone and Lequeux 1973). The unperturbed magnetic field, B, is taken to lie in the plane of the layer and in the y direction.

The ratios of pressures, α and β, are assumed to be constant throughout the layer. The local cosmic ray pressure, $P_{\rm CR}$, is observed to be 4×10^{-13} dyn (Meyer 1969; Shu 1974), the local, average magnetic field strength, B, determined from Faraday rotation is 2.2 ± 0.4 microgauss (Manchester 1974), and ρ, the density, equals the mean atomic weight, 2.2×10^{-24} g, times the mean interstellar density of $n=0.1$ to 1 cm^{-3}. For $n=1$ cm^{-3} (Jenkins and Savage 1974; Bohlin, Savage, and Drake 1978), we obtain $\alpha\approx0.18$ and $\beta\approx0.37$. These values are too low to give the observed scale height if $g_0=3.5\times10^{-9}$ cm s^{-1}. Values of $\alpha\approx\beta\approx1$ or larger have been used by others to model the Parker instability, and these will be used in the examples here as well. The values of α and β are not precisely known from observations; $\alpha=1$ corresponds to $B=5.2$ microgauss if $n=1$ cm^{-3} and $u=7$ km s^{-1}. This field strength is significantly larger than that determined by Faraday rotation. Values of $n=0.18$ cm^{-3} would give $\alpha=1$ if $B=2.2$ microgauss, but this does not explain the ~160 pc scale height of the diffuse cloud population, which has a larger mean density (Baker and Burton 1975). We shall use both $(\alpha,\beta)=(0.2,0.2)$ and $(\alpha,\beta)=(1.0,1.0)$ to bracket the numerical results within a likely range. Values of (α,β) much larger than these may occur in compressed regions (see Elmegreen 1982).

The equilibrium state is taken to be isothermal, because the large scale heating and cooling rates will be nearly uniform in an unperturbed galactic gas layer. The perturbations, however, are assumed to be adiabatic with index $\gamma(=d\ln P/d\ln\rho)$. There is no inconsistency with our use of adiabatic perturbations in a background gas layer that is isothermal (e.g., sound waves in the air have this property). A local increase in the interstellar gas density, n, causes the local cooling rate to increase, initially as n^2, but the heating rate from the unperturbed background of stars and random (Type I) supernovae will not change; thus the temperature in the higher density gas can be less than it is in the unperturbed background. Generally $\gamma\lesssim1$ for small changes in the interstellar density. Observations of the variation of gas temperature with density were compiled by Myers (1978). The mean value of γ from Figure 1 in Myers (1978) is 0.25, obtained by solving for $\gamma=1+d\ln(T)/d\ln(n)$. Here we are interested in the total gas pressure, including macroscopic motions. The total gas pressure is ρu^2 instead of nkT. The change in total pressure with density is not easily determined from observations. A theoretical study by Cowie (1980) showed that the macroscopic energy density in a large region decreases during compression because of an increase in the cloud-cloud collision rate. This energy is radiated away at the shock fronts between the colliding clouds; some of the energy may heat the gas elsewhere, as in the intercloud medium, but most of it will leave the galaxy altogether because the optical depth to such infrared cooling radiation is low. Cowie's study implies that $\gamma < 1$ for macroscopic motions. We can estimate the macroscopic γ from observations by comparing the ratio between the rms velocity in the interstellar medium and that in giant molecular clouds, $u/\Delta v$, to the ratio of densities between the interstellar medium and such clouds, n_0/n_c. Thus $\gamma = 1 + 2\ln(u/\Delta v)/\ln(n_0/n_c)$. Taking $u = 7$ km s^{-1}, $\Delta v = 4$ km s^{-1}, $n_0 = 1$ cm^{-3} and $n_c = 250$ cm^{-3} for typical values, we obtain $\gamma = 0.8$. We shall use $\gamma = 0.8$ for most of the numerical examples in this paper. The equations will be written in general form so any other value of γ may be used at a later time.

We analyze the growth of perturbations that are strictly two-dimensional. The gas is allowed to move parallel to the mean field, in the y-direction, and perpendicular to the plane, in the z-direction, but not in the orthogonal or x-direction. The density perturbations are like sheets or tubes (see Lerche 1967*a*) with infinite extent in the x-direction. Parker (1967) and Lerche (1967*b*) have shown that the gas field layer is unstable in both the y- and x-directions, and we expect this to be true in the self-gravitating case as well. The instability in the x-direction has a *short* wavelength; it may lead to a chaotic cloud structure in this direction (Parker 1967; Lerche 1967*b*; Asséo *et al.* 1978). The self-gravitational instability alone has a *long* wavelength in both the x- and y-directions (i.e., the Jeans length); its wavelength is comparable to the unstable wavelength along the y-direction of the pure Parker instability, namely $2\pi H$ for scale height H (e.g., compare Ledoux 1951 and Parker 1966). Thus the full, three-dimensional instability will grow by the Parker effect and by self-gravity in the y-direction (parallel to the unperturbed field), but it will fragment by the Parker instability alone in the x-direction. We are interested in how large clouds form by long-range instabilities, and we are not concerned with how these clouds may fragment as they form or after they form. For this reason, we need to investigate only the growth of perturbations parallel to B. This simplifies the problem greatly. *Short* wavelength growth perpendicular to B and in the galactic plane will not differ significantly from Parker's (1967) analysis when self-gravity is included.

b) *Modeling the Self-Gravitational Force*

The component of the self-gravitational acceleration of the perturbed gas layer that is directed toward the midplane increases with height above the plane, as does the acceleration from the stars. The component of this perturbed self-gravitational acceleration that is parallel to the plane decreases with height. To make a normal mode analysis of the instability, we integrate the equation of motion and the continuity equations over the height of a perturbation. This integration is also useful for two other reasons: (1) The equilibrium layer is exponential, so it has a density cusp at the midplane. The singularity resulting from this density cusp is removed by integration over the height of the perturbation. (2) The Parker type forces are largest away from the midplane, where the curvature of the magnetic field is largest (for a symmetric mode), whereas the self-gravitational forces are largest near the midplane, where the density perturbation is largest. These two different forces act in the same direction, and they both lead to the growth of a cloud in the midplane, but they differ in the height at which they are most effective. Integrating over the height of the perturbation gives an exact average of the combined forces for cloud formation.

III. THE DISPERSION EQUATION FOR THE PARKER-JEANS INSTABILITY WITH A GENERALIZED SELF-GRAVITATIONAL ACCELERATION

We first derive an equation for the Parker-Jeans instability in terms of generalized, perturbed self-gravitational accelerations from the gas, $\delta g_{\rm gas}$. Exact accelerations are introduced in § IV. The y and z axes are parallel and perpendicular to the galactic plane, respectively, and parallel and perpendicular to the unperturbed magnetic field, which is in the y-direction. The *unperturbed* self-gravitational acceleration of the gas in a direction perpendicular to the plane is taken as part of the total unperturbed acceleration in this direction, g_0; there is no need to separate out the gas and stellar contributions to this *unperturbed* acceleration.

The equation of motion for the gas in a layer with *total* gravitational acceleration $\boldsymbol{g}$, magnetic field $\boldsymbol{B}_T$ and pressures $\boldsymbol{P}_T$ and $\boldsymbol{P}_{\rm CRT}$ is (Parker 1966)

$$\rho_T \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\left(P_T + P_{\rm CRT} + \frac{B_T^2}{8\pi}\right) + \frac{1}{4\pi}\boldsymbol{B}_T\cdot\nabla\boldsymbol{B}_T + \rho_T\boldsymbol{g}, \qquad (6)$$

and the continuity equation is

$$\frac{\partial \rho_T}{\partial t} = -\nabla\cdot\rho_T\boldsymbol{v}. \qquad (7)$$

The cosmic rays move freely and quickly along the field, so (Shu 1974)

$$B_T\cdot\nabla P_{\rm CRT} = 0. \qquad (8)$$

The equilibrium solution to these equations is given by equations (1) and (2).

We introduce perturbations denoted by δ of the form

$$B_T = B + \delta B \qquad \rho_T = \rho + \delta\rho$$

$$P_T = P + \delta P \quad P_{\rm CRT} = P_{\rm CR} + \delta P_{\rm CR}, \tag{9}$$

and $\boldsymbol{v}$ is a perturbation. We assume as in Parker (1966) that the perturbed pressure varies as $\delta P = (\gamma P/\rho)\delta\rho$, so

$$\frac{\delta P}{\delta t} = -\boldsymbol{v}\cdot\nabla P - \gamma\rho u^2 \nabla\cdot\boldsymbol{v}. \tag{10}$$

Recall from § II that γ will be set equal to 0.8 for the numerical calculations in this paper. Also as in Parker (1966), we convert δB to a vector potential

$$\delta\boldsymbol{B} = \nabla\times\delta\boldsymbol{A}. \tag{11}$$

Since the interstellar gas is usually highly conductive, the magnetic field will follow the gas:

$$\frac{\partial\delta\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}). \tag{12}$$

We analyze only the growth of perturbations that are uniform along the x-direction, which is orthogonal to the y- and z-directions (see § IIa). Then δB is in the y-z plane, and δA is in the x-direction. From equations (11) and (12),

$$\frac{\partial}{\partial t}\partial A = -Bv_z. \tag{13}$$

It now follows from equation (8) that (Parker 1966):

$$\delta P_{\rm CR} = \frac{\delta A}{B}\frac{\partial}{\partial z}P_{\rm CR0} = -\frac{\delta A}{B}\frac{P_{\rm CR0}}{H}. \tag{14}$$

We isolate the gravitational terms that arise from the self-gravity of the gas and write

$$\boldsymbol{g} = -g(\text{stars}+\text{unperturbed gas})\,\hat{e}_z + \delta\boldsymbol{g}_{\rm gas}$$

$$= -\frac{u^2(1+\alpha+\beta)}{H}\hat{e}_z + \delta\boldsymbol{g}_{\rm gas}. \tag{15}$$

The equation of motion and the continuity equations are now written in terms of the perturbed quantities; the corresponding equilibrium equations are subtracted to give perturbation equations, and these perturbation equations are linearized for small perturbations. The resulting linearized perturbation equations are:

$$\rho\frac{\partial}{\partial t}v_y = -\frac{\partial}{\partial y}(\delta P + \delta P_{\rm CR}) + \left(\frac{\partial}{\partial y}\delta A\right)\frac{B}{8\pi H} + \rho\,\delta g_{{\rm gas},\,y} \tag{16}$$

$$\rho\frac{\partial}{\partial t}v_z = -\frac{\partial}{\partial z}(\delta P + \delta P_{\rm CR}) + \left(\frac{\partial}{\partial z}\delta A\right)\frac{B}{8\pi H} - \frac{B}{4\pi}\left(\frac{\partial^2}{\partial y^2}\delta A + \frac{\partial^2}{\partial z^2}\delta A\right) - \delta\rho\frac{u^2(1+\alpha+\beta)}{H} + \rho\,\delta g_{{\rm gas},\,z} \tag{17}$$

$$\frac{\partial}{\partial t}\delta\rho = v_z\frac{\rho}{H} - \rho\left(\frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z\right) \tag{18}$$

$$\frac{\partial}{\partial t}\delta P = v_z\frac{P}{H} - \gamma P\left(\frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z\right). \tag{19}$$

The coefficients are independent of t and y, so we assume generalized perturbations of the form

$$\delta\rho = \exp(i\omega t + iky)\,\delta\rho'(z), \tag{20}$$

and so on, for the other perturbed quantities. Thus $\partial/\partial t$ is replaced by $i\omega$ and $\partial/\partial y$ is replaced by ik.

There is no need to write all of the steps in the reduction of these equations here. It is evident that we can substitute $\delta\rho$ and δP from equations (18) and (19) into equation (16) to solve for v_y. Then $\delta\rho$ and δP can be put into equation (17), along with the newly derived v_y, and v_z and $\delta P_{\rm CR}$ can be eliminated using equations (13) and (14). The result of these operations is a single equation for δA and δg and their z-derivatives. This procedure was discussed by Parker (1966), and we follow through it again in Appendix B.

It is convenient to introduce the dimensionless growth rate and wavenumber,

$$\Omega = i\omega\frac{H}{u} \tag{21}$$

$$\nu = kH. \tag{22}$$

Thus the perturbation grows like $\exp(\Omega t u/H)$, and the wavelength parallel to the field is $2\pi H/\nu$.

The differential equation for $\delta A(z)$ now becomes, after dropping the primes,

$$\frac{HP_1}{B}\frac{\partial^2}{\partial z^2}\delta A + \frac{P_2}{HB}\delta A = \frac{H\delta g_{{\rm gas},\,z}}{u^2}(\gamma\nu^2 + \Omega^2) + \frac{Hi\nu q}{u^2}\delta g_{{\rm gas},\,y} + \frac{H^2 i\nu\gamma}{u^2}\frac{\partial}{\partial z}\delta g_{{\rm gas},\,y}. \tag{23}$$

The term $\exp(i\omega t + iky)$ has been dropped from both sides of the equation. Hereafter, δA and $\delta g_{\rm gas}$ are functions of only z, as is B. The quantities P_1, P_2, q, and

r are spatially constant:

$$P_1 = 2\alpha\gamma\nu^2 + \Omega^2(2\alpha+\gamma) \quad (24)$$

$$P_2 = r\nu^2 - \Omega^4 - \Omega^2(2\alpha+\gamma)(\nu^2+1/4) - 2\alpha\gamma\nu^4 \quad (25)$$

$$q = 1+\alpha+\beta-\gamma \quad (26)$$

$$r = (1+\alpha+\beta)q - \alpha\gamma/2. \quad (27)$$

Since the gravitational acceleration is the gradient of a potential, it has no curl,

$$H\frac{\partial}{\partial z}\delta g_{\text{gas},y} = H\frac{\partial}{\partial y}\delta g_{\text{gas},z} = i\nu\,\delta g_{\text{gas},z}. \quad (28)$$

Thus equation (23) becomes

$$H^2P_1\frac{\partial^2}{\partial z^2}\delta A + P_2\,\delta A = \frac{H^2B}{u^2}\left(\delta g_{\text{gas},z}\Omega^2 + i\nu q\,\delta g_{\text{gas},y}\right). \quad (29)$$

This is the desired equation. In § IV and Appendices A and B, we derive δg_{gas} as a function of $\delta\rho$, and then convert equation (29) to one written entirely in terms of δA and its derivatives.

The equation derived by Parker (1966) is the same as equation (29) except the right-hand side is equal to zero. The coefficients in his equation are independent of z so symmetric solutions are of the form

$$\delta A = \delta A_0 \sin(k_\perp z) \quad (30)$$

for perturbations of wavenumber $k_\perp$ and for constant δA_0. For dimensionless wavenumber

$$\zeta = k_\perp H, \quad (31)$$

the dispersion relation for the pure Parker instability is

$$-\zeta^2 P_1 + P_2 = 0. \quad (32)$$

Solutions to this equation are discussed in § V and Appendix C. We note that from equations (16), (17), and (18), and for symmetric perturbations,

$$v_y = -iv_{y0}e^{|z|/2H}\cos(\zeta z/H) \quad (33)$$

$$v_z = v_{z0}e^{|z|/2H}\sin(\zeta z/H) \quad (34)$$

$$\delta\rho = \delta\rho_0 e^{-|z|/2H}\cos(\zeta z/H), \quad (35)$$

for constants v_{y0}, v_{z0} and $\delta\rho_0$. Equation (35) gives the perturbed density that we use to derive δg_{gas}. The scale height of the density perturbation is $2H$, which we define to be L in what follows.

IV. SELF-GRAVITATIONAL FORCES IN THE MODEL INTERSTELLAR MEDIUM

The gravitational acceleration for an infinitely thin layer with mass column density $2\delta\rho L$ is

$$\delta g_{\text{gas},y} = 4\pi i G\,\delta\rho L \quad (z\approx 0) \quad (36)$$

$$\delta g_{\text{gas},z} = \pm 4\pi G\,\delta\rho L \quad (z \lessgtr 0)$$

$$\delta g_{\text{gas},z} = 0 \quad (z=0). \quad (37)$$

With $\delta\rho$ given by equation (35), however, the exact gravitational acceleration for our problem is a solution to the equation

$$\nabla\cdot\delta g_{\text{gas}} = -4\pi G\,\delta\rho(y,z). \quad (38)$$

This *exact* acceleration is written in a form analogous to equations (36) and (37) in Appendix A:

$$\delta g_{\text{gas},y}(y,z,t) = 4\pi i G\,\delta\rho(y,z,t)L\Gamma_y(z,k,k_\perp) \quad (39)$$

$$\delta g_{\text{gas},z}(y,z,t) = \pm 4\pi G\,\delta\rho(y,z,t)L\Gamma_z(z,k,k_\perp) \quad (z \gtrless 0). \quad (40)$$

The Γ functions are given by equations (A10) and (A11). At $z=0$, $\delta g_{\text{gas},y}$ is nonzero and $\delta g_{\text{gas},z}=0$. Also, as z goes to infinity, and for $k=k_\perp=0$, $\delta g_{\text{gas},z} = 4\pi G\,\delta\rho_0 L$, as expected for the acceleration toward the plane. Γ_z is always less than 0.

Equation (29) now reduces to

$$\frac{HP_1}{B}\frac{d^2\,\delta A}{dz^2} + \frac{P_2\delta A}{HB} = s_0\left(\frac{\delta\rho}{\rho_0}\right)\left(\Omega^2\Gamma_z - \nu q\Gamma_y\right), \quad (41)$$

where the dimensionless midplane density is defined to be

$$s_0 \equiv \frac{4\pi G\rho_0 HL}{u^2} = \frac{8\pi G\rho_0 H^2}{u^2}. \quad (42)$$

To obtain the final dispersion relation, the quantity $\delta\rho/\rho_0$ has to be written in terms of $\delta A/HB_0$. The procedure for this conversion involves an integration over the height of the perturbation and is discussed in Appendix B. We write the integrated gravitational force in terms of G functions, which are given explicitly by equation (B18). By definition,

$$G_{y,z} \equiv \frac{\int_0^{\pi/k_\perp} \frac{\rho}{\rho_0} \frac{\delta\rho}{\delta\rho_0} \Gamma_{y,z}\, dz}{\int_0^{\pi/k_\perp} \frac{\delta\rho}{\delta\rho_0}\, dz}. \tag{43}$$

The result of our reduction is the dispersion relation for the Parker-Jeans instability:

$$-\zeta^2 P_1 + P_2 = \frac{s_0 \nu^2 q}{\Omega^2 - \Omega_G^2}\left(\Omega^2 G_z - \nu q G_y\right). \tag{44}$$

The gravitational growth rate, Ω_G, is defined by the equation

$$\Omega_G^2 = s_0 \nu G_y - \gamma \nu^2. \tag{45}$$

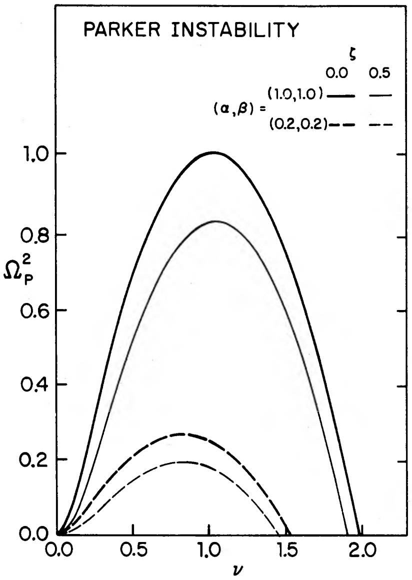

FIG. 1.—The square of the normalized growth rate of the pure Parker instability, Ω_P^2, is shown as a function of the normalized horizontal wavenumber, ν. The growth rates in this and in the following figures are normalized to u/H for sound speed u and scale height H, and the wavenumbers are normalized to $1/H$. Vertical wavenumbers of $\zeta = 0$ and 0.5 correspond to perturbation half-heights of ∞ and 500 pc, respectively, for $H = 160$ pc. The ratios of magnetic and cosmic ray pressures to gas pressure are α and β, and the effective ratio of specific heats is γ.

V. GROWTH RATES FOR THE PURE PARKER INSTABILITY, THE PURE SELF-GRAVITATIONAL INSTABILITY, AND THE PURE CONVECTIVE INSTABILITY

The change introduced by self-gravity to the pure Parker instability is best illustrated by analyzing each separate instability in turn. Figures 1–5 show the dispersion relation for each part of the instability. Such illustrations do not appear elsewhere in the literature, nor do the simple analytic expressions for these growth rates that are derived here and in the Appendices. A special limiting case to the Parker-Jeans instability ($\alpha = \beta \to 0$) is also shown in this section as a further illustration of the physical processes involved. We discuss the full solution to equation (44) in § VI.

a) The Pure Parker Instability ($s_0 = 0$)

The dispersion relation for the Parker instability without self-gravity is

$$-\zeta^2 P_1 + P_2 = 0,$$

where P_1 and P_2 were given by equations (24) and (25). The left-hand side is a second-order polynomial in Ω^2, and the coefficient of Ω^4 is -1, so we can write this dispersion relation as

$$-\left(\Omega^2 - \Omega_P^2\right)\left(\Omega^2 - \Omega_0^2\right) = 0. \tag{46}$$

The solutions for Ω^2, namely, Ω_P^2 and Ω_0^2, are written explicitly in terms of α, β, γ, ν, and ζ in Appendix C. The unstable mode corresponds to Ω_P^2, which is positive for instability (c.f. eq. [21]). The second solution, Ω_0^2, is always negative, and it corresponds to stable oscillations of the gas field layer around the galactic midplane.

Figure 1 shows Ω_P^2 versus ν for $\zeta = 0$ (*bold lines*) and $\zeta = 0.5$. These values of ζ correspond to perturbations extending to infinite distances from the midplane, and extending to ± 3.1 scale heights (± 500 pc if $H = 160$ pc), respectively. The solid lines correspond to the parameters $(\alpha, \beta, \gamma) = (1, 1, 0.8)$, and the dashed lines correspond to $(\alpha, \beta, \gamma) = (0.2, 0.2, 0.8)$.

For constant ζ, the squared growth rate of the Parker instability, Ω_P^2 shown in Figure 1, goes to 0 as $\nu \to 0$, it reaches a peak, Ω_P^2 (MAX.GROWTH), at wavenumber ν (MAX.GROWTH), and it crosses zero and goes negative (corresponding to stable wave motions) at critical wavenumber ν_{crit}. Perturbations with wavelengths shorter than $2\pi H/\nu_{crit}$ are stable.

The *peak* growth rates and corresponding wavenumbers, and the critical wavenumbers, are plotted as functions of ζ in Figure 2. Explicit formulae for these quantities are given in Appendix C.

The fastest growth of the Parker instability occurs for $\zeta = 0$. Such an infinitely extended perturbation is not always relevant for an investigation of cloud formation because the growth consists of purely lateral motions, parallel to the plane ($v_z = 0$). The concept that giant clouds form in magnetic valleys corresponds to the case $\zeta > 0$. Furthermore, the layer of gas that can form clouds may not extend much higher than 500 pc to 1000 pc because most of the mass of the interstellar medium at

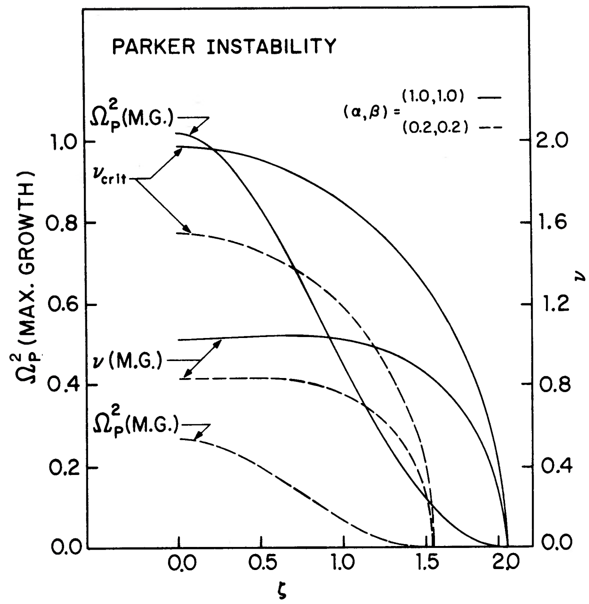

FIG. 2.—The square of the normalized growth rate of the pure Parker instability, Ω_P^2 (*left-hand axis*), maximized over horizontal wavenumber, ν, and plotted as a function of vertical wavenumber, ζ. The horizontal wavenumber at maximum growth, ν, and the critical wavenumber for instability, ν_{crit}, are plotted on the right-hand axis. See Fig. 1 for normalization factors and definitions of α, β, and γ.

these heights may be in the form of hot, low density coronal gas (Savage and deBoer 1979). A reasonable height for a perturbation may be ± 500 pc, corresponding to $\zeta = 0.5$. The maximum growth rates of the pure Parker instability for $\zeta = 0.5$ are Ω (MAX.GROWTH) = 0.44, 0.63, and 0.91 for $(\alpha, \beta, \gamma) = (0.2, 0.2, 0.8)$, $(0.5, 0.5, 0.8)$, and $(1, 1, 0.8)$, respectively. These growth rates correspond to e-folding times of 51, 36, and 25 million years, respectively, for $H = 160$ pc and $u = 7$ km s^{-1}. The fastest-growing wavelengths are 1.20, 1.12, and 0.97 kpc respectively, for these three cases and for $\zeta = 0.5$, and the critical wavelengths are 0.69, 0.64, and 0.53 kpc. Similar solutions with $\zeta = 0$ are given in Table 1 (for the $s_0 = 0$ case).

b) The Pure Jeans Instability ($\alpha = \beta = 0$)

For $\alpha = \beta = 0$, equation (44) reduces to a dispersion relation for the Jeans instability in an exponential gas layer. This is discussed in some detail in Appendix D. If $\gamma < 1$, the convective instability is still present, as discussed in § V*c*. We note here that for $\gamma = 1$, the convective instability is suppressed, and the purely self-gravitational dispersion relation becomes

$$\Omega^2(\Omega^2 - \Omega_G^2)\left[\Omega^2 + (\nu^2 + \zeta^2 + 1/4)\right] = 0. \quad (47)$$

There are three modes in this case: there is a steady-state solution with $\Omega^2 = 0$ (which is a remnant of the pure Parker instability), a second mode which is a gravitational instability with squared growth rate

$$\Omega^2 = \Omega_G^2 = s_0 \nu G_y - \nu^2, \quad (48)$$

and there is a stable mode that consists of oscillations around the midplane with frequency $(\nu^2 + \zeta^2 + 1/4)^{1/2}$. The oscillation is a density-symmetric wave running through the equilibrium gas layer. The restoring forces for the wave are gas pressure, P, and the gravitational force from stars, g_0.

The gravitational instability, $\Omega^2 = \Omega_G^2$, is analogous to the Jeans instability in a thin layer. The dispersion relation for the three-dimensional Jeans instability is, of course, (e.g., Spitzer 1978),

$$\omega^2 = k^2 u^2 - 4\pi G \rho, \quad (49)$$

but for an infinitely thin, self-gravitating sheet, this equation takes a different form,

$$\omega^2 = k^2 u^2 - 2\pi G \sigma k \quad (50)$$

for mass column density through the sheet, σ. This latter equation is derived in Elmegreen (1981).

The similarity between equations (50) and (48) becomes obvious if we convert the latter back into the physical units used in § III:

$$\omega^2 = k^2 u^2 - 4\pi G \rho_0 L k G_y. \quad (51)$$

The mass column density of the exponential layer here is $\sigma = 2\rho_0 H = \rho_0 L$. Thus the self-gravitating term here is $2G_y$ times as large as that in equation (50). The two equations become identical in the limit of $\zeta \to 0, \nu \to 0$ because then $G_y = 1/2$. In this limit of infinite wavelengths, the exponential plane acts like an infinitesimally thin plane, so the dispersion relations (50) and (51) become identical, as expected intuitively.

Figure 3 shows the squared growth rate of the purely self-gravitational mode, Ω_G^2, as a function of ν for $\zeta = 0.0$ (*bold lines*) and $\zeta = 0.5$, and for $\gamma = 0.8$. The two solutions are for different densities, $s_0 = 1$ and $s_0 = 10$. We use $\gamma = 0.8$ to facilitate comparisons between Ω_G^2 and the growth rate of the full Parker-Jeans instability. Note that Ω_G^2 is less for $\zeta = 0$ than it is for $\zeta = 0.5$ because the self-gravitational forces peak near the midplane. This is opposite to the situation for Ω_P^2, which is largest at $\zeta = 0$.

Figure 4 shows the peak growth rate (maximized over ν), and the horizontal wavenumber (ν) at peak growth as functions of the log of the vertical wavenumber, ζ. Again the tendency for Ω_G^2 to peak at $\zeta > 0$ is evident. For $\zeta \to 0$, the peak of Ω_G^2 over ν approaches a value independent of ζ; this case is for perturbations that extend for infinite heights above and below the plane.

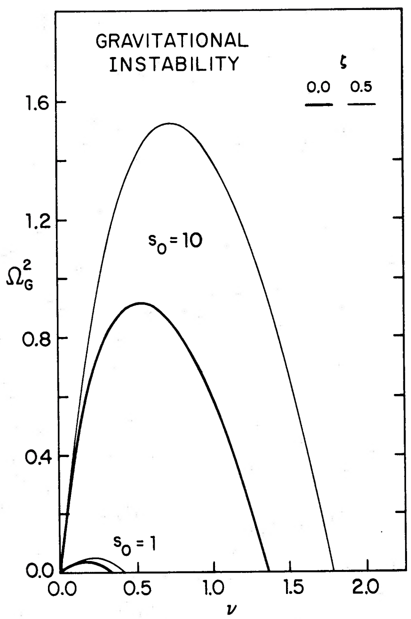

FIG. 3

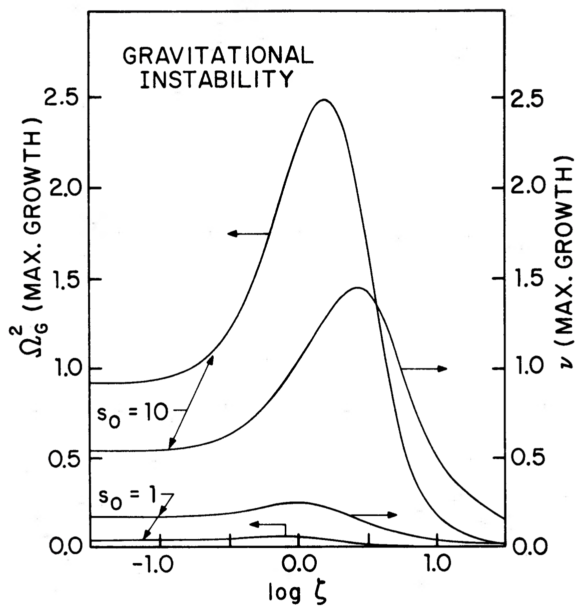

FIG. 4

FIG. 3.—The square of the normalized growth rate of the pure self-gravitational instability, Ω_G^2, plotted as a function of horizontal wavenumber ν. Values of vertical wavenumber, ζ, are indicated. The normalized density parameter, s_0, equals 1.8 times the midplane density, $n_0(\text{cm}^{-3})$, for a scale height of 160 pc and a sound speed of 7 km s^{-1}. The ratio of specific heats, γ, equals 0.8.

FIG. 4.—The square of the normalized growth rate of the pure self-gravitational instability, Ω_G^2 (*left-hand axis*), maximized over horizontal wavenumber, ν, is shown plotted as a function of the log of the vertical wavenumber, ζ. The horizontal wavenumber at maximum growth is shown on the right-hand axis. The dimensionless density parameter, s_0, equals 1.8 times the midplane density (in cm^{-3}), as in Fig. 3.

When ζ increases from 0, the mean self-gravitational force averaged over the height of the perturbation increases because the mean density averaged over that height increases, until $\zeta \approx 1$. Further increases in ζ decrease the self-gravitational force because the total mass of the perturbation begins to decrease rapidly for large ζ. The peak in Ω_G^2 is around $\zeta \approx 1$, which shows that only the lowest one or two scale heights of the gas are likely to condense by self-gravitational forces. We shall return to this point in our discussion of Figure 8 in § VI*b*.

c) *The Pure Convective Instability* ($\alpha = \beta = s_0 = 0$; $\gamma < 1$)

When $\alpha = \beta = s_0 = 0$, equation (44) reduces to a dispersion relation for pure convection. This reduction is discussed in Appendix D. The result is the squared growth rate for convective instabilities, $\Omega_{\text{conv}}^2(\nu)$, written as equation (D12); this is shown in Figure 5 for $\zeta = 0.0$ (*bold line*) and $\zeta = 0.5$, and for $\gamma = 0.8$.

The maximum growth rate occurs at $\nu = \infty$, and it is independent of ζ, H, and L:

$$\Omega_{\text{conv}}^2\,(\text{MAX.GROWTH}) = \frac{1-\gamma}{\gamma}. \qquad (52)$$

The instability criterion for convection is that γ must be less than 1.

A stratified interstellar medium *without* the Parker type destabilization from magnetic fields and cosmic rays ($\alpha = \beta = 0$), and *without* self-gravity ($s_0 = 0$) is still

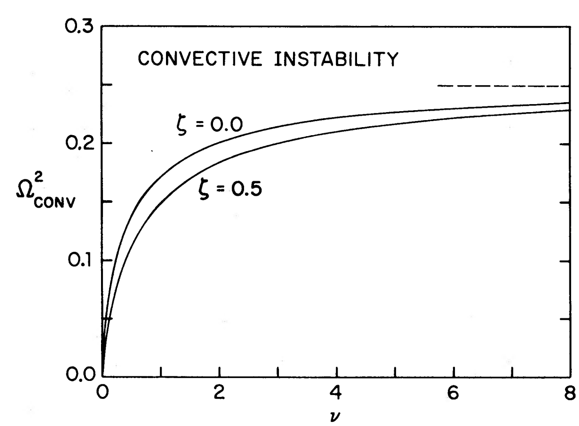

FIG. 5.—The square of the normalized growth rate of the pure convective instability, Ω_{CONV}^2, plotted as a function of vertical wavenumber, ν, for two values of the horizontal wavenumber, ζ. For this instability, $\alpha = \beta = s_0 = 0$, and $\gamma = 0.8$.

convectively *unstable* if $\gamma < 1$. The Schwarzschild criterion for local convective instability is that the background (equilibrium) temperature, T, and pressure, P, satisfy the inequality

$$\left.\frac{d\ln T}{d\ln P}\right|_{\rm Eq.} > \frac{\gamma-1}{\gamma}. \tag{53}$$

The equilibrium interstellar medium modeled here has a constant temperature with height, so

$$\left.\frac{d\ln T}{d\ln P}\right|_{\rm Eq.} = 0. \tag{54}$$

Thus $\gamma < 1$ leads to convective instability in the usual sense.

The Parker criterion for instability also may be written as an upper limit for γ, as shown by equation (C16). When $\alpha = \beta = 0$, this Parker criterion is again $\gamma < 1$ for instability. Thus the Parker instability is closely related to *convection*, and the Parker instability criterion is similar in some respects to the Schwarzschild criterion for convection.

We note that an interstellar medium with $\gamma < 1$ need not be *thermally* unstable. For example, if the cooling and heating functions for interstellar gas are of the usual form

$$\Lambda = \Lambda_0 n^2 T^\alpha$$

$$\Gamma = \Gamma_0 n, \tag{55}$$

then in thermal equilibrium, $\Lambda = \Gamma$, so $T^\alpha \propto n^{-1}$. But if $P \propto n^\gamma$, then $T \propto n^{\gamma-1}$ for an ideal gas, so $\gamma = 1 - 1/\alpha$; then $\gamma < 1$ if $\alpha > 0$. Thermal instability requires (Parker 1953; Field 1965):

$$\frac{\partial}{\partial T}(\Gamma - \Lambda) - \frac{n}{T}\frac{\partial}{\partial n}(\Gamma - \Lambda) > 0 \tag{56}$$

which corresponds to $\alpha < 1$ or $\gamma < 0$. Thus α between 0 and 1 (or $\gamma < 0$) corresponds to thermal instability and convective instability together (Dufouw 1970), while $\alpha > 1$ ($0 < \gamma < 1$) corresponds to convective instability alone. For the value of γ used in this paper, $\gamma = 0.8$, the interstellar medium will be *thermally stable* and *convectively unstable*.

d) Growth Rates for Small α and β and $\gamma = 0.8$

Solutions to the Parker-Jeans dispersion relation (44) for small but nonzero α and β and for $\gamma = 0.8$ are shown in Figure 6. This figure should be compared with Figures 3 and 5. Values of $\alpha = \beta = 0.01$ were chosen. The density parameter, s_0, equals 0.1, 1 and 10, and $\zeta = 0.5$.

We point out three aspects of these solutions. First of all, for large s_0 and for ν less than or equal to the wavenumber at peak growth, the growth rate is dominated by self-gravity; that is, equation (48) is valid and Figures 6 and 3 are identical. Thus the peak growth rate is determined by self-gravity for small α and β and large s_0, as expected.

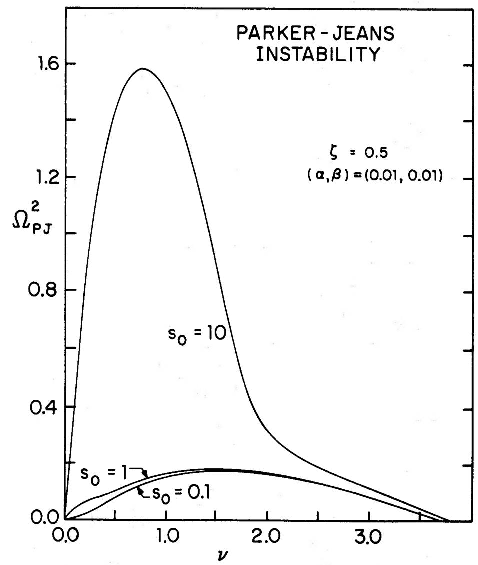

FIG. 6.—The square of the normalized growth rate for the combined Parker-Jeans instability, $\Omega^2_{\rm PJ}$, is shown as a function of the horizontal wavenumber, ν, for very small magnetic and cosmic ray pressures, and for three values of dimensionless midplane density, s_0. This figure shows the importance of convection in the combined instability, as discussed in § V*d*.

A second feature of Figure 6 is that Ω^2 stays positive for very large ν. Thus, very short wavelengths are still unstable by the Parker-convection mode. This instability occurs at short wavelengths for small (α, β) for two reasons: (1) convection is most rapid at short wavelengths (cf. § V*c*), and (2) the Parker instability is essentially a Rayleigh-Taylor instability, which of course grows fastest at short wavelengths. The Parker instability (not the convective instability) is stabilized at small enough wavelengths (corresponding to $\nu_{\rm crit}$) because the tension in the field lines exerts a large force for small field curvature. That is, the $B \cdot \nabla B$ term in Equation (6) leads to the existence of a critical wavenumber. (Using the notation of Appendix C, we note that this curvature force appears in the r and d terms, which exclusively determine $\nu_{\rm crit}$ and $\zeta_{\rm crit}$.) When B_0 gets small, the field lines bend easily at small wavelength, and the gas is unstable at large ν. Thus, the gas can be unstable at wavelengths parallel to the field that are much smaller than a Jeans length if B_0 is small. Incidentally, this curvature force also is absent for perturbations perpendicular to the field and in the galactic plane (i.e., in the x-direction), so such perturbations can have very short wavelengths as well. This was mentioned in § II. Both short wavelength instabilities can create chaotic cloud structure *inside* the large perturbations.

The third interesting aspect about Figure 6 is that the peak growth rate goes to a minimum but nonzero value of ~ 0.2 as $s_0 \to 0$. This is essentially the convective instability, since $\Omega_{\mathrm{conv}}{}^2\,(\mathrm{MAX.GROWTH}) = (1-\gamma)/\gamma = 0.25$ for $\gamma = 0.8$. Thus the self-gravitational and Parker type forces become dominated by the convective instability at small s_0, α, and β, except at very short wavelengths, where magnetic curvature forces eventually stabilize the gas.

VI. GROWTH RATES FOR THE PARKER-JEANS INSTABILITY

a) Limiting Cases

Combining equations (44) and (C2), we write the dispersion relation for the Parker-Jeans instability as

$$(\Omega^2 - \Omega_{\mathrm{P}}^2)(\Omega^2 - \Omega_0^2)(\Omega^2 - \Omega_G^2) = -\left(\Omega^2 - \frac{\nu q G_y}{G_z}\right) s_0 q \nu^2 G_z . \qquad (57)$$

We define the positive quantities

$$Q(\nu,\zeta) = -\frac{\nu q G_y}{G_z} \qquad (58)$$

$$R(\nu,\zeta,s_0) = -s_0 q \nu^2 G_z . \qquad (59)$$

The functions $G_y > 0$ and $G_z < 0$ are written in equation (B18), and q is from equation (26); s_0 is the dimensionless midplane density, and ν and ζ are dimensionless wavenumbers parallel and perpendicular to the plane (as defined previously). The growth rates, Ω_G, Ω_{P}, and Ω_0 are defined by equations (45) and (C3). Thus the dispersion relation becomes

$$\frac{(\Omega^2 - \Omega_{\mathrm{P}}^2)(\Omega^2 - \Omega_0^2)(\Omega^2 - \Omega_G^2)}{(\Omega^2 + Q)} = R . \qquad (60)$$

First consider the interesting limits to this equation. For $s_0 = 0$, self-gravity is unimportant; then R equals 0 and $\Omega_G^2 = -\gamma\nu^2$. The modes of oscillation in this case are the pure Parker instability, $\Omega^2 = \Omega_{\mathrm{P}}^2 > 0$, the Parker wave, $\Omega^2 = \Omega_0^2 < 0$, and a pure sound wave, $\Omega^2 = -\gamma\nu^2$.

For large density, $s_0 \to \infty$, so R and Ω_G^2 increase in proportion to s_0. Thus the right-hand side of equation (60) increases with s_0, so the left-hand side must increase with s_0 also. This can happen only if

$$\Omega^2 \to \Omega_G^2 + \text{constant} \quad (s_0 \to \infty)$$
$$\to s_0 \nu G_y . \qquad (61)$$

Thus *the instability becomes purely gravitational at large gas density*.

At very long wavelengths, $\nu \to 0$ while G_y and G_z remain finite and nonzero. Then $R \to 0$ as ν^2, and $Q \to 0$ as ν. The squared gravitational growth rate, Ω_G^2, goes to zero as ν, but, from equation (C3) or Figure 1, the squared Parker growth rate, Ω_{P}^2, goes to zero as ν^2, and the squared Parker wave frequency, $-\Omega_0^2$, stays finite as $\nu \to 0$. Thus, equation (60) shows that Ω^2 must go to zero as ν. In that case, the term $(\Omega^2 - \Omega_{\mathrm{P}}^2)(\Omega^2 - \Omega_0^2)/(\Omega^2 + Q)$ goes to a constant as $\nu \to 0$. Thus $\Omega^2 - \Omega_G^2$ must go to zero in proportion to R for long wavelengths, or

$$\Omega^2 \to \Omega_G^2 + \text{constant}\cdot\nu^2 \to s_0 \nu G_y \quad (\nu \to 0).$$

This shows that *long wavelength perturbations are dominated by self-gravity no matter what the gas density is* (as long as s_0 is not exactly zero).

b) Calculated Growth Rates

The dispersion relation (60) for the Parker-Jeans instability is cubic in Ω^2. Except for very low ν (see below), there is only one real positive solution. The positive solution is the square of the growth rate of the Parker-Jeans instability. It may be written explicitly in terms of the constant parameters in the problem, α, β, γ, and s_0, and in terms of the wavenumbers, ν and ζ_0. This explicit solution is:

$$\Omega_{\mathrm{PJ}}^2 = 2C_1 \cos\theta - D_1/3 , \qquad (62)$$

where

$$\theta = -\frac{1}{3}\arctan\left(\frac{C_2}{C_3}\right) + \frac{\pi}{3} , \qquad (63)$$

and

$$C_1 = (C_3^2 + C_2^2)^{1/6}$$

$$C_2 = \left(-\frac{D_2^3}{27} + \frac{D_2^2 D_1^2}{108} + \frac{D_1 D_2 D_3}{6} - \frac{D_3^2}{4} - \frac{D_3 D_1^3}{27}\right)^{1/2}$$

$$C_3 = \frac{1}{6}(-D_1 D_2 + 3D_3) + \frac{D_1^3}{27} \qquad (64)$$

$$D_1 = -(\Omega_G^2 + \Omega_{\mathrm{P}}^2 + \Omega_0^2)$$

$$D_2 = \Omega_G^2\Omega_{\mathrm{P}}^2 + \Omega_G^2\Omega_0^2 + \Omega_{\mathrm{P}}^2\Omega_0^2 - R$$

$$D_3 = -RQ - \Omega_G^2\Omega_{\mathrm{P}}^2\Omega_0^2 .$$

We have used the usual formula for solving cubic polynomials (e.g., see Abramowitz and Stegun 1970).

At very low $\nu(\lesssim 10^{-2})$, there are two real solutions to equation (60):

$$\Omega_1^2 = \Omega_G^2\left(1 + \frac{A}{\Omega_G^2}\right) \tag{65}$$

and

$$\Omega_2^2 = \Omega_P^2 - A, \tag{66}$$

where

$$A \approx \left(\frac{\Omega_G^2 + Q}{\Omega_G^2 - \Omega_0^2}\right) R \propto \nu^3. \tag{67}$$

As $\nu \to 0$, the first solution goes to Ω_G^2, and the second solution goes to zero; the second solution is always much smaller than Ω_G^2. Thus the long wavelength limit has a very slow instability that is a remnant of the pure Parker instability, and a much faster instability that is purely self-gravitational. In what follows we consider only the fastest instability ($\Omega_{PJ}^2 \sim \Omega_G^2$) at very small ν where there are two unstable solutions, and for larger ν, we discuss the only unstable mode there is, whose growth rate is given by equation (62).

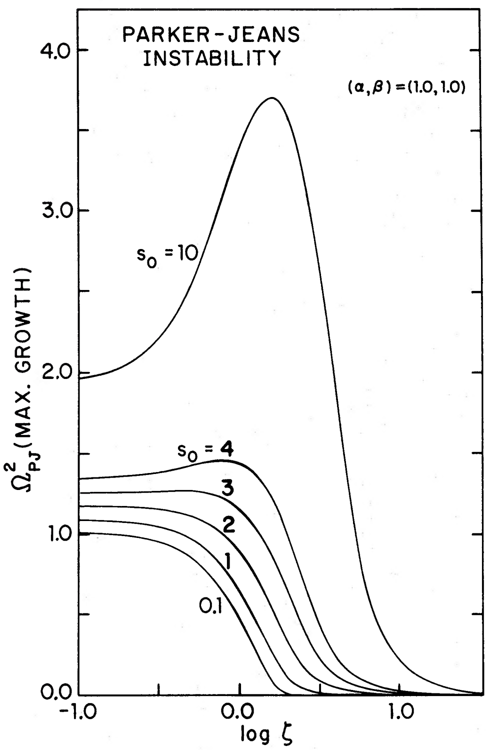

FIG. 8.—The square of the normalized growth rate for the combined Parker-Jeans instability, Ω_{PJ}^2, maximized over horizontal wavenumber, is plotted as a function of the log of the vertical wavenumber, ζ. Each curve corresponds to a different density; $(\alpha, \beta, \gamma) = (1, 1, 0.8)$ for this figure. This shows the tendency for the fastest-growing mode of the instability (where Ω_{PJ}^2 peaks) to change from one having an infinite extent in the vertical direction ($\zeta = 0$) to one having only a finite extent ($\zeta > 0$) when self-gravitational forces become important.

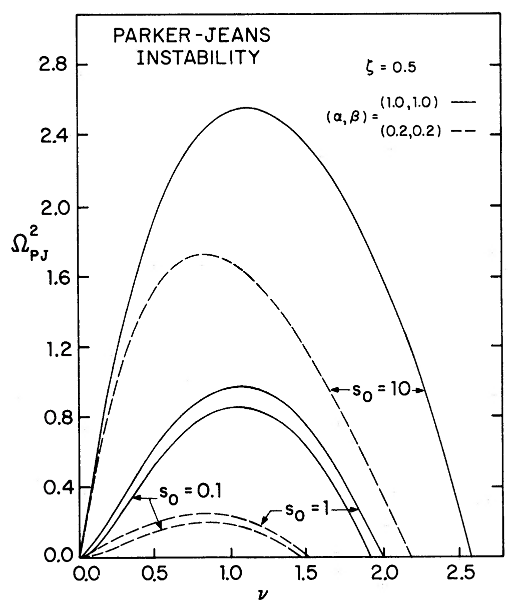

FIG. 7.—The square of the normalized growth rate for the combined Parker-Jeans instability, Ω_{PJ}^2, is plotted as a function of horizontal wavenumber, ν, for two values of the vertical wavenumber, ζ. The dimensionless midplane density is s_0, and the normalized magnetic and cosmic ray pressures are α and β. The increase in growth rate with increasing s_0, α, and β is evident. Self-gravity is too weak to affect the growth rate of the instability when $s_0 = 0.1$, but self-gravity begins to be important when s_0 is greater than or equal to 1.

The results are shown in Figures 7, 8, and 9. Figure 7 shows Ω_{PJ}^2 versus ν for $s_0 = 0.1$, 1, and 10, and for $\zeta = 0.5$. As in the rest of this paper, solid curves are for $(\alpha, \beta, \gamma) = (1, 1, 0.8)$ and dashed curves are for $(\alpha, \beta, \gamma) = (0.2, 0.2, 0.8)$. Figure 8 shows the peak values of Ω_{PJ}^2, maximized over ν, as functions of the log of the vertical wavenumber, ζ, for $(\alpha, \beta, \gamma) = (1, 1, 0.8)$ and for $s_0 = 0.1$, 1, 2, 3, 4, and 10. The curve for $s_0 = 0.1$ in Figure 7 is nearly identical to the curve for Ω_P^2 at the same α, β, γ, and ζ in Figure 1, as expected in the low density limit. (Figs. 2 and 8 show similar results for low s_0 as well, but the comparison is not straightforward in this case because of the different axes.) As s_0 increases, Ω_{PJ}^2 increases, approaching the pure gravitational solutions shown in Figures 3 and 4.

Figure 9 shows Ω_{PJ}^2 maximized over both wavenumbers ν and ζ and plotted as a function of the interstellar density parameter, s_0. We take the same two cases, $(\alpha, \beta, \gamma) = (1, 1, 0.8)$ and $(0.2, 0.2, 0.8)$. The two limiting cases for small and large s_0 are clear: the growth rate approaches the pure Parker solution for small s_0 ($\Omega_{PJ}^2 =$

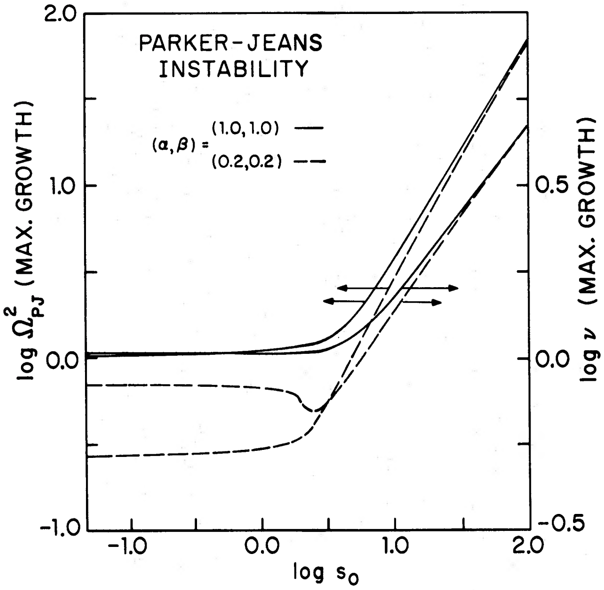

FIG. 9.—The log of the square of the normalized growth rate of the combined, Parker-Jeans instability, $\Omega_{\rm PJ}^2$, maximized over both vertical and horizontal wavenumbers (*left-hand axis*), is plotted as a function of the log of the normalized midplane density, s_0. The log of the normalized horizontal wavenumber at peak growth is shown on the right-hand axis. With the normalization assumed in this paper, the physical midplane density, n_0, equals $0.55 s_0$ cm^{-3}, the growth time equals $22.4/\Omega_{\rm PJ}$ million years, and the wavelength equals $1.01/\nu$ kpc. This figure shows the effect of self-gravity on the combined instability. For small s_0, $\Omega_{\rm PJ}^2$ and ν equal the corresponding constant values at peak growth of the pure Parker instability. For large s_0, $\Omega_{\rm PJ}^2$ and ν approach the pure self-gravitational values, which increase linearly with s_0. Self-gravity becomes important for these α and β when s_0 is between 3 and 6, corresponding to space densities between 1.6 cm^{-3} and 3 cm^{-3}. For n_0 between 5 cm^{-3} and 10 cm^{-3}, the peak growth rate of the instability is between 15 and 7 million years for a wide range in parameters (α, β, γ); see also Table 1.

$\Omega_{\rm P}^2 = $const.), and it approaches the pure gravitational solution for large s_0 ($\Omega_{\rm PJ}^2 = \Omega_G^2 = s_0 \nu G_y$).

The density at which $\Omega_{\rm PJ}^2$ begins to deviate significantly from $\Omega_{\rm P}^2$ is around $s_0 \approx 1$. This shows that s_0 *plays a role that is just as significant as* α *and* β *in driving the instability*, since $\alpha \approx 1$ and $\beta \approx 1$ correspond to similar thresholds for the importance of magnetic fields and cosmic rays.

Here we define a critical value of the density parameter to be the value of s_0 where the squared Parker-Jeans growth rate equals twice the pure Parker growth rate:

$$\Omega_{\rm PJ}^2(s_{\rm crit}) = 2\Omega_{\rm PJ}^2(s_0 = 0) = 2\Omega_{\rm P}^2. \qquad (68)$$

For $s_0 > s_{\rm crit}$, self-gravity nearly dominates the instability, while for $s_0 < s_{\rm crit}$, the convective and magnetic-Rayleigh-Taylor forces are most important. From Figure 9, we obtain

$$s_{\rm crit} = \begin{Bmatrix} 5.9 \\ 3.0 \end{Bmatrix} \text{ for } (\alpha, \beta, \gamma) = \begin{Bmatrix} 1, 1, 0.8 \\ 0.2, 0.2, 0.8 \end{Bmatrix}. \qquad (69)$$

When $\gamma = 1$, the corresponding values of $s_{\rm crit}$ are 5.7 and 2.2, which are not very different from the $\gamma = 0.8$ case.

The horizontal wavenumbers, ν, at peak growth (maximized over ν and ζ) are also shown in Figure 9 as functions of s_0. The wavenumber gets larger as s_0 increases (just as the Jeans length gets shorter as density increases). The dip in ν (MAX.GROWTH) for $(\alpha, \beta, \gamma) = (0.2, 0.2, 0.8)$ occurs because for small α, β, and s_0, the convective instability is relatively important, and this instability (as well as the Parker instability for small α and β) peaks at large ν (see Fig. 6). When gravity becomes influential, the wavenumber at maximum growth decreases to the self-gravitational value (i.e., self-gravity makes a long wavelength instability grow faster than the shorter wavelength instabilities driven by convection or weak Parker type forces). This self-gravitational value of ν increases with s_0, so the initial decrease in ν at small (α, β) eventually turns around and gives an increase in ν for larger s_0.

c) *Numerical Examples*

The dimensionless parameters Ω, ν, ζ, and s_0 are normalized in terms of the scale height for the gas, H, and the one-dimensional rms velocity dispersion in the interstellar medium, u. We choose $H = 160$ pc and $u = 7$ km s^{-1}. Then the growth time is

$$\tau = \frac{H}{u\Omega} = \frac{2.24 \times 10^7 \text{ yr}}{\Omega}, \qquad (70)$$

the horizontal and vertical wavelengths are, respectively,

$$\lambda_y = \frac{2\pi H}{\nu} = \frac{1.01}{\nu} \text{ kpc} \qquad (71)$$

$$\lambda_z = \frac{2\pi H}{\zeta} = \frac{1.01}{\zeta} \text{ kpc}, \qquad (72)$$

and the physical mass density in the midplane is

$$\rho_0 = \frac{u^2 s_0}{8\pi G H^2} = 1.20 \times 10^{-24} s_0 \text{ gm cm}^{-3}. \qquad (73)$$

Dividing this mass density by a mean atomic weight of 2.2×10^{-24} gm, the corresponding space density at the galactic midplane is

$$n_0 = 0.55 s_0 \text{ cm}^{-3}. \qquad (74)$$

Since self-gravity becomes important compared to the Parker effect when s_0 is around 2 to 6 (eq. [69]), an

TABLE 1

GROWTH TIMES AND WAVELENGTHS OF THE DOMINANT MODES IN THE PARKER-JEANS INSTABILITY

n_0 (cm^{-3})=	0	1	2	5	10	20
(α, β, γ)=	Growth Times ($\times 10^6$ yr)					
(0, 0, 1.0)	∞	62	34	16	9.6	5.9
(0.05, 0.05, 0.8) ...	53	49	29	15	8.8	5.5
(0.2, 0.2, 0.8)	43	39	27	14	8.6	5.4
(1.0, 1.0, 0.8)	22	21	19	12	8.0	5.2
(0.2, 1.0, 0.8)	30	20	18	12	7.9	5.1
(1.0, 0.2, 0.8)	36	32	25	14	8.5	5.3
(1.0, 1.0, 1.0)	25	23	22	13	8.6	5.4
(α, β, γ)=	Horizontal Wavelengths (kpc)					
(0, 0, 1.0)	...	3.18	1.84	0.98	0.63	0.43
(0.05, 0.05, 0.8) ...	1.03	2.00	1.39	0.79	0.53	0.37
(0.2, 0.2, 0.8)	1.21	1.26	1.26	0.77	0.52	0.36
(1.0, 1.0, 0.8)	0.99	0.98	0.93	0.70	0.50	0.35
(0.2, 1.0, 0.8)	0.65	0.66	0.69	0.62	0.46	0.33
(1.0, 0.2, 0.8)	1.41	1.40	1.22	0.77	0.53	0.37
(1.0, 1.0, 1.0)	1.14	1.13	1.11	0.82	0.58	0.40
(α, β, γ)=	Vertical Wavelengths (kpc)					
(0, 0, 1.0)	...	1.24	0.97	0.84	0.56	0.45
(0.05, 0.05, 0.8) ...	∞	1.23	0.87	0.62	0.49	0.40
(0.2, 0.2, 0.8)	∞	∞	0.90	0.61	0.48	0.38
(1.0, 1.0, 0.8)	∞	∞	1.50	0.65	0.48	0.38
(0.2, 1.0, 0.8)	∞	∞	1.14	0.56	0.43	0.34
(1.0, 0.2, 0.8)	∞	∞	1.08	0.64	0.50	0.39
(1.0, 1.0, 1.0)	∞	∞	1.86	0.71	0.52	0.41

interstellar gas layer with a density larger than 1 to 3 cm^{-3} (depending on α, β, and γ) will become unstable much faster than what would be determined by the Parker instability alone. This is the range of the mid-plane density when self-gravitational forces begin to dominate over magnetic and cosmic-ray forces during cloud formation (for the assumed range of α and β).

Table 1 lists the unstable growth times of the fastest-growing modes, and the wavelengths of these modes, for various n_0, α, β, and γ.

The free fall time over one scale height perpendicular to the galactic plane is $(\pi/2)[H/g(H)]^{1/2} = 18.7$ million years. Table 1 shows that the self-gravity of the gas makes the perturbation growth time less than the free-fall time for $n \gtrsim 2$ to 5 cm^{-3}, a result that is almost independent of α, β, and γ.

The classical Jeans growth time has the maximum value of $(4\pi G\rho)^{-1/2} = 23.3/n_0^{1/2}$ million years for density n_0. Our maximum growth times approach this value for large n_0 because, of course, self-gravity begins to dominate the other forces, but also because λ_z becomes comparable to or less than $2H$ for large n_0. Thus, the perturbation is nearly homogeneous for large n_0, so the classical Jeans analysis begins to apply. This convergence of our results to the classical Jeans result at large n_0 is a welcome sign that our procedures are valid for evaluating the exact, self-gravitational force in an exponential layer and for applying this force to obtain a dispersion relation.

VII. SUMMARY

The purpose of this paper was to include self-gravitational, magnetic, and cosmic-ray forces in a single theory for the large-scale stability of the interstellar medium. The result was a dispersion relation (eq. [60]) for small amplitude waves having the combined restoring forces. Explicit solutions to this equation were presented in both analytical (eqs. [62]–[64]) and graphical (Figures 7, 8 and 9) form.

The results showed that the relative importance of self-gravitational forces on the Parker (1966) instability can be measured by the dimensionless density parameter, s_0. For a scale height of 160 pc, and an unperturbed sound speed of 7 km s^{-1}, the physical space density in the midplane, n_0, equals $0.55 s_0$. The dimensionless density parameter measures the effect of self-gravitational forces in about the same way as the dimensionless magnetic field strength, α, and cosmic ray pressure, β, measure the dimensionless driving forces in the pure Parker instability. When s_0 approximately equals or exceeds $\alpha+\beta$ for $\alpha\sim\beta$, the self-gravitational forces become important or dominant compared to the magnetic and cosmic ray forces.

The growth times for the Parker-Jeans instability can be much less than the free fall time perpendicular to the galactic plane if n_0 exceeds $\sim 5\ \mathrm{cm}^{-3}$ for typical values of $\alpha\sim\beta\sim 1$. This rapid growth signifies the importance of the Parker-Jeans instability as a mechanism for forming giant cloud complexes.

APPENDIX A

THE GRAVITATIONAL ACCELERATION FROM A PERTURBATION IN DENSITY

Here we determine the gravitational acceleration directed toward a density perturbation in an exponential gas layer. Consider a generalized density perturbation of the form

$$\delta\rho(y,z)=\delta\rho_0 e^{ik(y-y_0)}e^{-|z|/L}\cos k_\perp z. \tag{A1}$$

This is symmetric around the midplane, and it corresponds to the density perturbation in the Parker instability if $L=2H$ (see § III). The acceleration in the y- or z-directions resulting from this perturbation, and measured at a point $y=Y$ and $z=Z$, may be obtained by integrating over infinitesimally thin strips that lie in the x-direction. The acceleration directed toward each strip is

$$\frac{2G\,\delta\rho\,dy\,dz}{\left[(y-Y)^2+(z-Z)^2\right]^{1/2}}. \tag{A2}$$

The fraction of this total acceleration that is in the y-direction is $(y-Y)/[(y-Y)^2+(z-Z)^2]^{1/2}$. Thus the acceleration at (Y,Z) in the y-direction is

$$\delta g_{\mathrm{gas},\,y}(Y,Z)=2G\,\delta\rho_0\int_0^\infty e^{-z/L}\cos k_\perp z\,dz\int_{-\infty}^{\infty}\frac{\cos k(y-y_0)(y-Y)}{\left[(y-Y)^2+(z-Z)^2\right]}dy$$
$$+2G\,\delta\rho_0\int_0^\infty e^{-z/L}\cos k_\perp z\,dz\int_{-\infty}^{\infty}\frac{\cos k(y-y_0)(y-Y)}{\left[(y-Y)^2+(z+Z)^2\right]}dy. \tag{A3}$$

These integrals may be solved by substitution of $\xi=(z-Z)/L$ in the first one and $\xi=(z+Z)/L$ in the second one, and by substituting $x=(y-Y)/L$ in both of them. The resulting term in $\cos k(xL+Y-y_0)$ can be expanded into

$$\cos k(Y-y_0)\cos kxL-\sin k(Y-y_0)\sin kxL, \tag{A4}$$

and the integral containing $\cos(kxL)\,dx$ goes to zero. The remaining term in $\sin kxL$ integrates to

$$\int_{-\infty}^{\infty}\frac{x\sin(kxL)}{x^2+\xi^2}dx=\pi e^{-kL|\xi|}. \tag{A5}$$

The acceleration becomes

$$\delta g_{\mathrm{gas},\,y}(Y,Z)=-2\pi G\,\delta\rho_0 L\sin k(Y-y_0)\Re(I_y) \tag{A6}$$

where $\Re(I_y)$ is the real part of the integrals:

$$I_y=\int_0^\infty \exp(-\xi+ik_\perp L\xi+ik_\perp Z-kL\xi-Z/L)\,d\xi+\int_0^{Z/L}\exp(\xi-ik_\perp L\xi+ik_\perp Z-kL\xi-Z/L)\,d\xi$$
$$+\int_{Z/L}^\infty \exp(-\xi+ik_\perp L\xi-ik_\perp Z-kL\xi+Z/L)\,d\xi. \tag{A7}$$

This equation may be simplified by direct integration of the exponentials, and by selection of the real part.

The result is

$$\delta g_{\mathrm{gas},\,y}(Y,Z)=4\pi i G\,\delta\rho L\Gamma_y. \tag{A8}$$

Similarly, for the z component of the acceleration,

$$\delta g_{\mathrm{gas},\,z}(Y,Z)=4\pi G\,\delta\rho L\Gamma_z. \tag{A9}$$

The Γ terms are

$$\Gamma_y=\frac{1}{2}\left[A_+ + A_-\frac{e^{-|z|(kL-1)/L}}{\cos(k_\perp z)}-B_- k_\perp L\tan(k_\perp|z|)\right] \tag{A10}$$

$$\Gamma_z=\frac{1}{2}\left[A_- - A_-\frac{e^{-|z|(kL-1)/L}}{\cos(k_\perp z)}-B_+ k_\perp L\tan(k_\perp|z|)\right], \tag{A11}$$

where

$$A_\pm=\frac{kL+1}{(kL+1)^2+(k_\perp L)^2}\pm\frac{kL-1}{(kL-1)^2+(k_\perp L)^2} \tag{A12}$$

$$B_\pm=\frac{1}{(kL+1)^2+(k_\perp L)^2}\pm\frac{1}{(kL-1)^2+(k_\perp L)^2}. \tag{A13}$$

APPENDIX B

REDUCTION OF THE PERTURBED, SELF-GRAVITATIONAL ACCELERATION FROM THE GAS, δg_{gas}, TO TERMS INVOLVING ONLY δA

We begin with the equations of motion and the continuity equations, integrate these equations over the height of a perturbation, and then solve for the perturbed density parameter, $\delta\rho$, as a function of the perturbed magnetic vector potential, δA. We show in detail how the dispersion relation for the Parker-Jeans instability, equation (44), may be derived.

Consider the equation of mass continuity (18):

$$\frac{\partial}{\partial t}\delta\rho=v_z\frac{\rho}{H}-\rho\left(\frac{\partial}{\partial y}v_y+\frac{\partial}{\partial z}v_z\right). \tag{B1}$$

Now, introduce explicit variations of the form (cf. eqs [20], [34], and [35]):

$$\delta\rho=\delta\rho_0\exp(\Omega tu/H+i\nu y/H-|z|/L)\cos(\zeta z/H) \tag{B2}$$

$$v_y=-iv_{y0}\exp(\Omega tu/H+i\nu y/H+|z|/L)\cos(\zeta z/H) \tag{B3}$$

$$v_z=v_{z0}\exp(\Omega tu/H+i\nu y/H+|z|/L)\sin(\zeta z/H), \tag{B4}$$

where $\delta\rho_0$, v_{y0}, and v_{z0} are constants. These perturbations give a symmetric density and flow pattern around the midplane. When substituted into the pure Parker instability, they give a magnetic vector potential that is sinusoidal in the z-direction (cf. eq. [31]):

$$\delta A=\delta A_0\exp(\Omega tu/H+i\nu y/H)\sin(\zeta z/H). \tag{B5}$$

We introduce the dimensionless vector potential

$$a=\frac{\delta A}{HB}=\frac{\delta A_0}{HB_0}\exp(\Omega tu/H+i\nu y/H+|z|/L)\sin(\zeta z/H). \tag{B6}$$

Then, using equation (13), which is conveniently written as

$$\frac{v_z}{u}=-\Omega a, \tag{B7}$$

the mass continuity equation becomes

$$\Omega\,\delta\rho_0 E\cos(\zeta z/H)=-\nu\rho_0\frac{v_{y0}}{u}E\cos(\zeta z/H)-\frac{\Omega}{2}\rho_0 aE\sin(\zeta z/H)+\zeta\rho_0 aE\cos(\zeta z/H). \tag{B8}$$

Here we have used the fact that $L\equiv 2H$ and $\rho=\rho_0 e^{-z/H}$. The term E is an abbreviation for the exponential,

$$E\equiv\exp(\Omega tu/H+i\nu y/H-|z|/L). \tag{B9}$$

Now we integrate equation (B8) over z from $z=0$ to $z=\pi/k_\perp$, which is one-half of a wavelength from the plane. The result is simply

$$\frac{\delta\rho_0}{\rho_0}=\frac{-\nu}{i\Omega}\frac{v_{y0}}{u}. \tag{B10}$$

This result states that the mass column density in the perturbed layer near the midplane increases only because of the convergent flow along the y-direction.

Now consider the perturbed equation of motion in the y-direction; i.e., parallel to the plane (eq. [16]):

$$\rho\frac{\partial}{\partial t}v_y=-\frac{\partial}{\partial y}(\delta\rho+\delta\rho_{CR})+\left(\frac{\partial}{\partial y}\delta A\right)\frac{B}{8\pi H}+\rho\,\delta g_{gas,\,y}. \tag{B11}$$

We eliminate δP and δP_{CR} using equations (19) and (14), set

$$\delta g_{gas,\,y}=4\pi iG\,\delta\rho L\Gamma_y \tag{B12}$$

from equation (A8), and then use equations (3) and (4) to eliminate P and B in favor of ρ and u. Then we use variations of the form (B2) to (B4) to obtain

$$\rho_0\frac{v_{y0}}{u}\left(1+\frac{\gamma\nu^2}{\Omega^2}\right)E\cos(\zeta z/H)=\rho_0\frac{a\nu}{i\Omega}\left[q+\frac{\gamma}{2}\right]E\sin(\zeta z/H)$$
$$-\frac{\rho_0\zeta\nu a\gamma}{i\Omega}E\cos(\zeta z/H)+\frac{s_0\delta\rho_0\Gamma_y}{i\Omega\rho_0}Ee^{-|z|/H}\cos(\zeta z/H), \tag{B13}$$

where the dimensionless density is

$$s_0=\frac{4\pi G\rho_0 LH}{u^2}. \tag{B14}$$

This equation preserves all the z dependence of the original perturbed equation of motion in the y-direction. Now we integrate equation (B13) over the height of the perturbation to obtain, after some cancellation,

$$\frac{v_{y0}}{u}\left(1+\frac{\gamma\nu^2}{\Omega^2}\right)=\frac{2\zeta a\nu q}{i\Omega}+\frac{s_0\,\delta\rho_0 G_y}{i\Omega\rho_0}. \tag{B15}$$

We digress for a moment to define the G functions. Integrating equation (B13) over z leads to a useful function

$$G_y = \frac{\int_0^{\pi/k_\perp} \Gamma_y e^{-|z|/H} E \cos(\zeta z/H)\, dz}{\int_0^{\pi/k_\perp} E \cos(\zeta z/H)\, dz}. \tag{B16}$$

Later we shall use the equivalent function for the z-component of the equation of motion, so we introduce it here:

$$G_z = \frac{\int_0^{\pi/k_\perp} \Gamma_z e^{-|z|/H} E \cos(\zeta z/H)\, dz}{\int_0^{\pi/k_\perp} E \cos(\zeta z/H)\, dz}. \tag{B17}$$

These G functions are solved to be

$$G_{y,z}(\nu,\zeta) = \left[\!\left[\left(\frac{3}{2} A_{\pm} - 2B_{\mp}\zeta^2 \right) \left[\frac{1+\exp\left(-\frac{3\pi}{2\zeta}\right)}{9+4\zeta^2} \right] \pm \frac{A_-}{4} \left\{ \frac{1-\exp\left[-\frac{\pi}{\zeta}(1+\nu)\right]}{1+\nu} \right\} \right]\!\right] \cdot \left[\frac{1+4\zeta^2}{1+\exp\left(-\frac{\pi}{2\zeta}\right)} \right], \tag{B18}$$

where $A_{\pm}$ and $B_{\pm}$ were defined in equations (A12) and (A13). Here the y component of G is evaluated using the top symbol of a $\pm$ or $\mp$ combination, and the G_z component uses the bottom symbol.

Now we return to equation (B15). We substitute into this equation the value for $\delta\rho_0/\rho_0$ found in equation (B10) to obtain

$$\frac{v_{y0}}{u}\left(1+\frac{\gamma\nu^2}{\Omega^2}-\frac{s_0\nu G_y}{\Omega^2}\right)=\frac{2\zeta a\nu q}{i\Omega}. \tag{B19}$$

Thus

$$\frac{\delta\rho_0}{\rho_0}=\frac{2\zeta a\nu^2 q}{\Omega^2+\gamma\nu^2-s_0\nu G_y}. \tag{B20}$$

This value of $\delta\rho_0/\rho_0$ shall be used momentarily to convert δg to δA in the Parker-Jeans equation (29).

Finally we consider the z-component of the perturbed equation of motion; this component was used to obtain equation (29). Again, v_z in the z-component equation of motion is replaced by $-i\omega\delta A/B_0$, using equation (13), $\delta\rho$ is replaced by its expression from the continuity equation *before* integration over z (i.e., eq. [B8]), and v_y is replaced by its expression from the equation of motion in the y-direction *before* integrating over z. After some reduction, we obtain equation (29) again

$$\rho\left(\frac{H}{B}P_1\frac{\partial^2}{\partial z^2}\delta A+\frac{P_2}{HB}\delta A\right)=\frac{H\rho}{u^2}\left(\delta g_{\mathrm{gas},z}\Omega^2+i\nu q\,\delta g_{\mathrm{gas},y}\right). \tag{B21}$$

This time we have preserved all of the z dependences of the original equation of motion; i.e., ρ has not been cancelled from each side, nor have we multiplied each side by B to obtain the form given earlier (eq. [29]), which was useful then because its left-hand side was equivalent to Parker's (1966) equation. Equation (B21) is the normalized equation of motion in the z-direction with time derivatives replaced by $\Omega u/H$, and $\partial/\partial y$ replaced by $i\nu/H$, and with $\delta\rho$, v_y, etc. substituted from other equations, preserving their explicit z-dependences.

Now we substitute spatial variations of the form given by equations (B2)–(B4) to obtain:

$$\left(-\zeta^2P_1+P_2\right)\rho_0 aE\sin(\zeta z/H)=s_0\delta\rho_0\left(\Omega^2\Gamma_z-\nu q\Gamma_y\right)e^{-|z|/H}E\cos(\zeta z/H). \tag{B22}$$

One of the reasons for our integration over z becomes evident from this expression (B22): the Parker type forces (on the left-hand side) peak where the field curvature is greatest, and this occurs at $z=\pi/(2k_\perp)$. The self-gravitational forces (on the right) peak where the mass density is largest, and this occurs at $z=0$. The *mean* driving force over the entire height of the perturbation is of greatest interest in determining the growth rate for the perturbation. Our integration over height gives just this mean force.

We integrate equation (B22) over z from $z=0$ to $z=\pi/k_\perp$ and substitute $\delta\rho_0/\rho_0$ from equation (B20) to obtain

$$-\zeta^2 P_1 + P_2 = \frac{s_0 \nu^2 q\left(\Omega^2 G_z - \nu q G_y\right)}{\Omega^2+\gamma\nu^2 - s_0 \nu G_y}. \tag{B23}$$

This is the result stated earlier (eq. [44]).

APPENDIX C

GROWTH RATES FOR THE PARKER INSTABILITY WITHOUT SELF-GRAVITY ($s_0=0$)

Here we write an explicit form for the two solutions of the equation

$$-\zeta^2 P_1 + P_2 = 0, \tag{C1}$$

originally derived by Parker (1966). The polynomials P_1 and P_2 were given previously in equations (24) and (25); since they are second order in Ω^2, we write

$$-\zeta^2 P_1 + P_2 = -\left(\Omega^2-\Omega_{\rm P}^2\right)\left(\Omega^2-\Omega_0^2\right). \tag{C2}$$

There are two solutions to equation (C1), $\Omega_{\rm P}^2$ and Ω_0^2. We find after some algebraic reduction that these solutions are given by the equations

$$\Omega_{\rm P,0}^2 = -a\left(\nu^2+\zeta^2+1/4\right) \pm b\left[\left(\nu^2-\nu_1^2\right)\left(\nu^2-\nu_2^2\right)\right]^{1/2}, \tag{C3}$$

$$a=\alpha+\frac{\gamma}{2}, \tag{C4}$$

$$b=\alpha-\frac{\gamma}{2}, \tag{C5}$$

and where $\Omega_{\rm P}^2$ corresponds to the + sign and Ω_0^2 to the − sign. The wavenumbers ν_1 and ν_2 depend only on α, β, γ, and ζ, and are given by

$$\nu_{1,2}^2 = \frac{1}{(2c-2)}\left\{-\left[\frac{c}{2}+d+\zeta^2(2c-1)\right] \pm \left[\left(\zeta^2-\zeta_1^2\right)\left(\zeta^2-\zeta_2^2\right)\right]^{1/2}\right\}; \tag{C6}$$

the subscripts 1 and 2 correspond to the + and − signs. The wavenumbers ζ_1 and ζ_2 depend only on α, β, and γ and are given by

$$\zeta_{1,2}^2 = d - \frac{c}{2}(1+4d) \pm \frac{1}{2}(1+4d)\left[c(c-1)\right]^{1/2}. \tag{C7}$$

The constants are

$$c=\frac{a^2}{2\alpha\gamma} \tag{C8}$$

$$d=\frac{r}{2\alpha\gamma}, \tag{C9}$$

where r was defined in equation (27); d is the only term that contains the cosmic ray pressure, β. The constants a, b, c, d, and r depend only on α, β, and γ.

For constant ζ, the growth rate peaks at a value of ν found by setting $d\Omega_P^2/d\nu=0$ in equation (C3), solving for ν, and then using this ν back in the equation to obtain Ω^2. The result is

$$\Omega_P^2(\text{MAX. GROWTH})=-a\left[\frac{1}{2}\left(\nu_1^2+\nu_2^2\right)+\zeta^2+1/4\right]-\left(\nu_1^2-\nu_2^2\right)\left(\frac{\alpha\gamma}{2}\right)^{1/2} \tag{C10}$$

and the corresponding wavenumber squared is

$$\nu^2(\text{MAX. GROWTH})=\tfrac{1}{2}\left(\nu_1^2+\nu_2^2\right)+a\left(\nu_1^2-\nu_2^2\right)/(8\alpha\gamma)^{1/2}. \tag{C11}$$

The critical wavenumber for instability is found by setting $\Omega_P^2=0$ in equation (C3) and solving for ν. It is

$$\nu_{\text{crit}}=\left(d-\zeta^2\right)^{1/2}. \tag{C12}$$

The perturbation is unstable if $\nu<\nu_{\text{crit}}$, as shown in Figure 1. Instability also requires

$$\zeta<\zeta_{\text{crit}}\equiv d^{1/2}. \tag{C13}$$

Perturbations that are too small in the z-direction are stable.

The usual criterion for the Parker instability, as given by Parker (1966), for example, is that the interstellar medium is unstable if

$$d>0. \tag{C14}$$

Since $d=r/(2\alpha\gamma)$, this is the same as

$$r=(1+\alpha+\beta)(1+\alpha+\beta-\gamma)-\alpha\gamma/2>0 \tag{C15}$$

(the term r defined here is the same as the Y term in Parker 1966, Appendix III). Parker and others write this instability criterion as an upper limit to γ:

$$\gamma<\frac{(1+\alpha+\beta)}{(1+1.5\alpha+\beta)}. \tag{C16}$$

Equation C16 is the same as requiring $d>0$ or $\zeta_{\text{crit}}^2>0$.

APPENDIX D

GROWTH RATES FOR THE PURE GRAVITATIONAL INSTABILITY ($\alpha=\beta=0$) AND PURE CONVECTIVE INSTABILITY ($\alpha=\beta=s_0=0, \gamma<1$)

I. GRAVITATIONAL INSTABILITY

The equations of motion and continuity from § III can be reduced to a single equation for v_z and its z-derivatives in the same manner as it was reduced to an equation for δA and its derivatives. When $B_0\neq 0$, these two reductions give identical dispersion relations for perturbations of the assumed form. When $B_0=0$, δA is undefined so the equations must be written in terms of v_z. In this way, we obtain the following z-integrated dispersion relation for a model interstellar medium with no magnetic fields or cosmic ray pressure ($\alpha=\beta=0$):

$$-\zeta^2P_1'+P_2'=\frac{s_0\nu^2(1-\gamma)}{\Omega^2+\gamma\nu^2-s_0\nu G_y}\left[\Omega^2G_z-\nu(1-\gamma)G_y\right]. \tag{D1}$$

This result is identical to the limit of the Parker-Jeans equation (44) at $\alpha=\beta=0$. The coefficients P_1' and P_2' are the same as the limits of our previously defined P_1 and P_2 (eqs. [24] and [25]) except that the fraction 1/4 in P_2 should be replaced by $(H/L)^2$ for perturbation scale height L and unperturbed scale height H. The ratio L/H need not be 2 when $B_0=0$; it can be 1, for example. Thus we write

$$P_1'=\Omega^2\gamma \tag{D2}$$

$$P_2'=-\Omega^4-\Omega^2\gamma\left[\nu^2+(H/L)^2\right]+\nu^2(1-\gamma). \tag{D3}$$

The left-hand side of equation (D1) may be reduced to the same form as before,

$$-\zeta^2P_1+P_2=-\left(\Omega^2-\Omega_P^2\right)\left(\Omega^2-\Omega_0^2\right), \tag{D4}$$

and $\Omega_{P,0}^2$ is given again by equation (C3), but now $a=\gamma/2$ and $b=-\gamma/2$, as in the limit of equations (C4) and (C5) for $\alpha=0$. With $\alpha\to 0$, however, c and d from equations (C8) and (C9) *diverge*. The appropriate expression for $\nu_{1,2}^2$ when $\alpha=0$ is

$$\nu_{1,2}^2=-\left[\left(\frac{H}{L}\right)^2+\frac{d}{2c}+\zeta^2\right]\pm\left\{\left(\frac{d}{c}\right)\left[\left(\frac{H}{L}\right)^2+\frac{d}{4c}+\zeta^2\right]\right\}^{1/2} \tag{D5}$$

instead of equation (C6). The ratio d/c has the same value it did before in the limit of $\alpha=\beta=0$, namely

$$\frac{d}{c}=\frac{4}{\gamma^2}(1-\gamma). \tag{D6}$$

Now that Ω_P^2 and Ω_0^2 are defined for the case $\alpha=\beta=0$, we can rewrite equation (D1) in the form of equation (60) and then solve it for Ω^2 by equations (62) through (64).

II. PURE CONVECTIVE INSTABILITY

Without magnetic fields, cosmic ray pressure, or self-gravity, the linearized perturbation equations of motion become

$$\rho\frac{\partial}{\partial t}v_y=-\frac{\partial}{\partial y}\delta P \tag{D7}$$

$$\rho\frac{\partial}{\partial t}v_z=-\frac{\partial}{\partial z}\delta P-\delta\rho\frac{u^2}{H}. \tag{D8}$$

The continuity equations (18)–(19) remain unchanged. With perturbations of the form assumed elsewhere in this paper,

$$\delta\rho=\delta\rho_0\exp\left(\Omega ut/H+i\nu y/H+i\zeta z/H-z/L\right) \tag{D9}$$

$$v=v_0\exp\left(\Omega ut/H+i\nu y/H+i\zeta z/H+z/H-z/L\right), \tag{D10}$$

these perturbation equations can be reduced to the single dispersion equation:

$$-\zeta^2P_1'+P_2'=0, \tag{D11}$$

where P_1' and P_2' are given by equations (D2) and (D3).

Equation (D11) also may be obtained simply as the limit of equation (44) for $\alpha=\beta=s_0=0$. The solution for $\Omega^2(\nu,\zeta)$ is

$$\Omega_{\rm CONV}^2=-\frac{\gamma}{2}\left(\nu^2+\zeta^2+1/4\right)\pm\frac{\gamma}{2}\left[\left(\nu^2+\zeta^2+1/4\right)^2+\frac{4\nu^2(1-\gamma)}{\gamma^2}\right]^{1/2} \tag{D12}$$

if $L=2H$. The unstable mode corresponds to the + sign. This unstable solution is plotted in Figure 3. The maximum of Ω^2_{CONV} occurs at $\nu=\infty$; it has the value

$$\Omega^2_{CONV}\,(\mathrm{MAX.GROWTH})=\frac{1-\gamma}{\gamma}, \qquad \mathrm{(D13)}$$

as obtained previously by Dufouw (1970).

REFERENCES

Abramowitz, M. and Stegun, I. A. 1970, in *Handbook of Mathematical Functions* (New York: Dover), p. 17.
Asseo, E., Cesarsky, C. J., Lachieze-Ray, M., and Pellat, R. 1978, *Ap. J. (Letters)*, **225**, L21.
Baker, P. L., and Burton, W. B. 1975, *Ap. J.*, **198**, 281.
Bohlin, R. C., Savage, B. D., and Drake, J. F. 1978, *Ap. J.*, **224**, 132.
Cowie, L. L. 1980, *Ap. J.*, **236**, 862.
Dufouw, R. J. 1970, Ph.D. thesis, California Institute of Technology.
Elmegreen, B. G. 1979, *Ap. J.*, **231**, 372.
______. 1981, in *Formation of the Planetary Systems*, ed. A. Brahic, (Toulouse: Cepadues Editions), in press.
______. 1982, *Ap. J.*, **253**, 655.
Falgarone, E., and Lequeux, J. 1973, *Astr. Ap.*, **25**, 253.
Field, G. B. 1965, *Ap. J.*, **142**, 531.
Fujimoto, M. 1968, *Ap. J.*, **152**, 391.
Goldreich, P., and Lynden-Bell, D. 1965, *M.N.R.A.S.*, **130**, 7.
Jenkins, E. B., and Savage, B. D. 1974, *Ap. J.*, **187**, 243.
Ledoux, P. 1951, *Ann. d'Ap.*, **14**, 438.
Lerche, I. 1967*a*, *Ap. J.*, **148**, 415.
Lerche, I. 1967*b*, *Ap. J.*, **149**, 553.
Manchester, R. N. 1974, *Ap. J.*, **188**, 637.
Meyer, P. 1969, *Ann. Rev. Astr. Ap.*, **7**, 1.
Mouschovias, T. Ch. 1974, *Ap. J.*, **192**, 37.
Myers, P. 1978, *Ap. J.*, **225**, 380.
Oort, J. 1965, in *Stars and Stellar Systems*, Vol. **5**, *Galactic Structure*, ed. A. Blaauw and M. Schmidt (Chicago: University of Chicago Press), p. 455.
Parker, E. N. 1953, *Ap. J.*, **117**, 431.
______. 1966, *Ap. J.*, **145**, 811.
______. 1967, *Ap. J.*, **149**, 517.
Roberts, W. W. 1969, *Ap. J.*, **158**, 123.
Safronov, V. S. 1960, *Ann. d'Ap.*, **23**, 979.
Savage, B. D., and de Boer, K. S. 1979, *Ap. J. (Letters)*, **230**, L77.
Shu, F. 1974, *Astr. Ap.*, **33**, 55.
Spitzer, L., Jr. 1978, in *Physical Processes in the Interstellar Medium* (New York: Wiley).
Toomre, A. 1964, *Ap. J.*, **139**, 1217.
Zweibel, E. G., and Kulsrud, R. M. 1975, *Ap. J.*, **201**, 63.

BRUCE G. ELMEGREEN: Department of Astronomy, Columbia University, New York, NY 10027