THE ASTROPHYSICAL JOURNAL, 250:432–449, 1981 November 15


# THE GROWTH OF ANISOTROPIC STRUCTURES IN A FRIEDMANN UNIVERSE

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*Received 1981 March 11; accepted 1981 June 2*

## ABSTRACT

Motivated by observational evidence for the existence of flattened or striated structures of very large scale (~100 Mpc) in the universe, we study the evolution of nonspherical structures. We set up equations to describe the evolution of a rotating, dust ellipsoid in an expanding universe. We derive solutions for the shape and density evolution as the ellipsoid expands along with a Friedmann background universe and show that deviations from spherical symmetry are preserved by adiabatic expansion. Analytic expressions are derived to describe the nonlinear evolution of pressure-free ellipsoids. We calculate details of "pancake" formation which are relevant for realistic theories of galaxy and cluster formation. In particular, the density collapse at turnaround and the collapse velocity at pancake formation are found to be reduced relative to the spherical case if the initial fluctuations are anisotropic: this may affect kinematic determinations of Ω in the local supercluster and also enable galaxies with massive halos to form during the fragmentation of pancakes whose dynamics are dominated by massive neutrinos. Significant deviations from spherical symmetry in the baryon irregularity spectrum might also be naturally associated with primordial isothermal inhomogeneities in a Friedmann universe and their possible origin at the epoch of "grand unification." An exact model for a general relativistic pancake collapse is given, based on an exact solution to Einstein's equations found by Szekeres. This collapse does not pass through a series of ellipsoids and has no symmetries.

*Subject headings:* cosmology — galaxies: general

## I. INTRODUCTION

The large-scale structure of the universe reveals very considerable deviations from spherical symmetry. Galaxies, clusters of galaxies, and superclusters all possess complex structure and exhibit overt asymmetries. One would like to know whether such asymmetry must be attributed to the initial conditions of the very early universe, or whether it could have developed naturally during the course of the evolution of the large-scale structure. In this paper, we shall present some calculations that describe the development of shape eccentricities in the expanding universe.

Some specific observational motivation comes from the well-known flattening of the Local supercluster and the recent evidence for very large-scale (~100 Mpc) holes and both filamentary and sheetlike structure in the galaxy distribution. The lack of evidence for any significant rotation on large scales (e.g., in many elliptical galaxies and in galaxy clusters) and the existence of anisotropy on scales where gravitational collapse seems unlikely to have played any role are indicative that the anisotropy may possibly be an intrinsic property associated with the origin of the structure itself, rather than a consequence of tidal interactions at a relatively late stage in its evolution. Finally, a new difficulty arises with the possible existence of a finite rest mass of ~30 eV for the electron neutrino which helps to reconcile the deuterium abundance with predictions of cosmological nucleosynthesis (Schramm and Steigman 1981) and is stimulating revived interest in the adiabatic fluctuation theory of galaxy formation (Doroshkevich *et al.* 1980). There is still at least one major problem in this theory, since the adiabatic density fluctuation spectrum peaks at the neutrino Jeans mass $\sim 4\times 10^{15}\,(m_\nu/30\text{ eV})^{-2}\,M_\odot$ (Bond, Efstathiou, and Silk 1980). This results in the neutrinos acquiring the thermal energy appropriate to galaxy clusters, which therefore inhibits the formation of massive galactic halos of neutrinos. We shall show that anisotropic collapse may provide one means of resolving this problem.

Many calculations have been performed to study the evolution of linearized density perturbations in an expanding universe (e.g., Bardeen 1981). These have been extended to follow the behavior of nonlinear pressureless density inhomogeneities with spherical symmetry (e.g., Kursov and Ozernoi 1975; Silk 1974). In a dense universe ($\Omega \geq 1$), all inhomogeneities eventually recollapse if they initially possess negative gravitational binding energy. However, in a low ($\Omega \ll 1$) density universe, the background expands at a rate faster than inhomogeneities can undergo free-fall collapse, and inhomogeneities that possess a density contrast less than a critical value ($\propto \Omega^{-1}$) will eventually freeze into the background ($\delta\rho/\rho \to$ constant). While these calculations are probably adequate for modeling the development of large-scale inhomogeneities at $z \gg 1$, the assumption of spherical symmetry provides a very poor approximation to a realistic gravitational collapse in the nonlinear regime, and moreover precludes any possibility of ascertaining the range of initial states from which highly asymmetric structures may have evolved.

Various authors have discussed mechanisms which tend to result in the development of aspherical structures in the universe. Binney and Silk (1979) examined the effect of tidal interactions between neighboring protogalaxies. When the inhomogeneities become nonlinear, tidal interactions result in elongation and the formation of prolate structures. This mechanism is only effective on relatively small scales, perhaps appropriate to elliptical galaxies, where a sufficiently close neighbor may be present to promote the effect. An extensive series of studies by Zel'dovich and his coworkers has developed the pancake theory, in which galaxies form after the collapse into thin sheets and ensuing fragmentation of mass on cluster and supercluster scales. The pancake collapse is a manifestation of the Lin, Mestel, and Shu (1965) instability, which results in the increase of eccentricity during collapse. This effect requires that the collapsing object is dense enough to become bound along at least one axis. Objects of very large scale or small density contrast will still be undergoing expansion: if sufficiently asymmetric, their morphology may reflect the anisotropy of their initial conditions or early growth.

In what follows, we shall discuss the extent to which large deviations from spherical symmetry can arise in the very large-scale mass distribution. We analyze the evolution of the shape of a density inhomogeneity in the expanding universe as it evolves by gravitational instability. We show how deviations from spherical symmetry are preserved in the linear regime. The nonlinear coupling of the shape anisotropy to the density inhomogeneity and to any possible rotation will be derived. To describe their fully nonlinear evolution, we model inhomogeneities by homogeneous ellipsoids of arbitrary initial eccentricity in an expanding background. The density contrast at the epoch when an axis first attains its maximum expansion is evaluated, and we also compute the collapse velocity when the pancake forms. Two applications are relevant here. One is to assess the effects of anisotropy on determinations of $\Omega$ utilizing distortions from the Hubble flow in the Local supercluster (Silk 1974). A second is to provide an estimate of whether neutrino-dominated pancakes can fragment into galaxies surrounded by massive halos that consist of neutrinos. Finally, we also calculate a particular general relativistic model that illustrates the nonlinear growth of highly anisotropic inhomogeneities.

A specific outline of the remainder of this paper is as follows. In § II, we set up and solve equations that describe the evolution of a rotating ellipsoidal dust inhomogeneity in an expanding universe. We calculate the coupled evolution of the eccentricity and overdensity together with the effects of pressure and rotation. We also discuss the possible origin and evolution of aspherical inhomogeneities in the early universe. In § IV, we analyze the pancake-like collapse of self-gravitating spheroids embedded in the expanding universe. We derive the dependence of the collapse parameters on the initial eccentricity. Finally, in § V, we give a particular example of a general relativistic nonspherical dust collapse.

## II. LINEAR THEORY

### *a*) *Formation*

Let $\hat{x}^\mu = (t, \hat{x}^i)$ be comoving coordinates which describe an isotropic and homogeneous Friedmann universe. The spacetime metric is

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + \Sigma^2(r)\left(\sin^2\theta\, d\phi^2 + d\theta^2\right)\right] \tag{1}$$

when $\Sigma(r) = \sin r$, $r$, or $\sinh r$ according as the curvature is positive, null, or negative. The scale factor $a(t)$ is determined by Einstein's equations.

We shall be primarily interested in a pressureless, ($p = 0$), background cosmology with zero curvature. Therefore, the Einstein equations give

$$\ddot{a} = -\frac{4\pi G}{3}\rho_B a.$$

The solution is

$$\rho_B = \frac{1}{6\pi G t^2}; \qquad a = t^{2/3}. \tag{2}$$

We shall be interested in the behavior of perturbations to this smooth background. The inhomogeneities of physical interest are smaller than the particle horizon at all times after recombination, and therefore intrinsically general relativistic effects are irrelevant to their evolution. We can legitimately adopt a Newtonian description in a coordinate system which is expanding with the overall Hubble flow.

In the comoving $\{\hat{x}^\mu\}$ system, the Newtonian hydrodynamical equations of continuity, motion, and gravitation are easily obtained. They can be simplified to remove the expansion of the background universe by adopting inertial coordinates $x^\mu = (t, x^i)$, where

$$x^i \equiv a(t)\hat{x}^i. \tag{3}$$

They become

$$\frac{\partial \rho}{\partial t} + \frac{\partial V_i}{\partial x_i}\rho = 0, \tag{4}$$

$$\frac{dV_i}{dt} = -\rho^{-1}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} - \frac{4\pi G}{3}\rho_B x_i, \tag{5}$$

$$\nabla^2\phi = 4\pi G(\rho - \rho_B), \tag{6}$$

where $\rho$ and $p$ are the density and pressure perturbations over the background,

$$V_i = \frac{\dot{a}}{a}x_i + a u_i, \tag{7}$$

$$\frac{d}{dt} \equiv \partial_t + V_i \partial/\partial x_i, \tag{8}$$

and $u_i$ is the coordinate component of the three-velocity. Similar transformations have been used by Irvine (1965), Nariai (1969), Drury and Stewart (1976), and Kursov and Ozernoi (1975).

Nariai and Fujimoto (1972) have calculated the form of equations (4–6) in a noninertial reference frame $\{x'_\mu\}$ rotating with angular velocity $\Omega_i$ relative to the $\{x_i\}$ frame. The form of equations (4) and (6) remains the same, but Coriolis terms appear in equation (5); it becomes

$$\frac{dV'_i}{dt} + 2\varepsilon_{ijk}\Omega_j V'_k = -\rho^{-1}\frac{\partial p}{\partial x'_i} - \left[\left(\frac{4\pi G}{3}\rho_B - \Omega^2\right)\delta_{ij} + \Omega_i\Omega_j - \varepsilon_{ijk}\frac{d\Omega_k}{dt}\right], \tag{9}$$

where $V'_i$ is the velocity in the rotating reference frame. Henceforth we shall drop the primes on the variables $x'_i$, $V'_i$.

Following Nariai and Fujimoto we set up the equations describing the evolution of a uniform rotating ellipsoid in an expanding cosmological background. The velocity field is

$$V_i = \alpha_{ij}(t)x_j, \tag{10}$$

and for an ellipsoidal geometry,

$$E(x_j, t) \equiv 1 - \sum_{i=1}^{3}\left(\frac{x_i}{\alpha_i}\right)^2 = 0, \tag{11}$$

where $\alpha_i(t)$ is the $i$th principal axis. If the ellipsoidal shape is maintained we have

$$\frac{dE}{dt} = 0, \tag{12}$$

and the mass, $M$, of the ellipsoid is

$$M=\frac{4\pi}{3}\rho\alpha_1\alpha_2\alpha_3=\text{constant}. \tag{13}$$

The velocity gradients $\alpha_{ij}$ can be decomposed in the standard way to

$$\alpha_{ij}\equiv\frac{1}{3}\left(\frac{\dot\alpha_1}{\alpha_1}+\frac{\dot\alpha_2}{\alpha_2}+\frac{\dot\alpha_3}{\alpha_3}\right)\delta_{ij}+\sigma_{ij}+\varepsilon_{ijk}\omega_r, \tag{14}$$

where $\sigma_{ij}$ is the shear tensor

$$\sigma_{ij}\equiv\frac{1}{2}(\alpha_{ij}+\alpha_{ji})-\frac{1}{3}\alpha_{kk}\delta_{ij}, \tag{15}$$

with $\sigma^2\equiv2\sigma_{ij}\sigma^{ij}$, and $\omega_k$ the vorticity vector

$$\omega_i\equiv\varepsilon_{ijk}\alpha_{jk}. \tag{16}$$

If we integrate equation (6) over the region $E(x_i, t)$ we obtain the gravitational potential of the ellipsoid:

$$\phi=\pi G(\rho-\rho_B)\alpha_1\alpha_2\alpha_3(U_{ij}x_ix_j-I), \tag{17}$$

where $U_{ij}=\text{diag}(U_i)$ and is given, with $I$, by elliptic integrals (MacMillan 1958),

$$U_i=\int_0^\infty\frac{dx}{(\alpha_i^2+x)\psi^{1/2}(x)}, \tag{18}$$

$$I=\int_0^\infty\frac{dx}{\psi^{1/2}(x)}, \tag{19}$$

and $\psi(x)=\Pi_{i=1}^3(\alpha_i^2+x)$.

If we assume the rotation velocity possesses only an $x_3$ component, then from equation (9) we obtain the equations of motion for the principal axes of the rotating ellipsoid:

$$\ddot\alpha_1=-\frac{4\pi G}{3}\rho_B\left(1-\frac{3}{2}\alpha_1\alpha_2\alpha_3U_1\right)\alpha_1-\frac{3}{2}GMU_1\alpha_1+\frac{8L^2}{(\alpha_1+\alpha_2)^3}, \tag{20}$$

$$\ddot\alpha_2=-\frac{4\pi G}{3}\rho_B\left(1-\frac{3}{2}\alpha_1\alpha_2\alpha_3U_2\right)\alpha_2-\frac{3}{2}GMU_2\alpha_2+\frac{8L^2}{(\alpha_1+\alpha_2)^3}, \tag{21}$$

$$\ddot\alpha_1=-\frac{4\pi G}{3}\rho_B\left(1-\frac{3}{2}\alpha_1\alpha_2\alpha_3U_3\right)\alpha_3-\frac{3}{2}GMU_3\alpha_3, \tag{22}$$

$$\frac{\ddot\rho}{\rho}-\frac{4}{3}\left(\frac{\dot\rho}{\rho}\right)^2-4\pi G\rho-\sigma^2+\frac{8L^2}{\alpha_1\alpha_2(\alpha_1+\alpha_2)^2}=0, \tag{23}$$

where $L$ is the (conserved) angular momentum of the ellipsoid. The first terms on the right hand side of equation (20)–(22) describe the background expansion effects, while the last terms in equations (20) and (21) incorporate the rotational support which opposes gravity.

The form of equation (23) allows us to discern the qualitative effects of rotation and shear anisotropy upon the process of gravitational instability. If $\sigma=L=0$ in equation (23), then it may be solved exactly. The density reaches a maximum value equal to $9\pi^2/16$ of that in the flat background universe described by equation (2), and after that recollapse occurs. It is clear from equation (23) that rotation ($L$) acts in the opposite sense to self-gravitation ($G\rho$) and so makes it easier to resist gravitational collapse. Rotating objects can attain a larger density contrast over the background before collapse occurs. Likewise, shear ($\sigma^2$) reinforces the effect of gravity and makes gravitational binding and collapse easier, essentially by providing more degrees of freedom for the recollapse (all axes need not collapse simultaneously).

### *b*) *The Evolution of Shape and Density*

One would like to ascertain how likely it is that asymmetric structures on very large scales originated from objects which possessed significant deviations from sphericity at the end of the radiation era, $z \approx 10^3$. We therefore need to know how eccentric shapes evolve as the universe expands.

Let us first use equations (20)–(22) to ascertain the manner in which a shape ($\alpha_1 \neq \alpha_2 \neq \alpha_3$) perturbation behaves in an expanding universe. If we introduce the fractional axes

$$\beta_i = \frac{\alpha_i}{a}, \tag{24}$$

then equations (20)–(22) become, in the absence of rotation ($L \equiv 0$),

$$\ddot{\beta}_i + \frac{4}{3t}\dot{\beta}_i - \frac{2}{9t^2}\beta_i = -\frac{2}{9t^2}\left(1 - \frac{3}{2}\beta_1\beta_2\beta_2 a^3 U_1\right)\beta_i - \frac{a^3\beta_i U_i}{3t^2}. \tag{25}$$

To second order in $\beta_i$ these equations become

$$p_i'' + \frac{1}{2}p_i' = -\frac{\beta p_i}{10}, \tag{26}$$

$$\beta'' + \frac{1}{2}\beta' - \frac{3}{2}\beta = -\frac{5\beta^2}{2} - \frac{3}{4}\sum_1^3 p_i^2, \tag{27}$$

where

$$\beta \equiv \sum_{i=1}^{3} \beta_i, \tag{28}$$

$$p_i \equiv \beta_i - \frac{1}{3}\beta, \tag{29}$$

and

$$' \equiv \frac{d}{dt}; \quad \tau = \ln a = \frac{2}{3}\ln t. \tag{30}$$

Equations (26) and (27) therefore describe the nonlinear evolution of the shape anisotropy, $p_i$, and $\beta$ which describes the density contrast to first order; by equation (13) we see

$$\frac{\delta\rho}{\rho_B} \equiv \frac{\rho - \rho_B}{\rho_B} = -\beta. \tag{31}$$

A zero-order solution for $\{p_i, \beta\}$ is obtained by neglecting the right-hand side of equations (26) and (27). This yields

$$p_i^{(0)} = A_i^{(0)} t^{-1/3} + B_i^{(0)} = A_i^{(0)} e^{-\tau/2} + B_i^{(0)}, \tag{32}$$

$$\beta^{(0)} = C(0) t^{2/3} + D(0) t^{-1/3} = C^{(0)} e^{\tau} + D^{(0)} e^{-3\tau/2}. \tag{33}$$

These are the well-known growing and decaying modes of perturbation theory, first found by Lifshitz (1946). To obtain the second-order coupling between the shape ($p_i$) and density ($\beta$) perturbations we solve for the first-order $p_i^{(1)}$ and

$\beta^{(1)}$ given by

$$p_i^{(1)''}+\frac{1}{2}p_i^{(1)'}=-\frac{\beta^{(0)}}{10}p_i^{(0)}, \tag{34}$$

$$\beta^{(1)''}+\frac{\beta^{(1)}}{2}-\frac{3\beta^{(1)}}{2}=-\frac{5\beta^{(0)^2}}{2}-\frac{3}{4}\sum_{i=1}^{3}p_i^{(0)^2}. \tag{35}$$

This gives

$$p_i^{(1)}=A_i^{(0)}e^{-\tau/2}+B_i^{(0)}-\frac{A_i^{(0)}}{5}C^{(0)}e^{\tau}-A_i^{(0)}\frac{D^{(0)}}{30}e^{-2\tau}-\frac{B_i^{(0)}}{15}D^{(0)}e^{-3\tau/2}, \tag{36}$$

$$\beta^{(1)}=C^{(0)}e^{\tau}+D^{(0)}e^{-3\tau/2}-\frac{5C^{(0)^2}}{7}e^{2\tau}-\frac{5D^{(0)^2}}{12}e^{-3\tau}+e^{-\tau/2}\left(\frac{10}{3}C^{(0)}D^{(0)}+\Sigma A_i^{(0)}B_i^{(0)}\right)+\frac{3}{4}\Sigma A_i^{02}e^{-\tau}+\frac{1}{2}\Sigma B_i^2. \tag{37}$$

From these expressions we can ascertain the couplings between different growing and decaying modes. To first order, shape and distortions remain constant. The only *growing* modes in the shape anisotropy are driven by the density perturbation $\beta$. Likewise, the rapid $t^{4/3}$ term in $\beta$ is a second-order curvature mode driven by the first-order term $\sim C^{(0)}t^{2/3}$.

The ratio of the shear to the Hubble expansion factor is given by

$$\left(\frac{\sigma}{H}\right)_0\approx\frac{t}{a}\frac{d}{dt}(ap_0)\approx\text{constant},\qquad\text{to first order}, \tag{38}$$

$$\left(\frac{\sigma}{H}\right)_1\approx\text{constant}+\eta t^{2/3},\qquad \eta\ll 1,\ \text{to second order}. \tag{39}$$

The evolution in eccentricity is given by $p_i\approx(\delta x-\delta y)/a$.

### *c) Generalization to an Open Universe*

We can repeat the previous analysis for the late stages of an open universe of negative curvature. In this phase the expansion factor and density evolution are well described by

$$a=t, \tag{40}$$

$$\rho=\frac{\lambda}{4\pi G t^3};\qquad \dot{\lambda}\equiv 0. \tag{41}$$

In this case the analogs of equations (26) and (27) are

$$p_i''+\frac{p_i'}{2}=-\frac{\lambda p_i}{3}e^{-\tau}\left(1+\frac{\beta}{5}\right), \tag{42}$$

$$\beta''+\beta'=\lambda e^{-\tau}\left(\frac{2\beta}{3}-5\beta^2-\frac{1}{2}\sum_1^3 p_i^2\right), \tag{43}$$

where now $t=e^{\tau}$.

The second-order solutions are

$$p_i^{(1)}=A_i^{(0)}+B_i^{(0)}t^{-1}+\lambda A_i^{(0)}\left(1+\frac{C^{(0)}}{5}\right)\frac{\ln t}{t}+\frac{\lambda}{12t^2}\left(B_i^{(0)}+\frac{A_i^{(0)}D^{(0)}}{5}+\frac{B_i^{(0)}C^{(0)}}{5}\right)+\frac{\lambda B_i^{(0)}}{90}D^{(0)}t^{-3}, \tag{44}$$

$$\beta^{(1)}=C^{(0)}+D^{(0)}t^{-1}+\left(5\lambda C_0^{(0)^2}+\frac{\lambda}{2}\Sigma A_i^2+\frac{2C_0^{(0)}\lambda}{3}\right)\frac{\ln t}{t}+\left(\frac{D^{(0)}\lambda}{3}-5C^{(0)}D^{(0)}\lambda-\frac{\lambda B_i^{(0)^2}}{2}\right)-\frac{1}{6t^3}\left(5D^{(0)^2}\lambda+\frac{\lambda B_i^{(0)^2}}{2}\right). \tag{45}$$

All growing modes are absent, and the shape and density perturbations freeze in at constant values. The shear to Hubble contrast is (eq. [38])

$$\frac{\sigma}{H} \approx \text{constant}. \tag{46}$$

These calculations indicate that small deviations from sphericity are preserved by the expansion of the universe. We shall see in § III that the final parameters of the nonlinear evolution are determined by these small initial eccentricities.

### *d*) *The Effects of Rotation and Pressure*

From equation (25) we can obtain the following equations for the principal axes when rotation and pressure terms are included. The latter is parametrized by the sound speed in the perturbation

$$v_s^2 \equiv \frac{dp}{d\rho}. \tag{47}$$

In the flat cosmological background this gives, to first order,

$$\ddot{\beta}_1 + \frac{4}{3t}\dot{\beta}_1 - \frac{2}{9t^2}\beta = -\frac{2L^2}{a^4}(\beta_1+\beta_2) - \frac{2v_s^2\beta_1}{a^2}, \tag{48}$$

$$\ddot{\beta}_2 + \frac{4}{3t}\dot{\beta}_2 - \frac{2}{9t^2}\beta = -\frac{2L^2}{a^4}(\beta_1+\beta_2) - \frac{2v_s^2\beta_2}{a^2}, \tag{49}$$

$$\ddot{\beta}_3 + \frac{4}{3t}\dot{\beta}_3 - \frac{2}{9t^2}\beta = -\frac{2v_s^2\beta_3}{a^2}. \tag{50}$$

It is easier to work in the variables $\beta$, $\gamma$, $\varepsilon$, where

$$\gamma \equiv \beta_1 - \beta_2,$$

$$\varepsilon \equiv \beta_1 + \beta_2,$$

which satisfy

$$\ddot{\gamma} + \frac{4}{3t}\dot{\gamma} = -\frac{4L^2\varepsilon}{t^{8/3}} - \frac{2v_s^2\gamma}{t^{4/3}}, \tag{51}$$

$$\ddot{\varepsilon} + \frac{4}{3t}\dot{\varepsilon} - \frac{4}{9t^2}\beta = -\frac{2v_s^2\varepsilon}{t^{4/3}}, \tag{52}$$

$$\ddot{\beta} + \frac{4}{3t}\dot{\beta} - \frac{2}{3t^2}\beta = -\frac{2v_s^2\beta}{t^{4/3}}, \tag{53}$$

#### i) *Irrotational Case: (L=0)*

We solve equation (53) to obtain the effect of pressure damping on the density perturbation:

$$\beta = t^{-1/6}\mathcal{C}_{\pm 5/2}(3\sqrt{2}\,v_s t^{1/3}), \tag{54}$$

where $\mathcal{C}$ are linear combinations of cylindrical functions ($J, Y, H^{(1)}, H^{(2)}$). At late times $\mathcal{C}_\nu(x) \approx \nu^{-1/2}$ (oscillatory terms), and so asymptotically $\beta \approx t^{-1/3}$.

Also equation (51) solves as

$$\gamma = t^{-1/6}\mathcal{C}_{\pm 1/2}(3\sqrt{2}\,v_s t^{1/3}) \tag{55}$$

and

$$\gamma \approx t^{-1/3} \quad \text{as } t \to \infty ; \tag{56}$$

therefore,

$$\varepsilon \approx t^{-1/3} \quad \text{as } t \to \infty$$

from (51).

The effects of pressure are the familiar ones: density perturbations are damped out on small scales, and disparities in the scale factors of the ellipse are also attenuated at the same rate.

ii) *Pressure-Free Case* ($v_s = 0, L \neq 0$)

The effects of rotation only couple directly to the $\gamma$ equation, and we can solve for equations (51)–(53) exactly to obtain

$$\beta = At^{2/3} + Bt^{-1}, \tag{57}$$

$$\varepsilon = Ct^{-1/3} + D + \frac{2At^{2/3}}{3} + \frac{2Bt^{-1}}{3}, \tag{58}$$

$$\gamma = Et^{-1/3} + F - 6L^2Ct^{-1} - 18L^2 Dt^{-2/3} - 8L^2A \ln t - \frac{6}{5}L^2Bt^{-5/3}. \tag{59}$$

To the order of approximation calculated, the rotation has no effect upon the rate of growth of the density amplitude. It only directly affects the evolution of axes orthogonal to the rotation axis and so couples to $\gamma$ but not $\varepsilon$. The effects die away in the course of time because the centrifugal potential driving asymmetries is smaller than the potential deriving from $\delta\rho$.

## III. THE ORIGIN OF ASYMMETRIC STRUCTURES

Since our calculations indicate that sizeable asymmetries can be preserved before inhomogeneities enter the nonlinear phase of their evolution we should inquire into the likelihood of their existence. This involves discussing the shape evolution through the radiation era of the big bang model and is complicated by the presence of pressure gradients and the effects of general relativity. A few general points can be made. If we suppose the overall expansion dynamics are still described by the Friedmann model and density gradients are small then the shape evolution problem amounts to solving for the shear tensor $\sigma_{ab}$, where on scale $\lambda$, using Ellis (1973), we have

$$\sigma_{ab} \approx a^{-3}\left\{\Sigma_{ab}(\lambda) + \int_0^t a^3(t)\left[{}^{(3)}R_{ab}(\lambda) - \tfrac{1}{3}{}^{(3)}R(\lambda)\delta_{ab}\right] dt\right\}.$$

Here ${}^{(3)}R_{ab}$ is the Ricci three-curvature of the universe. Its isotropic part contributes the well-known decaying mode of anisotropy, $\sigma \propto a^{-3}$; however, the anisotropic curvature drives a distortion mode that need not decay over finite time intervals, and its time dependence can only be calculated from the remaining Einstein equations. In practice, one is interested in more specialized problems: theories of galaxy formation focus attention upon two types of primordial density inhomogeneity—adiabatic and isothermal (Barrow 1980*a*). We can argue that the likelihood of large deviations from sphericity surviving the radiation era may be linked to whether the inhomogeneity is isothermal or adiabatic.

### *a*) *Adiabatic Inhomogeneities*

It has been shown that photon diffusion and radiative viscosity erase adiabatic inhomogeneities on scales less than $M_D \approx 10^{13}\Omega^{-5/4} M_\odot$ by the end of the radiation era (Silk 1968). Any adiabatic perturbations of this order which were nonspherical would be damped by radiative viscosity slightly more efficiently ($\propto[1-e]^2$). Any adiabatic inhomogeneity of mass exceeding $M_D$ might in principle be strongly aspherical. However, if the baryons in an adiabatic fluctuation of scale $\lambda > ct$ are very nonspherical the photons must share this geometry because of the thermal coupling. For a fluctuation power spectrum $|\delta_k|^2 \propto k^n$ with $n < 1$, this asymmetry would create a non-Friedmann cosmology on the horizon scale $\lambda \gg ct$. The metric perturbations induced by strongly anisotropic adiabatic inhomogeneities are significant, and unless $n \geq 1$ one would not expect nonspherical inhomogeneities in the matter distribution to emerge from the radiation era as *small* deviations from a smooth Friedmann background universe. In fact, Zel'dovich has argued for $n=1$, corresponding to constant curvature metric fluctuations. Constraints from the anisotropy of the cosmic background radiation and from the large-scale matter distribution favor a somewhat greater value of $n$ (Wilson and Silk 1981). The existence of any 'equipartition' principle demanding that shape and density perturbations contribute equally to the metric perturbations on some scales may allow stronger conclusions to be drawn.

### *b) Isothermal Inhomogeneities*

The situation is much more interesting if density inhomogeneities are isothermal ($\delta\rho_\gamma \equiv 0, \delta\rho_m \neq 0$). In this case for an arbitrary fluctuation spectrum, the baryon distribution can possess very large asymmetries on scales exceeding the horizon which have no significant effect upon the expansion dynamics and the Friedmann background because $\delta\rho_m \ll \rho_\gamma$ in the radiation era.

Isothermal perturbations arise from spatial variations in the baryon number of the very early universe ($t \lesssim 10^{-4}$ s) (see Barrow 1980*b*). Recent developments in gauge theories of the strong and electroweak interactions make it attractive to conclude that the baryon number of the universe arose from nonequilibrium $C$, $CP$, and baryon nonconserving interactions during the first $\sim 10^{-35}$ s of its expansion history. If the cosmological dynamics were strongly anisotropic and slightly inhomogeneous at this early time then baryon number would be distributed in an anisotropic and inhomogeneous fashion at $t \approx 10^{-35}$ s (Barrow and Turner 1981; Bond, Kolb, and Silk 1981).

This distribution would be preserved in the quark sea until baryons condense at $t \approx 10^{-4}$ s. At this time their gravitational effect upon the Friedmann background metric would be completely negligible (the *smooth* radiation density dominates by a factor $\gtrsim 10^6$). The shape distortions would then be preserved in the baryon distribution until the end of the radiation era, $t \approx 10^{12}$ s. They would be expected to have effects upon two sets of observable parameters. Primordial nucleosynthesis would be affected through the baryon distribution. Regions of high baryon density destroy more deuterium and burn it to helium$-$4. There would arise a spatial distribution of these light elements that would mirror the distribution of baryons. Also large asymmetries in the baryon distribution at recombination would increase the expected fluctuation level of the microwave background radiation on small angular scales, and they would be more readily detectable than spherical overdensities (Wilson and Silk 1981).

Finally we should mention that there have been recent conjectures that extended sheet or stringlike structures might rather naturally emerge from the very early universe (Kibble 1976, 1980; Zel'dovich 1980). It is possible that spontaneous symmetry breaking at $\sim 10^{-35}$ s could ensure that, when the universe cools below some critical temperature $T_c \approx 10^{14}$ GeV, the Higgs field gains a nonzero vacuum expectation value and is partitioned into vacuum domains with unusual geometry—sheets or strings (depending upon the topology of the manifold of degenerate vacua). The possibility of sheets of vacuum energy appears to be excluded by the isotropy of the microwave background; even one would contribute $\sim 10^6$ as much energy density as all known forms of matter today; see Zel'dovich, Kobzarev, and Okun (1974). However, one-dimensional strings do not suffer from such a constraint (the thickness should be of order of the Compton wavelength of the Higgs mass $\sim 10^{14}$ GeV responsible for the symmetry breakdown). These objects should remain thermally decoupled from radiation during the early expansion history and would impose a nonspherical, linear geometry on the matter distribution at recombination.

Similarly, one could imagine that the initial spectrum of inhomogeneities was not of the traditional adiabatic or isothermal type at all. For example, suppose the structure emerging from the Planck era, $t_p \approx 10^{-43}$ s, was formed out of a network of gravitational discontinuities in the metric and its first derivative; that is, a network of intersecting gravitational shock fronts. The tesselated character of the gravitational field would then fashion inhomogeneities into linear and sheetlike structures during the subsequent expansion.

## IV. NONLINEAR THEORY

The calculations of § II indicate how small initial asymmetries evolve in the linear and almost linear regime. Eventually the overdensity of the ellipsoid will become considerable because the linearly growing modes ($\delta\rho/\rho \approx t^{2/3}$) attain amplitudes of order unity, and the ellipsoid will begin to recollapse in at least one direction. The ellipsoid will tend toward an increasingly pancake or cigarlike configuration depending upon its initial degree of oblateness or prolateness.

Consider a spheroidal perturbation in an expanding universe (the following analysis can readily be generalized to the case of an ellipsoidal perturbation). Provided that the density within the spheroid is always assumed to be uniform, and initially (at $t_0$) exceeds that of the background by an amount $\delta\rho(t_0) \equiv \delta \cdot \rho(t_0)$, the subsequent evolution can be described approximately by

$$x = a - \tfrac{3}{2} A (a - a_e), \tag{60}$$

$$z = a - \tfrac{3}{2} B (a - a_e). \tag{61}$$

Here, the principal axes are $x_0 X(t)$ and $z_0 Z(t)$, $a(t)$ is the background scale factor, $a_e(t)$ is the scale factor for a universe of mean density $\rho+\delta\rho$ at $t_0$, and $A$ and $B$ are functions of the initial eccentricity defined by equations (62) and (63) (MacMillan 1958). They are equal to the $U_1$ and $U_3$ defined by equation (18) when $\alpha_1=\alpha_2>\alpha_3$. We relabel them here for convenience:

$$A=e^{-2}\left[e^2-1+(1-e^2)^{1/2}\sin^{-1}e/e\right], \tag{62}$$

$$B=2e^{-2}\left[1-(1-e^2)^{1/2}\sin^{-1}e/e\right], \tag{63}$$

and $1-e^2=(z_0/x_0)^2$. Initial conditions are $X=Z=a=a_e=1$ at $t=t_0$. The initial velocity is equal to the Hubble velocity at $t_0$. White and Silk (1979) have shown that equation (1) provides a reasonably accurate model for the nonlinear as well as the linear evolution of a homogeneous spheroidal perturbation, followed until the first $(Z)$ axis recollapses (pancake formation). The parametric equations satisfied by $a(t)$ and $a_e(t)$ are

$$a=\frac{\Omega}{2(1-\Omega)}(ch\phi-1), \qquad \frac{t}{t_0}=\frac{3}{4}\frac{\Omega}{(1-\Omega)^{3/2}}(sh\phi-\phi) \tag{64}$$

and

$$a_e=\frac{1}{2\delta}(1-\cos\theta), \qquad \frac{t}{t_0}=\frac{3}{4\delta^{3/2}}(\theta-\sin\theta). \tag{65}$$

At the instant, denoted by subscript $i$, when the first $(Z)$ axis turns around, $\dot{Z}=0$, we have

$$\dot{a}_i(1-2/3B)=\dot{a}_{ei}. \tag{66}$$

It is convenient to define the free-fall time for a spherical perturbation of density excess $\delta$

$$t_{\rm ff}\equiv\left(\frac{3\pi}{2\delta^{3/2}}\right)t_0, \tag{67}$$

so that

$$t=(t_{\rm ff}/2\pi)(\theta-\sin\theta). \tag{68}$$

Hence, equation (66) can be rewritten using equation (63) as

$$a_i(1-2/3B)=\frac{\sin\theta_i}{1-\cos\theta_i}\frac{\pi}{\delta t_{\rm ff}}\left(\frac{a_i}{\dot{a}_i}\right). \tag{69}$$

We consider first the evolution of an ellipsoid to a pancake configuration in an Einstein-de Sitter background with background evolution

$$a=(t/t_0)^{2/3}. \tag{70}$$

Equation (69) can now be further simplified, using equations (65) and (70) to give

$$a_i=(t_i/t_0)^{2/3}=\delta^{-1}[3/4(\theta_i-\sin\theta_i)]^{2/3} \tag{71}$$

and

$$1-2/3B=\frac{\sin\theta_i}{1-\cos\theta_i}\left[\frac{3}{4}(\theta_i-\sin\theta_i)\right]^{1/3}. \tag{72}$$

The density contrast at turn-around is

$$\delta_i = a_i^3/(X_i^2 Z_i), \tag{73}$$

where from equations (60) and (65),

$$X_i = a_i\left(a - \frac{3}{2}A\right) + \frac{3}{4}\frac{A}{\delta}(1-\cos\theta_i) \tag{74}$$

and

$$Z_i = a_i\left(a - \frac{3}{2}B\right) + \frac{3}{4}\frac{B}{\delta}(1-\cos\theta_i). \tag{75}$$

One obtains

$$\delta_i = \frac{(3\pi/4)(2/3B)(2/3A)^2(\theta_i - \sin\theta_i)^2\pi^2}{\left\{(1-\cos\theta_i)/2 - (1-2/3B)\left[3/4(\theta_i - \sin\theta_i)\right]^{2/3}\right\}\left\{(1-\cos\theta_i)/2 + (2/3A - 1)\left[3/4(\theta_i - \sin\theta_i)\right]^{2/3}\right\}^2}. \tag{76}$$

This expression yields the familiar limit $\delta_i = (3\pi/4)^2$ for the spherical case, when $A = B = 2/3$ and $\theta_i = \pi$. In general, one has to first solve equation (72) for $\theta_i$ when an arbitrary initial flattening is specified; equation (76) then provides the density contrast at turn around for the $Z$-axis.

One finds that compared to a spherical fluctuation with the same initial overdensity $\delta$, the turn-around occurs earlier, at about $0.14\ t_{\text{ff}}$ for the case of extreme initial flattening, as opposed to $0.5\ t_{\text{ff}}$ for spherical collapse. The density contrast at turn-around is reduced relative to $(3\pi/4)^2$ by up to a factor of about 3.

Consider next the epoch of pancaking, denoted by subscript $p$, when $Z = 0$, and

$$(1-\cos\theta_p)/2\delta = a_p(1-2/3B). \tag{77}$$

Making use of equations (68) and (70), we can rewrite equation (77) as

$$1 - 2/3B = \frac{1-\cos\theta_p}{2}\left[\frac{3}{4}(\theta_p - \sin\theta_p)\right]^{-2/3}. \tag{78}$$

This gives $\theta_p$ for arbitrary flattening. One can now calculate the parameters of interest at the epoch of pancaking, notably the pancake radius and collapse velocity. We find that

$$X_p = (1 - A/B)a_p, \tag{79}$$

where $a_p$ is obtained from equation (77), and

$$\dot{Z}_p = -\frac{3B}{2}\dot{a}_p\left\{1 - 2/3B - \frac{\sin\theta_p}{1-\cos\theta_p}\left(\frac{3}{4}\right)^{1/3}(\theta_p - \sin\theta_p)\right\}^{1/3}. \tag{80}$$

The velocity in the plane of the pancake is obtained from

$$\dot{X}_p = \frac{A\dot{Z}_p}{B} + \dot{a}_p(1 - A/B). \tag{81}$$

The collapse velocity down the $z$-axis can be expressed as

$$v_z = -(\dot{a}_p/a_p)r_p\phi, \tag{82}$$

where $r_p = r_0 x_p$ is the pancake radius, and

$$\phi \equiv \left[1 - \frac{3}{2}\frac{\sin\theta_p}{(1-\cos\theta_p)^2}(\theta_p - \sin\theta_p)\right] B(1-e^2)^{1/2}. \tag{83}$$

Solving equation (78) for $\theta_p$ leads to the following conclusions: the pancake is still expanding in the $X$-direction if the initial flattening is $z_0/r_0 < 0.44$, for which value we see that the collapse velocity is just reduced to the Hubble velocity in the plane of the pancake $\dot{a}_p/a_p r_p (\phi \approx 1)$ and the radial collapse factor $X_p/a_p = 0.6$. For more extreme initial flattening, the collapse velocity is reduced considerably: by a factor $\phi^{-1} = 2.5$ for $z_0/r_0 = 0.24$ and 6.1 for $z_0/r_0 = 0.12$. The radial collapse factor $X_p/a_p$ in these cases amounts to 0.8 and 0.97, respectively. The dimensionless collapse rate in the plane of the pancake is

$$\frac{\dot{X}_p}{X_p} = \frac{\dot{a}_p}{a_p}\left[1 - A + \frac{3A}{2}\frac{\sin\theta_p(\theta_p - \sin\theta_p)}{(1-\cos\theta_p)^2}\right]. \tag{84}$$

A useful relation is

$$X_p \dot{Z}_p^2 = \frac{3B^2}{2t_0^2}\left\{(1-2/3B)^{3/2} - \frac{\sin\theta_p}{[2(1-\cos\theta_p)]^{1/2}}\right\}^2. \tag{85}$$

Write $\theta_p = 2\pi - \varepsilon$, with $\varepsilon = 0$ for spherically symmetric collapse. Then expanding in powers of $\varepsilon$, equation (78) yields

$$B = \frac{2}{3}\left[1 + \varepsilon^2 (12\pi)^{-2/3}\right] + \theta \varepsilon^4 (12\pi)^{-4/3},$$

and equation (85) reduces to

$$X_p \dot{Z}_p^2 = \frac{2}{3t_0^2}\left[1 - \frac{\varepsilon^2}{8} + \theta\left(\frac{\varepsilon^3}{12\pi}\right)\right]^2.$$

Hence, for moderately small initial eccentricities, this product is independent of the initial degree of flattening. This simplification occurs because $X_p \propto 1 - A/B \propto e^2$, and the dominant contribution to $\dot{Z}_p \approx [3B\dot{a}_p \sin\theta_p / 2^{3/2} (1-\cos\theta_p)^{1/2}](1-2/3B)^{-1/2}$ is $\propto e^{-1}$.

To generalize the preceding to $\Omega < 1$ universes, let us denote by $t_1$ the epoch at which $z = \Omega^{-1}$. For $z \geq \Omega^{-1}$ an open ($\Omega < 1$) universe is well approximated by the flat Einstein-de Sitter behavior, but for $z < \Omega^{-1}$ the evolution is curvature-dominated, as in equations (40), (41). Any inhomogeneities of large amplitude that have turned around both the $X$ and $Z$ axes before $z \approx \Omega^{-1}$ will evolve exactly as in the $\Omega = 1$ case since their dynamics become independent of the background for $z < \Omega^{-1}$. However, axes which have not turned around by a redshift of $\Omega^{-1}$ will be affected by the change in background expansion. For example, suppose the $Z$-axis turns around when $z > \Omega^{-1}$. It will continue to collapse independently of the background for $z \leq \Omega^{-1}$, but if the $X$-axis has not turned around before $z \approx \Omega^{-1}$, it can become frozen into the background expansion which is proceeding so rapidly that collapse need never occur along the $X$-axis. Notice also that ellipsoids do not have to possess negative binding energy in order to collapse down one axis. Even though $Z \to 0$, the $X$-axis may continue to expand indefinitely, $X \to \infty$, or at least until other physical processes, like fragmentation, intervene. Assume that at $t_p$ the universe is open, the density parameter at this epoch being denoted by $\Omega_p$. Consequently, we infer that

$$t_1/t_p = \Omega_p. \tag{86}$$

If $t_1 \ll t_p$, it follows moreover that

$$a_p = \left(\frac{t_1}{t_0}\right)^{2/3}\left(\frac{t_p}{t_1}\right) = \Omega_p^{-1/3}\left(\frac{t_p}{t_0}\right)^{2/3}. \tag{87}$$

Equation (78) generalizes to

$$1-\frac{2}{3B}=\Omega_p^{1/3}\left(\frac{1-\cos\theta_p}{2}\right)\left[\frac{3}{4}(\theta_p-\sin\theta_p)\right]^{-2/3}, \tag{88}$$

and equation (80) becomes

$$\dot{Z}_p=-\frac{3B}{2}\dot{a}_p\left\{1-\frac{2}{3B}-\Omega_p^{1/3}\frac{\sin\theta_p}{1-\cos\theta_p}\left(\frac{3}{4}\right)^{1/3}(\theta_p-\sin\theta_p)^{1/3}\right\}. \tag{89}$$

Equation (21) can be rewritten as

$$\frac{\dot{X}_p}{X_p}=\frac{\dot{a}_p}{a_p}\left\{1-A\left[1+\Omega_p^{1/2}\frac{\sin\varepsilon}{(1-\cos\varepsilon)^{1/2}2^{1/2}}\left(1-\frac{2}{3B}\right)^{-3/2}\right]\right\}, \tag{90}$$

in fair agreement (at least for a limited range of values of $\varepsilon$ or $e$ and $\Omega_p$) with the numerical interpolation given by White and Silk (1979):

$$\frac{\dot{X}_p}{X_p}=\frac{\dot{a}_p}{a_p}\left\{1-1.1\Omega_p^{0.5}\left[(1-e^2)^{-1/2}-1\right]^{-1.3}\right\}. \tag{91}$$

At turn-around, the derived expressions for $\theta_i$ and $\delta_i$ generalize to

$$1-\frac{2}{3B}=\frac{\sin\theta_i}{1-\cos\theta_i}\left[\frac{3}{4}\Omega_i(\theta_i-\sin\theta_i)\right]^{1/3}, \tag{92}$$

$$\delta_i=\frac{(3\pi/4)^2a(2/3B)(2/3A)^2(\theta_i-\sin\theta_i)^2(1-\cos\theta_i)^{-3}8/\pi^2\Omega_i}{\left[1-3\sin\theta_i(\theta_i-\sin\theta_i)/(1-\cos\theta_i)^2\right]\left[1-3\sin\theta_i(\theta_i-\sin\theta_i)/(1-\cos\theta_i)^2(2/3A-1)/(1-2/3B)\right]^2}. \tag{93}$$

As in the spherical case, the density contrast at turn-around is enhanced by a factor $\sim\Omega_i^{-1}$ relative to that for the flat cosmological model.

Finally, it is of interest to compare these results for homogeneous spheroidal inhomogeneities with the Zel'dovich approximation for pancake formation, which White and Silk (1979) show can be written in the form

$$X=a\left[1-(a-1)\frac{3}{10}A\delta\right], \tag{94}$$

$$Z=a\left[1-(a-1)\frac{3}{10}B\delta\right]. \tag{95}$$

These expressions are valid for an Einstein-de Sitter universe: the generalization to $\Omega<1$ is straightforward, but does not concern us here. The numerical factor is chosen so that equations (94) and (60) coincide in the linear regime when $a\gg1$ (or $t\gg t_0$). One infers from equations (94) and (95) that, for example, equation (79) is unchanged and $\dot{Z}_p=-\dot{a}_p$, leading to a collapse velocity on a scale $R_p$ of

$$V_z=-R_p(\dot{a}_p/a_p)(1-e^2)^{1/2}(1-A/B)^{-1}. \tag{96}$$

As before, the collapse velocity can be reduced appreciably for sufficiently great initial flattening.

One should also note that the initial conditions, which specify $\delta$ and eccentricity, are equivalent to specifying $\delta$ and an appropriate velocity perturbation. For example, one can show from equations (94) and (95) that (with $H\equiv\dot{a}/a$)

$$\frac{\Delta H}{H}\equiv(\dot{Z}-\dot{X})/\dot{a}=-\frac{3}{5}a(B-A)\delta. \tag{97}$$

Since $\Delta H/H\propto a$, as is the rate of growth of the density contrast in the linear regime, it follows that any anisotropic velocity perturbation, such as a plane-wave perturbation, is equivalent to specifying an initial eccentricity $B-A\approx$

$(\Delta H/H)_0 \delta^{-1} \approx 1$. An additional minor point to note in making numerical comparisons is that the velocity field in the homogeneous spheroidal perturbation is of the form $v_z = z_0 \dot{Z}(t)$. Hence, a more relevant value of $v_z$ may be the volume-averaged mean-square value: $\langle v_z^2 \rangle^{1/2} = 3^{-1/2} v_z$.

## V. A SIMPLE RELATIVISTIC COLLAPSE

In §§ I–II we have examined the Newtonian dynamics of a self-gravitating ellipsoid embedded in a background universe which expands at a rate determined by Einstein's equations. It is well known that the special case of a spherical dust inhomogeneity can be described by an exact solution of the Einstein equations due to Tolman (Tolman 1934; Bondi 1947) and the results of linear perturbation theory can be easily recovered (Silk 1977).

If one neglects the effects of the cosmological background and treats the inhomogeneity as a separate universe, then the behavior of aspherical dust inhomogeneities can be traced using some general relativistic solutions found by Szekeres (1975). These exact solutions of Einstein's equations describe the evolution of anisotropic and inhomogeneous dust clouds, ($p=0$) possessing no spatial symmetries (they have no Killing vectors; Bonnor, Sulaiman, and Tomimura 1977). They are determined by spacetime metrics of the form (Bonnor and Tomimura 1976)

$$ds^2 = dt^2 - Q^2(x, y, z, t)\, dx^2 - R^2(t)\left(dy^2 + h^2(y)\, dz^2\right), \tag{98}$$

where

$$Q = AR + T \tag{99}$$

and $A(x, y, z)$, $T(x, t)$, $h(y)$ are spatially varying quantities determined by the Einstein equations. The function $R(t)$ satisfies the Friedmann equation

$$2R\ddot{R} + \dot{R}^2 = -k; \qquad k = 0, \pm 1, \tag{100}$$

and $h(y)$ satisfies

$$\frac{d^2 h}{dy^2} = -kh. \tag{101}$$

The solutions fall into two classes. One generalizes the Tolman models away from spherical symmetry by allowing each spherical shell of dust to have a different center. The other class generalizes the homogeneous but anisotropic Kantowski-Sachs (K-S) universes (Kantowski and Sachs 1966) to provide models of strongly anisotropic inhomogeneities. The particular solution of interest to us is of closed K-S type where the metric functions of equations (99), (100), and (101) have the form

$$A = [\sigma(x) \cos z + \nu(x) \sin z] \sin y + \omega(x) \cos y, \tag{102}$$

$$R = 2K \sin^2 \frac{\theta}{2}, \qquad t = K(\theta - \sin\theta), \tag{103}$$

$$T = \beta(x)\left[\frac{\theta}{2} \cot\left(\frac{\theta}{2}\right) - 1\right] + \mu(x) \cot\frac{\theta}{2}, \tag{104}$$

$$8\pi G\rho = \frac{6KA - \beta(x)}{QR^2}, \tag{105}$$

where $K>0$; $0<t<2\pi K$; $0 \le x \le 2\pi$; $0<y<\pi$, and $0<z<\pi$.

In the $(y, z)$-plane the model expands and then recollapses exactly as a closed Friedmann universe. Along the $x$-axis the model at first collapses and then expands out to infinity as $R \to 0$. The solution (45–48) reduces to the homogeneous isotropic Friedmann and the homogeneous, anisotropic K-S solutions in special cases:

$$\text{Friedmann: } \sigma = \omega = \beta = \mu = 0; \qquad \nu = 1; \tag{106}$$

$$\text{K-S: } \sigma = \nu = \omega = 0; \qquad \beta, \mu \text{ constant}. \tag{107}$$

In general, one of the five arbitrary functions $\beta$, $\mu$, $\sigma$, $\nu$, $\omega$ can be transformed to $\pm 1$, leaving the dynamics determined by the four remaining functions of one variable and the constant $K$. The model is physical[1] when we choose

$$\beta < 0, \qquad \mu > 0, \qquad \pi\beta + \mu \leq 0. \tag{108}$$

We can now compare the evolution of a dust ball described by equations (98)–(105) with that of a ball described by the Friedmann solution (eq. [106]). A comparison of physical quantities at the same time instant will effectively describe the behavior of an anisotropic inhomogeneity embedded in a Friedmann background.

The $y, z$ axes reach a maximum when $\dot{R}=0$ at $\theta=\pi$, and at that moment the orthogonal scale factors are

$$R(\pi) = 2K, \tag{109}$$

$$Q(\pi) = 2AK - \beta, \tag{110}$$

$$\dot{Q}(\pi) = -K\left(\mu + \frac{\pi\beta}{2}\right) \geq \frac{K\pi\beta}{2}. \tag{111}$$

We can use our coordinate freedom to set $\beta$ constant.

We can easily calculate the density contrast over the Friedmann background using equation (105). If we denote the Friedmann background density by $\rho_B$ we have at turn-around ($\theta=\pi$)

$$\rho_B(\pi) = \frac{4}{3GK^2\pi^2}, \tag{112}$$

and the density contrast is

$$\left(\frac{\rho}{\rho_B}\right)_{\theta=\pi} = \frac{9\pi^2}{16}(1-x)(1-3x)^{-1}, \tag{113}$$

where $x=\beta/6AK$. When the inhomogeneity is spherical ($x=0$), we recover the standard result. When the anisotropy of the perturbation is very large, $|x| \to \infty$, then the density contrast approaches a smaller value (compare eq. [76]),

$$\underset{|x|\to\infty}{\mathrm{Lt}} \left(\frac{\rho}{\rho_b}\right)_{\pi} = \frac{3\pi^2}{16}. \tag{114}$$

When the anisotropy is small, $|x|<1$, we see an analogous effect to that calculated in §§ II, IV. A smaller turn-around density contrast is required than in the case of spherical inhomogeneities

$$\left(\frac{\rho}{\rho_b}\right)_{\pi} \approx \frac{9\pi^2}{16}(1-2x), \qquad |x|<1. \tag{115}$$

Eventually the imploding $Q$-axis is reversed into expansion. Suppose the $Q$ turn-around ($\dot{Q}=0$) occurs close to the $R$-axis maximum, say when $\theta$ reaches

$$\theta^* = \pi - \varepsilon, \tag{116}$$

where $|\varepsilon|$ is small. We have that $\dot{Q}=0$ when

$$\varepsilon = \frac{\mu - \pi/2}{2AK+\beta}, \tag{117}$$

and at that moment the density contrast is

$$\left(\frac{\rho}{\rho_b}\right)_{\theta^*} = \frac{3(6AK+1)(\theta^* - \sin\theta^*)^2}{16Q(\theta^*)\sin^4\frac{\theta^*}{2}} \approx \frac{9\pi^2}{16}\left(1+2x-\frac{3\varepsilon\mu x}{2}-\frac{8\varepsilon}{3\pi}\right)+0(x^2, \varepsilon x),$$

[1] This avoids singularities in $0<t<2\pi K$.

so by equation (117)

$$\left(\frac{\rho}{\rho_b}\right)_{\theta*} \approx \frac{9\pi^2}{16}(1-10x). \tag{118}$$

Thus, equations (115) and (118) give the overdensities relative to the Friedmann background at the turning points of the expansion in the $x$ and $y$-$z$ dimensions. The spherically symmetric results are obtained when $x \equiv 0$.

To study the final stages of the pancake collapse, $t \to 2\pi K$, $R \to 0$, $Q \to \infty$, it is convenient to change the time variable to

$$\tau \equiv 2\pi K - t. \tag{119}$$

Suppose the Szekeres dust cloud begins at $\theta = \pi$, then the initial conditions for the pancake collapse are

$$R_0 = R(\pi) = 2K, \tag{120}$$

$$Q_0 = Q(\pi) = 2KA - \beta, \tag{121}$$

at $\tau = \pi K$. The initial eccentricity, $e_0$, is given by

$$e_0^2 \equiv 1 - R_0^2/Q_0^2 \tag{122}$$

$$= 1 - A^{-2}(1-3x)^{-2}. \tag{123}$$

As $\tau \to 0$, the metric (98) approaches

$$ds^2 = dt^2 - \left(\frac{4K}{3\tau}\right)^{2/3} dx^2 - \left(\frac{9K\tau^2}{3}\right)^{2/3}(dy^2 + \sin^2 y\, dz^2), \tag{124}$$

and the density diverges as

$$\rho = \frac{6A - \beta K^{-1}}{24\pi G\tau}. \tag{125}$$

It is interesting to compare the form of this collapse to the Newtonian ellipsoids. Hutchins (1976) has found that numerical studies of irrotational collapsing ellipsoids can be modeled by a relation of the form

$$\frac{\dot{\rho}}{\rho} = \lambda(3\pi G\rho)^{1/2}, \tag{126}$$

where

$$\lambda = 10^{0.12}(1-e_0)^{1.41}\left(\frac{\rho}{\rho_0}\right)^{1/2}, \tag{127}$$

and $e_0$, $\rho_0$ are the initial eccentricities and densities, respectively. For the collapse of a sphere we have $\lambda = 2/3$. For the Szekeres collapse we have

$$\lambda_{SZ} = \left(\frac{\rho}{6\rho_0}\right)^{1/2} \frac{(1-x)^{-1/2}(1-3x)^{-1/2}}{-3x}, \tag{128}$$

where $\rho_0 = \rho(\pi)$. Therefore, when $|x|$ is small, we have

$$\Lambda_{SZ} \approx \frac{1}{\beta}\left(\frac{\rho}{6\rho_0}\right)^{1/2}\left[1 - \frac{(1-e_0^2)^{-1/2}}{A}\right], \quad |x| < 1, \tag{129}$$

and when anisotropy and inhomogeneity is large,

$$\lambda_{SZ} \sim \frac{1}{\beta\sqrt{2}}\left(\frac{\rho}{\rho_0}\right)^{1/2}, |x| \to \infty. \tag{130}$$

Finally we note that the Szekeres solutions possess straightforward Newtonian analogs (Lawitzky 1980). None of our results are consequences of general relativistic effects alone.

## VI. SUMMARY

We have examined the evolution of nonspherical inhomogeneities in an expanding Friedmann universe. This generalizes earlier analytic studies away from the restrictive and unrealistic assumption of spherical symmetry during the nonlinear stages of gravitational binding and collapse. Specifically, we followed the evolution of a self-gravitating dust ellipsoid embedded in a Friedmann background. The evolution of the density contrast and shape anisotropy were calculated when the rotating ellipsoid is expanding with the cosmological background. We discover that deviations from spherical symmetry are preserved by the expansion of the universe. When the ellipsoid becomes nonlinear it partially decouples from the background expansion and collapses anisotropically to a pancake configuration first identified by Lin, Mestel, and Sha (1965). We generalize their analysis to include the effects of the expanding background universe and provide simple analytic formulae for the density contrast at turn-around, the epoch of pancaking, and the infall velocity anisotropy at pancaking. We also give an exact relativistic description of nonspherical gravitational collapse.

Various connections with models of galaxy and cluster formation are indicated. One is to kinematic determinations of $\Omega$ from the local Hubble flow in the Virgo supercluster and to generalize the nonlinear spherical model of Silk (1974). Perhaps the most important is to the adiabatic fluctuation theory of galaxy formation in the presence of massive neutrinos. The pancake theory of galaxy formation has been in disfavor largely because of the excessive fluctuations predicted in both the matter and radiation distributions on scales above the damping mass. Massive neutrinos provide a timely release from these quandaries, although the recent detection of a quadrupole moment in the cosmic microwave background radiation may pose a new problem (Silk and Wilson 1981). One major conceptual difficulty in accepting the pancake theory with massive neutrinos is that it becomes difficult to see how galaxy halos can form. For if the pancake collapse results in the neutrinos acquiring random motions appropriate to that of a rich cluster of galaxies, the neutrinos will have too much kinetic energy to form galaxy halos. We have shown that the collapse velocity at pancake formation can be reduced by a substantial factor in the presence of initial anisotropy relative to the velocity acquired in spherically symmetric free-fall. In the presence of neutrinos, one species of which has a mass of $\sim 30$ eV, the minimum half-thickness of adiabatic fluctuations is $\sim 10(1+z)^{-1}$ Mpc (Bond, Efstathiou, and Silk 1980). Collapse along this axis effectively yields the minimum collapse velocity: the pancake radius can be up to a factor $\sim 10$ larger according to our analysis. It seems likely that an effective value of the collapse velocity (averaging over the pancake volume) can easily be reduced to several hundred km s$^{-1}$ for a pancake on the scale of the Local supercluster. If galaxies form at pancake collapse, it seems possible for them to acquire substantial halos of neutrinos, although no detailed dynamical calculations have been performed. One might also speculate that halo formation would tend to be inhibited in more isotropic superclusters, where higher velocities may have been attained by the collisionless component.

We also discuss the likely origin and evolution of significant asphericities in the baryon distribution of the universe and show how adiabatic and isothermal perturbations may differ in this respect. Such primeval anisotropic structures may be inevitable in the primordial baryon distribution if initial protogalactic perturbations are isothermal in character. These asymmetries might explain the presence of developed nonspherical structures of supergalactic dimensions as vestigial remnants of chaotic conditions in the very early universe at the time of baryon generation.

This work was supported in part by the Miller Institute for Basic Research (J. D. B. and J. S.), Christ Church, Oxford and the Department of Astrophysics, Oxford University (J. D. B.). The research of J. S. was supported in part by NSF grant 79-15244. Some of this work was completed when J. S. was an S.R.C. senior visitor at the Department of Astrophysics, Oxford University. We acknowledge helpful discussions with J. R. Bond and S. D. M. White.

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