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Translated by R. B. Rodman

The nonlinear instability of a Maclaurin disk of stars surrounded by a giant halo

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(Submitted December 17, 1979)
Astron. Zh. **58**, 524–527 (May–June 1981)

An analysis is given of the stability of a galaxy, modeled as a rotating Maclaurin disk, against nonradial "affine" perturbations in the presence of an extended, nonoscillating halo of uniform density. A criterion is obtained for linear stability of the disk. In the nonlinear case the criterion holds only up to a certain value of the critical angular rotational velocity; beyond that value the nonlinear stabilization effect gradually falls behind the linear stabilization. If the angular rotational velocity has its maximum possible value (no velocity dispersion), the disk will be unstable even when surrounded by a massive halo.

PACS numbers: 98.50.Cz

1. Research carried out in the past few years has shown that when studying the dynamics of giant spiral galaxies, one should take their spherical components into account. Among the flat model spiral galaxies in dynamical equilibrium, the phase model in the form of a circular Maclaurin disk is of special interest.[1] Several authors have investigated its linear stability.[2-4] Antonov and the author[5] have made a nonlinear analysis of the stability of such a model against perturbations that preserve the homogeneity of the corresponding space density, and have established that the criterion for stability of nonradial oscillations having an arbitrary amplitude takes exactly the same form as in the linear approximation.

It is natural to expect that a disk which itself is stable will remain so if it is embedded in a halo,[6] even in the nonlinear case. The latter conjecture will be substantiated below. But when a halo is introduced we may enter the domain of large angular rotational velocities for the disk, and our previous arguments and estimates[5] might no longer be entirely valid. We would point out that Kalnajs[2] has also discussed the oscillations of a disk accompanied by a halo, but he did not obtain an explicit analytic form for the linear stability criterion for nonradial oscillations. We shall here analyze the stability of a Maclaurin disk against nonradial perturbations in the presence of a nonoscillating (passive) spherical subsystem of uniform density; we take into account the distinction between the linear and nonlinear cases.

Since all the oscillations to be considered will take place in the plane of the disk itself, the gravitational potential of the surrounding halo may be expressed in the form $\Phi(x, y)=-(2\pi/3)G\rho_0 r^2$ (ρ_0=const, $r^2=x^2+y^2$), which excludes the z component. Thus the halo may take the shape of a sphere or an ellipsoid of revolution. The gravitational potential of the equilibrium Maclaurin disk in its own (x, y) plane will be $U(x, y) = -\pi^2 G\sigma_0 r^2/4R_0$, where σ_0 is the surface density at the center of the disk and R_0 is the disk radius. Introducing the dimensionless parameter $\mu=\Phi/U$ and the normalization $(4\pi/3)G\rho_0+\pi^2 G\sigma_0/2R_0=1$, we may write the gravitational potentials as

$$U(x,y)=-\frac{r^2}{2(1+\mu)},\quad \Phi(x,y)=-\frac{\mu r^2}{2(1+\mu)}. \qquad (1)$$

2. Nonradial oscillations will cause a circular disk to take on an elliptical shape in the disturbed state. Let $a(t)$, $b(t)$ denote the major and minor semiaxes of the disturbed disk, respectively; then provided the volume density remains homogeneous,[7] the surface density of the disk will be given by $\sigma(x, y, t) = \text{const}(1-x^2/a^2-y^2/b^2)^{1/2}$.

To establish a criterion for the linear stability of nonradial oscillations, we shall make use of the principle of varying the energy functional[15] $E=M\overline{v^2}/2+W_\mu+\Psi_{dh}$, where M is the total mass of the disk, the superior bar represents a phase average,

$$W_\mu=-\frac{2R_0{}^3M}{5\pi(1+\mu)}\int_0^\infty\frac{ds}{\sqrt{s(a^2+s)(b^2+s)}} \qquad (2)$$

denotes the potential energy of the elliptical disk, and

$$\Psi_{dh}=-\iint\Phi(x,y)\sigma(x,y,t)\,dx\,dy=\frac{\mu M}{2(1+\mu)}\overline{r^2} \qquad (3)$$

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is the energy of disk—halo interaction. Since the kinetic energy is independent of μ, its minimization will give the same result as before,[5] and we will have

$$\frac{E}{M}\geqslant\frac{(2c_3-e^2n^2)^2}{4m^2}+\frac{e^4}{4}n^2+\frac{2(m^2+n^2)}{(m^2-n^2)^2}\xi$$

$$-\frac{4mn}{(m^2-n^2)^2}\sqrt{\xi^2-4c_2}+\frac{1}{M}(W_\mu+\Psi_{dh})\equiv H(m,n,e,\mu).$$

Here $c_3=2R_0^2\Omega/5$, with Ω the ratio of the centroid velocity to the circular velocity of a star ($0\leqslant\Omega\leqslant1$), $m=(\overline{x^2})^{1/2}+(\overline{y^2})^{1/2}$, $n=(\overline{x^2})^{1/2}-(\overline{y^2})^{1/2}$, $\xi=2\sqrt{c_2}+2c_3e^2n^2-e^4n^4$, $c_2=(1-\Omega^2)\cdot{}^2R_0^8/625$. We shall later need the following stationary values of the parameters:

$$m_0=\frac{2R_0}{\sqrt{5}},\quad n_0=0,\quad \xi_0=2\sqrt{c_2},\quad H_0=\frac{(2\mu-1)R_0^2}{5(1+\mu)}. \tag{5}$$

Regarding n as small and expanding the function H in powers of n^2 to an accuracy of order n^2, we obtain

$$H(m,\ n,\ e,\ \mu)=H_1(m,\ \mu)+n^2H_2(m,\ e,\ \mu)+o(n^3), \tag{6}$$

where

$$H_1(m,\mu)=\frac{\mu m^2}{4(1+\mu)}-\frac{4R_0^3}{5\sqrt{5}\,m(1+\mu)}+\frac{4R_0^4}{25m^2}, \tag{7}$$

$$H_2(m,e,\mu)=\frac{e^4}{4}+\frac{6R_0^2\Omega}{5m^2}e^2-\frac{16R_0^3}{5\sqrt{5}\,m^3}\sqrt{\Omega(1-\Omega^2)}\,e$$

$$+\frac{12R_0^4(1-\Omega^2)}{25m^4}-\frac{R_0^3}{5\sqrt{5}\,m^3(1+\mu)}+\frac{\mu}{1+\mu}. \tag{8}$$

It will readily be seen that $H_1(m,\ \mu)\geq H_1(m_0,\ \mu)$, and in H_2 we may set $m=m_0$. We now have to make use of the fact that the critical state corresponds to the point where there exist two coalescent roots of the equation $H_2(m_0,\ e,\ \mu)=0$ with respect to argument e. Solving this equation subject to the condition $(\partial H_2/\partial e)_{m=m_0}=0$, we obtain the required critical value of Ω:

$$\Omega^*(\mu)=\sqrt{1-\frac{(1-2\mu)(16\mu+19)^2}{486(1+\mu)}}. \tag{9}$$

According to Eq. (9), if $\mu>1/2$ the disk will be linearly stable against nonradial oscillations for arbitrary Ω. In the case $\mu=0$ we arrive at the standard result $\Omega^*=\sqrt{125/486}$. If $\mu>0$ we will have $\Omega^*(\mu)>\Omega^*(0)$.

3. Now let us see whether Eq. (9) holds true for oscillations of large amplitude. For this purpose it is convenient to begin the analysis with the inequality (34) of our previous paper.[5] In the halo case this relation will take the form

$$H-H_0\geqslant\frac{R_0^4}{125m^2}\left[\frac{1-\Omega^2}{1-\alpha}\left(19-\frac{21}{19}\alpha-\frac{\alpha^2}{19}\right)-64\Omega^2\alpha+19\Omega^2+1\right]$$

$$-\frac{P(\alpha)}{(1+\mu)m}+\frac{(1-2\mu)R_0^2}{5(1+\mu)}+\frac{\mu(1+\alpha)m^2}{4(1+\mu)}\equiv\Pi, \tag{10}$$

an expression which is obtained from the inequality (4) above by minimizing with respect to ξ. In the relation (10), $\alpha=n^2/m^2$ ($0\leq\alpha\leq1$) and

$$P(\alpha)=-\frac{(1+\mu)mW_\mu}{M}=\frac{4R_0^3}{5\sqrt{5}\pi}\int_0^\infty\frac{dq}{(1+q)\sqrt{q(1-\varepsilon)}}, \tag{11}$$

$$\varepsilon=\alpha\left(\frac{1-q}{1+q}\right)^2.$$

First we shall show that if the disk is in weak rotation, with $\Omega\leq\sqrt{125/486}$, then the halo will act as a stabilizing factor whatever the amplitude of the oscillations. That is to say, the part of the expression Π responsible for the presence of the halo,

$$\frac{\mu}{1+\mu}\left[\frac{1}{m}P(\alpha)+\frac{(1+\alpha)}{4}m^2-\frac{3R_0^2}{5}\right], \tag{12}$$

will be positive for arbitrary $\alpha\in[0,\ 1]$. In fact, the bracketed quantity in the expression (12) is a monotonically increases function of α and its smallest value corresponds to the point $\alpha=0$, so that we will have

$$\frac{4R_0^3}{5\sqrt{5}\,m}+\frac{m^2}{4}-\frac{3R_0^2}{5}=\left(m-\frac{2R_0}{\sqrt{5}}\right)^2\left(\frac{1}{4}+\frac{R_0}{\sqrt{5}\,m}\right)>0.$$

Because of the halo the value of Ω may now increase to 1. According to Eq. (9), in the case $\mu=1/2$ we can have a disk rotating at the maximum possible angular velocity, $\Omega=1$. Let us test the behavior of nonlinear oscillations in this case. The expression (10) for Π will become

$$\Pi(m,\alpha)=\frac{4R_0^4}{625m^2}(25-16\alpha)-\frac{2}{3m}P(\alpha)+\frac{(1+\alpha)}{12}m^2. \tag{13}$$

We readily see that $d^2P/d\alpha^2>0$, $\partial^2\Pi/\partial\alpha^2<0$ for any $\alpha\in[0,\ 1]$; that is the function $\Pi(m,\ \alpha)$ will be convex with respect to α. It therefore suffices to determine the signs of $\Pi(m,\ \alpha)$ at the points $\alpha=0$ and $\alpha=1$. For $\alpha=0$, the quantity

$$\Pi(m,0)=\frac{4R_0^4}{25m^2}-\frac{8R_0^3}{15\sqrt{5}m}+\frac{m^2}{12}$$

$$=\left(m-\frac{2R_0}{\sqrt{5}}\right)^2\left(\frac{1}{12}+\frac{R_0}{3\sqrt{5}m}+\frac{R_0^2}{5m^2}\right)>0. \tag{14}$$

When $\alpha=1$, we find

$$P(1)=\frac{2R_0}{5\sqrt{5}\pi}\int_0^\infty\frac{dq}{q}=+\infty,\qquad \Pi(m,1)=-\infty. \tag{15}$$

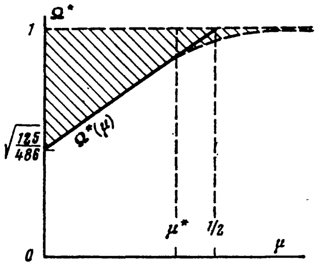

FIG. 1. Critical value Ω^* of the angular rotational velocity as a function of the parameter $\mu=\Phi/U$. The domain of nonlinear instability is shaded; the dashed curve represents the $\Omega^*(\mu)$ branch in the nonlinear case.