When may an unstable gravitating disk be considered infinitely thin?

V. L. Polyachenko and A. M. Fridman

Institute of Terrestrial Magnetism, the Ionosphere, and Radio-Wave Propagation, Siberian Branch, USSR Academy of Sciences, Irkutsk,
and Astronomical Council, USSR Academy of Sciences, Moscow

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The model of an infinitely thin gravitating disk is admissible for purposes of stability analysis only if a massive halo is present. Constraints are imposed on the basic disk and halo parameters such that (*a*) the most highly unstable perturbation wavelengths will remain much greater than the disk thickness (the thin-disk approximation) and (*b*) the contribution of the halo to the perturbed gravitational potential may be neglected. Disk and halo density distributions with respect to thickness are derived.

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1. FORMULATION OF THE PROBLEM

It is no accident that interest has lately heightened in analyzing the stability of a gravitating disk. In many models of astrophysical objects a disk is the dominant feature. The flat component of a spiral galaxy, the late evolutionary phase of the protoplanetary cloud, Saturn's rings, and in the past few years accretion disks around compact masses, and pancake cosmologies—this but a partial list of the objects that have been represented in the form of gravitating disks.

In attempting to determine whether such an object is stable, as the most straightforward way of testing one of its possible routes of evolution, one is unavoidably faced with the need to simplify the model further. The least complicated and naturally the most popular method of doing this has been to consider the model of an infinitely thin disk—a disk whose thickness h is many times smaller than the wavelengths λ of the perturbations that develop ($h \ll \lambda$). However, Goldreich and Lynden-Bell[1] proved in 1965 that a gravitating disk will reach maximum instability (provided the temperature anisotropy $\alpha = T_\perp/T_\parallel$ is not too great, where $T_\perp$, $T_\parallel$ denote the temperature of the medium across and along the rotation axis, respectively) for wavelengths comparable with the disk thickness[1)] ($\lambda \sim h$). This last condition means that the infinitely thin model disk in its customary form is not suitable for the most unstable wavelengths—for the very wavelengths that build up at the fastest rate, so that their growth overtakes that of all other modes. Whether this circumstance compels us to analyze the stability only of model disks of finite thickness is a much more laborious question to resolve.[2]

Our aim in this letter is to identify the conditions such that the infinitely thin disk model may legitimately be accepted to test stability and to investigate the fundamental concomitant problems of how nonlinear density waves[3–9] and weak turbulence[10–12] will evolve. We are persuaded that these problems can be studied in the massive halo.

What relationships, then, should hold among the basic parameters of the two components (disk and halo) in order that we may treat the disk as infinitely thin in deciding whether it is stable?

2. VERTICAL DENSITY DISTRIBUTION OF LIGHT GASEOUS COMPONENT AND MASSIVE STELLAR HALO

For definiteness consider a gaseous disk embedded in a halo of stars. Assign subscript 0 to stationary quantities; let subscripts g and $*$ designate quantities referring to the gaseous and stellar components, respectively.

We shall assume that the stellar halo is far denser than the gas. Thus our first assumption

$$\rho_{0*}/\rho_{0g} \gg 1 \tag{1}$$

allows us to neglect the density of the gaseous component in the Poisson equation for stationary quantities. Further, we shall regard the system as so extended along the plane z = 0 that the gravitational potential varies much more steeply in the z direction than along the radius r; that is, $d^2\psi_0/dz^2 \gg d^2\psi_0/dr^2$. In view of this last condition the Poisson equation will become

$$\frac{d^2\Psi_0}{dz^2} = 4\pi G\rho_{0*}. \tag{2}$$

Along the z axis the stellar component will have the equation of equilibrium

$$\frac{d\Psi_0}{dz} = -\frac{1}{\rho_{0*}}\frac{dP_{0*}}{dz}. \tag{3}$$

For a barotropic stellar component

$$P_{0*} = P_{0*}(\rho_{0*}) \tag{4}$$

and the condition (3) may be written in the form

$$\frac{d\Psi_0}{dz} = -\frac{c_{\parallel*}^2}{\rho_{0*}}\frac{d\rho_{0*}}{dz}, \tag{5}$$

where $c_{\parallel*}^2 = dP_{0*}/d\rho_{0*}$ represents the square of the stellar velocity dispersion along the z axis. We shall henceforth assume that $c_{\parallel*}^2$ depends weakly on z (in comparison with the density ρ_{0*}), so that

$$\frac{1}{c_{\parallel*}^2}\frac{dc_{\parallel*}^2}{dz} \bigg/ \frac{1}{\rho_{0*}}\frac{d\rho_{0*}}{dz} \ll 1. \tag{6}$$

On differentiating Eq. (5) with respect to z and using the condition (6) as well as the Poisson equation (2), we obtain the Emden equation

$$u'' + be^u = 0. \tag{7}$$

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A prime here indicates differentiation with respect to z;

$$u(z) = \ln \rho_{0*},\quad b = 4\pi G/c_{\parallel *}^2. \tag{8}$$

We seek a solution to Eq. (7) in the form

$$u(z) = \ln [A/\mathrm{ch}^2 az] \tag{9}$$

[ch x ≡ cosh x], where the two constants A, a [Eq. (7) is a second-order equation] can be evaluated by substituting Eq. (9) into Eq. (7):

$$A = \rho_{0*}(0),\quad a = 1/h_* = \sqrt{2\pi G\rho_{0*}(0)}/c_*. \tag{10}$$

From Eqs. (8)-(10) we obtain the standard solution

$$\rho_{0*}(z) = \rho_{0*}(0)/\mathrm{ch}^2(z/h_*). \tag{11}$$

For a barotropic gaseous component

$$P_{0g} = P_{0g}(\rho_{0g})$$

and its equation of equilibrium along the z axis will be

$$\frac{d\Psi_0}{dz} = -\frac{c_g^2}{\rho_{0g}}\frac{d\rho_{0g}}{dz}, \tag{12}$$

where $c_g^2 = dP_{0g}/d\rho_{0g}$ is the squared velocity of sound in the gas. Equations (5), (12) have equal left-hand members, so we may equate the right-hand sides to obtain

$$\rho_{0g}(z)/\rho_{0g}(0) = [\rho_{0*}(z)/\rho_{0*}(0)]^{c_{\parallel *}^2/c_g^2}, \tag{13}$$

or, with Eq. (11),

$$\rho_{0g}(z)/\rho_{0g}(0) = [\mathrm{ch}^2(z/h_*)]^{-c_{\parallel *}^2/c_g^2}. \tag{14}$$

In the special case $c_{\parallel *}^2 = c_g^2$ the density distribution of the gas will exactly replicate that of the stars. Ordinarily the stellar halo will have a higher temperature than the gas, so that $c_{\parallel *}^2 > c_g^2$. In this event the gaseous disk will have a smaller characteristic thickness h_g than the stellar component: $h_g < h_*$ [see Eq. (23) below].

3. AN ISOLATED GASEOUS DISK CANNOT BE CONSIDERED INFINITELY THIN (IN SENSE $k_0 h \ll 1$)

Let us now see why in the absence of a massive stellar halo the most highly unstable modes in a gaseous disk (or even in a stellar disk, if the temperature anisotropy is not too great) will correspond to wavelengths comparable with the disk thickness ($\lambda \sim h_g$). The most unstable mode k_0 is determined by the condition $\partial\omega^2/\partial k = 0$ that the dispersion curve reach a minimum (see Chapter V, Sec. 2.2 of the Russian edition of our book[3]), where $\omega^2 = \varkappa^2 - 2\pi G\sigma_{0g}k + k^2c_g^2$,

$$k_0 = \pi G\sigma_{0g}/c_g^2, \tag{15}$$

$$\sigma_{0g} = \int_{-\infty}^{\infty} \rho_{0g}(z)\,dz, \tag{16}$$

and the other symbols have their customary meaning.[3]

In the event no stellar halo is present, the potential will be determined solely by the gaseous component, so that, by analogy to Eq. (11),

$$\rho_{0g}(z)/\rho_{0g}(0) = \mathrm{ch}^{-2}(z/h_g), \tag{17}$$

where

$$h_g = c_g/\sqrt{2\pi G\rho_{0g}(0)}. \tag{18}$$

Substituting Eq. (17) into Eq. (16) we obtain

$$\sigma_{0g} = 2h_g\rho_{0g}(0). \tag{19}$$

Equations (15), (18), (19) now yield

$$k_0 h_g = 1. \tag{20}$$

This last relation demonstrates that the approximation of an infinitely thin disk will not be admissible in the neighborhood of the wave vector $k = k_0$, which corresponds to the most highly unstable mode.

4. HOW WILL A STELLAR HALO CHANGE MATTERS?

If the disk is surrounded by a halo of stars, we will have a different state of affairs. To determine the surface density of the gaseous disk in this case, we use Eq. (14):

$$\sigma_{0g} = \rho_{0g}(0)\,h_* \int_{-\infty}^{\infty} \mathrm{ch}^{-2\nu} x\,dx,\quad x = z/h_*,\quad \nu = c_{\parallel *}^2/c_g^2. \tag{21}$$

The integral in Eq. (21) is readily evaluated from the standard expression[13]

$$\int_{-\infty}^{\infty} \mathrm{ch}^{-2\nu} x\,dx = B\left(\tfrac{1}{2},\nu\right) = \Gamma\left(\tfrac{1}{2}\right)\Gamma(\nu)\,\Gamma^{-1}\left(\tfrac{1}{2}+\nu\right).$$

Indeed, if $\nu \gg 1$ we may use the asymptotic relation[14]

$$\Gamma(az+b) \sim \sqrt{2\pi}e^{-az}(az)^{az+b-1/2}$$

to obtain finally

$$\sigma_{0g} = \sqrt{\pi}\rho_{0g}(0)\,h_*\frac{c_g}{c_{\parallel *}}\ (\text{for } c_{\parallel *}^2/c_g^2 \gg 1). \tag{22}$$

It is evident from Eq. (22) that the gas disk will have a characteristic thickness[2)]

$$h_g \simeq h_*\frac{c_g}{c_{\parallel *}}. \tag{23}$$

With Eqs. (10), (15), (23) we now find[2] that

$$k_0 h_g = \frac{\sqrt{\pi}}{2}\frac{\rho_{0g}(0)}{\rho_{0*}(0)} \ll 1. \tag{24}$$

Thus if the stellar component is denser than the gaseous component, the stability of the gaseous disk may legitimately be investigated by regarding it as infinitely thin, provided a certain further condition holds. We proceed now to derive that condition.

5. FUNDAMENTAL THEOREM

To evaluate the inequality (24) let us employ Eq. (15), which has been obtained from the dispersion relation describing small oscillations of a gaseous disk (Chapter V, Sec. 2.2 of our book[3]) in the absence of effects from a stellar component. We have therefore to derive the condition such that the contribution Ψ_1 of the stellar component to the disturbed gravitational potential may be neglected.

The ratios of the disturbed to the undisturbed surface density for a gaseous disk and for a disk of stars are given by

$$(\sigma_1/\sigma_0)_g = k^2\Psi_1/(\omega^2 - \varkappa^2 - k^2c^2)_g, \tag{25}$$

$$(\sigma_1/\sigma_0)_* = k^2\Psi_1 I_*/(\omega^2 - \varkappa^2 - k^2c_\perp{}^2)_*. \tag{26}$$

In accord with the inequality (24) we regard the gaseous disk as infinitely thin, so that the reduction factor I_*, representing a thickness correction, enters only for the stellar disk.

It is pertinent to point out here that although Toomre[15] and Shu[16] derived their reduction factors on the assumption that $|k|h < 1$, direct calculation confirms that the correct asymptotic behavior is obtained in the opposite limiting case $|k|h \gg 1$. In fact, using the relation[3]

$$\Psi_1 = -2\pi G\sigma_1/|k| \tag{27}$$

between Ψ_1 and σ_1, we find for $|k|h \gg 1$ that

$$I = 2/|k|h. \tag{28}$$

Substituting the expressions (27), (28) into Eq. (26) and recognizing that $\rho_0 = \sigma_0 h$, we obtain

$$\omega^2 = \varkappa^2 + k^2c_s{}^2 - \omega_0{}^2, \qquad \omega_0{}^2 = 4\pi G\rho_0. \tag{29}$$

Equation (29) is a dispersion relation describing small oscillations in a rotating, gravitating cylinder in a plane perpendicular to the generatrix as $|k|h \to \infty$ (see Bisnovatyi-Kogan et al.[17] and Chapter II of our book[3]).

In light of the arguments above, the reduction factor in Eq. (26) may be used in either of the two limiting cases: 1) $|k|h_* \ll 1$; 2) $|k|h_* \gg 1$.

Consider the first case ($|k|h_* \ll 1$). In the leading (zeroth) order with respect to $|k|h_*$ the reduction factors I_* of Shu[16] and Toomre[15] take the same value, $I_* = 1$. For a two-component medium of gas and stars, the disturbed potential and density will be related, according to Eq. (27), by

$$\Psi_1 = -\frac{2\pi G}{|k|}(\sigma_{1*} + \sigma_{1g}). \tag{30}$$

This equation shows that the contribution of the stellar component to the disturbed potential may be neglected if

$$\sigma_{1g} \gg \sigma_{1*} \tag{31}$$

or, using Eqs. (25), (26) with the value[18] $I_* = 1$, if the condition

$$\sigma_{0g}/c_g{}^2 \gg \sigma_{0*}/c^2_{\perp*} \tag{32}$$

holds. Introducing the anisotropy coefficient

$$\alpha = c^2_{\perp*}/c^2_{\parallel*} \tag{33}$$

of the stellar disk and making use of Eq. (23), we finally obtain the following condition such that the stellar component in the disturbed gravitational potential may be neglected:

$$1 \gg \frac{\rho_{0g}(0)}{\rho_{0*}(0)} \gg \frac{h_g}{\alpha h_*} \quad \text{for} \quad |k|h_* \ll 1. \tag{34}$$

In the opposite case ($|k|h_* \gg 1$), if we retain only the first term in the expansion of I_* with respect to $1/|k|h_*$, which is the same for both types of reduction factors,[15,16] we will have

$$I_* = 2/|k|h_*, \tag{35}$$

and the condition (31) will become

$$1 \gg \frac{\rho^2_{0g}(0)}{\rho^2_{0*}(0)} \gg \frac{h_g{}^2}{\alpha h_*{}^2} \quad \text{for} \quad |k|h_* \gg 1. \tag{36}$$

From the foregoing discussion we may infer the following

T h e o r e m. A necessary and sufficient condition for the approximation of an infinitely thin disk to exist in the domain of unstable wave vectors is the presence of a stellar halo with parameters satisfying the inequalities (34) and (36).

6. SCOPE OF THE MODEL

Clearly if the anisotropy $\alpha \gtrsim 1$, the inequalities (36) will follow automatically from the inequalities (34).

Even though we are here concerned with the legitimacy of the approximation of an infinitely thin g a s e o u s disk, the same problem also arises for a s t e l l a r disk in which the anisotropy of the stellar velocity dispersion is not too great.[3)] It is natural to resolve the issue in this case by the analogous requirement of a massive stellar halo and to introduce into the conditions (34), (36) the additional parameter α_1, which will aid in satisfying them if $\alpha_1 > 1$.

To illustrate, let us take the gaseous disk of the Galaxy. Since $\alpha > 1$ in the Galaxy ($\alpha \approx 3$), we shall use only the condition (34), which we rewrite as follows [the left-hand inequality (34) is satisfied; we are now interested only in the right-hand inequality]:

$$\sigma_{0g}/\sigma_{0*} \gg c_g{}^2/c^2_{*\perp}.$$

Recent data indicate[19,20] that $\sigma_{0g}/\sigma_{0*} \approx 4\% = 1/25$, $c_g \approx 7$ km/sec, $c_{*\perp} \approx 35$ km/sec; that is,

$$\sigma_{0g}/\sigma_{0*} \approx (c_g/c_{*\perp})^2 \approx 1/25.$$

Evidently, then, the condition (34) does not hold: the stellar component makes the same contribution as the gaseous component to the perturbed galactic gravitational potential.

It therefore is not legitimate to neglect the contribution of the stellar component to the perturbed gravitational potential and to represent the unstable gaseous disk of the Galaxy as being infinitely thin.

We leave it to the reader to convince himself that in certain other widely used models it is highly problematical whether the conditions (34) and (36) will be satisfied.

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[1)] Goldreich and Lynden-Bell gave this proof for the case where the disk $\omega^2 \approx 0$. Equation (20) of this letter proves their assertion in the general case.

[2)] The estimate (23) actually holds whatever value the parameter $\nu \equiv c_{\parallel*}{}^2/c_g{}^2$ may have. Equation (23) can be obtained directly from Eq. (14)

[Eqs. (14) and (21) hold for all values of ν]. If $x \equiv z/h_* < 1$, Eq. (14) implies that

$$\frac{1}{\mathrm{ch}^{2\nu}x} = \left(\frac{2}{e^x + e^{-x}}\right)^{2\nu} \approx (1 + x^2/2)^{-2\nu} \approx (e^{x^2/2})^{-2\nu} = e^{-\nu x^2}.$$

Thus for $x < 1$ we will have $\rho_g(x) \approx \rho_g(0)\,e^{-\nu x^2}$, showing that the characteristic thickness of the gas layer is determined by the condition $x \approx 1/\sqrt{\nu}$, which is equivalent to Eq. (23) [together with Eq. (21')].

3) One can readily see that for a stellar disk $k_0 h_{*1} \sim 1/\alpha_1 \ll 1$ if $\alpha_1 \gg 1$ (the quantities α_1, h_{*1} are not to be confused with α, h_* for a stellar halo).

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Nonlinear evolution of spherical stellar systems

V. L. Polyachenko

Institute of Terrestrial Magnetism, the Ionosphere, and Radio-Wave Propagation, Siberian Branch, USSR Academy of Sciences, Irkutsk

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A universal method is proposed for computer simulation of collisionless model systems conforming to a specified distribution function. The method is applied to show that the growth of instabilities in a spherically symmetric system whose members have nearly radial trajectories will impart to the system a distinctly ellipsoidal deformation.

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1. A modification of direct N-body methods will be applied in this letter to study the nonlinear evolutionary phase of an inhomogeneous model stellar system. A real system, such as a spherical distribution of stars described by the function $f_0(E, L)$ (where E, L denote the energy and angular momentum of a star), will be represented as a system of N mutually interacting points. Since N is necessarily bounded by some value N_{max} (in our case $N_{max} \lesssim 300$; for larger numbers of particles, other types of modeling may be used[1-3]) and in such a model system one ought not exaggerate the role of "collisions" when the particles undergo close encounters, we shall take the two-body interaction force to be Newton's law, which "cuts off" in the simplest fashion at short distances.

Usually the cutoff radius c is specified to be a few percent of the radius R of the system. Such a value for the cutoff clearly should not have much effect on the accuracy with which we can describe large-scale processes of interest to us (in particular, long-wave instability), with characteristic scales of order R. According to Newton's second law every particle in the model system will move under the forces exerted by all the other particles. If we stipulate the initial coordinates and velocities of the N particles and then solve the corresponding system of 3N second-order equations on a computer, we can, in principle, determine what the whole subsequent evolution of the model system will be.

In general outline that is the plan which, for example, Ostriker and Peebles[4] have followed in analyzing the instability of disk configurations for model galaxies. We shall take the same approach below. Our treatment dif-