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HELIUM-SHELL FLASHING IN LOW-MASS STARS AND PERIOD CHANGES IN MIRA VARIABLES

P. R. WOOD AND D. M. ZARRO
Mount Stromlo and Siding Spring Observatories, Research School of Physical Sciences, Australian National University

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ABSTRACT

Helium-shell flashes have been studied in stars with a total mass M of 0.8, 1.0, 2.0, and 3.0 $M_\odot$ and hydrogen-depleted core masses M_c in the range 0.53–0.9 $M_\odot$. Particular attention was focused on the variation in surface luminosity resulting from the flash, since the peak surface luminosity during the flash cycle plays an important role in determining the termination point of asymptotic giant branch evolution. It was found that both the peak in surface luminosity at the flash (L_P) and the quiescent luminosity maximum (L_Q) occurring between each shell flash increase linearly with core mass. However, L_P and L_Q are shown to be independent of total stellar mass in the range of masses studied because envelope convection does not penetrate far enough into the star. Formulae are given for the dependence on M_c of L_P, L_Q, the interpulse period, and the length of the interval during which the surface luminosity at the flash exceeds the quiescent maximum. Approximate formulae for the time dependence of the surface luminosity are also given. Rates of period change in the Mira variables R Hya, R Aql, and W Dra are compared with rates of change predicted by theoretical flash calculations and, as a result, constraints on the luminosity of these stars are derived via the theoretical relation between luminosity and rate of change of luminosity. The luminosities thus derived for R Hya and R Aql are compared with luminosities derived from pulsation theory.

Subject headings: stars: interiors — stars: long-period variables — stars: pulsation

I. INTRODUCTION

Low- and intermediate-mass stars ($M \lesssim 9\ M_\odot$) evolving up the giant branch for the second time are known to have thermally unstable helium-burning shells (Schwarzschild and Harm 1965; Wiegert 1966) which periodically undergo brief phases of fierce burning (shell flashes), thereby causing rapid changes in surface luminosity. Between each flash, where the star spends most of its lifetime, the helium shell remains quiescent while the hydrogen-burning shell provides most of the energy output of the star.

There are a number of ways in which the second or asymptotic giant branch (AGB) phase of evolution may be terminated. The termination mechanisms fall into two distinct categories: (1) gradual reduction of the mass of the hydrogen-rich envelope to less than 0.002 $M_\odot$ owing to consumption by the hydrogen-burning shell and stellar-wind mass loss (Kwok, Purton, and Fitzgerald 1978), or (2) rapid ejection of the hydrogen-rich envelope by a dynamical instability of the envelope (Lucy 1967; Paczyński and Ziolkowski 1968; Roxburgh 1967), by relaxation oscillations associated with fundamental mode pulsation (Smith and Rose 1972; Wood 1974; Tuchman, Sack, and Barkat 1979) or by radiation pressure ejection of the whole envelope (Faulkner 1970; Sparks and Kutter 1972). Termination mechanisms of the first kind will operate most significantly in the quiescent interflash phase since this is where each AGB star spends most of its lifetime. Theoretical calculations show that ejection mechanisms of the second kind are encountered when the luminosity of the star exceeds some critical value. Therefore, if the surface luminosity during a helium-shell flash exceeds the quiescent luminosity maximum preceding the flash, the ejection mechanism will be initiated during the flash peak phase.

In the giant branch evolutionary scenarios of Wood and Cahn (1977) and Tuchman, Sack, and Barkat (1979), the rapid ejection mechanism plays a crucial role in terminating AGB evolution. It is therefore necessary to know the height of the luminosity pulse produced by a helium-shell flash in order to establish accurately the endpoint of AGB evolution. Results of existing calculations of helium-shell flashes are confusing with regard to the height of the surface luminosity peak produced during the flash. For example, shell flash calculations of Schwarzschild and Harm (1967), Sweigart (1971, 1973), Gingold (1974), Christy-Sackmann and Paczyński (1975), Paczyński (1977), and Schonberner (1979) all show a surface luminosity peak at the shell flash which is higher than the quiescent luminosity maximum, whereas in the calculations of Weigert (1966), Iben (1975), and Gingold (1975) the flash luminosity peak does not exceed the quiescent luminosity maximum. Since these two groups of calculations involve overlapping ranges of total mass and core mass, there is no immediately apparent reason for the difference in flash height between the two groups. However, some possibilities are discussed later in the light of the present calculations.

The confusing picture presented above led Wood and Cahn (1977) (hereafter referred to as WC) and Tuchmann, Sack, and Barkat (1979, hereafter TSB) to make totally different assumptions about the height of the surface luminosity flash peak. The peak was assumed to be less than the quiescent maximum by WC, while TSB assumed that the peak rose above the quiescent maximum to a height which increased strongly as the total mass *decreased.* The aim of the present study is to provide a more accurate picture of AGB evolution by studying the systematic variation in helium-shell flash characteristics with total mass and core mass. This should allow better quantitative predictions to be made by models such as those of WC and TSB.

II. COMPUTATIONAL DETAILS

The calculations were performed with a modified version of the Mount Stromlo evolution code (Faulkner 1968; Faulkner and Wood 1972) in which the energy flux was defined at zone centers rather than zone boundaries in order to prevent numerical instabilities (Sugimoto 1970). The complete star was included in the Henyey models (i.e., no envelope was fitted at the outer boundary of the core) since gravitational-internal energy production in the envelope could, in principle, alter the surface luminosity significantly during a fast shell flash. In some models, gravitational-internal energy production was neglected in the outer layers with $T < 2.5 \times 10^4$ K because of troubles with convergence, but test calculations showed that this omission did not significantly alter the height of the surface luminosity peak of the shell flash.

Radiative opacities were derived from tables of Cox and Stewart (1970), conductive opacities from Hubbard and Lampe (1969), and nuclear reaction rates from Fowler, Caughlan, and Zimmerman (1975). An envelope abundance $(X, Z) = (0.68, 0.02)$ was assumed. Convection was treated using the mixing-length theory as detailed by Böhm-Vitense (1958) with a mixing length of one pressure scale height. Provision was made to allow overshooting of convection from the envelope down into the hydrogen-depleted regions although, for the low-mass stars considered, very little mixing of this type occurred. No scheme to artificially shift the hydrogen-burning shell between flashes was used since, as shown by Iben (1976), such schemes can significantly alter the nature of the shell flash. The nuclear reactions $^{14}\text{N}(\alpha, \gamma)^{18}\text{F}(\beta^+\nu)^{18}\text{O}(\alpha, \gamma)^{22}\text{Ne}$ were not generally included in the calculations, but occasional tests showed that the behavior of the surface luminosity and the extent of convective mixing were only slightly altered by including these reactions (see Fig. 2).

Each series of models was initiated by constructing a star with a hydrogen-depleted, degenerate core of given mass and an envelope mass of the size necessary to make up the required total stellar mass. The mass separation of the two nuclear burning shells was chosen to be slightly larger than in evolving double-shell source models so that there was an initial evolutionary phase during which the He shell provided a large fraction of the energy output of the star. During this phase, time steps were relatively large since the helium shell is thicker than the hydrogen shell, and the core had a chance to come to a state near thermal equilibrium. Often a single strong shell flash occurred in this early phase, after which the luminosity steadily increased until gentle oscillations in the nuclear output occurred which grew in amplitude to become fully developed shell flashes after about 15 cycles. In most models about 200 mesh points were used, and a total of about 400,000 models was computed.

III. COMPUTATIONAL RESULTS

a) Behavior of Surface Luminosity during a Typical Flash Cycle

The behavior of the surface luminosity during a typical fully developed flash cycle is shown in detail in Figure 1. The main, quiescent part of the cycle is spent in steady burning by the predominant hydrogen shell as the surface luminosity recovers to a maximum *A* just before onset of a helium-shell flash. Immediately following the flash, the hydrogen-burning shell is rapidly extinguished, causing a sudden drop in surface luminosity which is halted at *B* when the energy released by helium burning begins to diffuse out of the core to the stellar surface. The pulse in

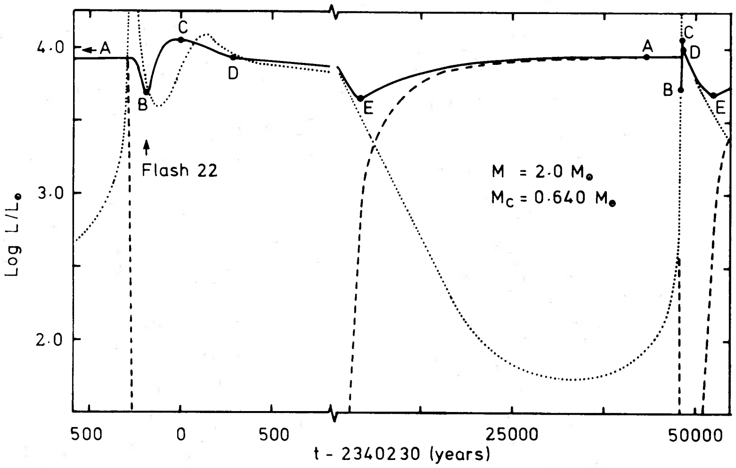

FIG. 1.—Variation with time during a flash cycle of surface luminosity (*continuous line*), hydrogen-burning luminosity (*dashed line*), and helium-burning luminosity (*dotted line*).

surface luminosity resulting from diffusion of energy released by the primary helium-shell flash gives rise to the maximum at *C*. Further diffusion of residual energy from the primary flash provides a significant part of the surface luminosity until *D*, after which the luminosity is provided mainly by the exponentially decaying helium-burning shell until the minimum at *E*. All flash cycles have a second weaker flash of the helium shell similar to that shown in Figure 1. The secondary flash usually produces a small hump of height $\delta \log L < 0.02$ on the exponential tail from *D* to *E*, particularly for core masses $\lesssim 0.7\ M_\odot$. The rise in luminosity from *E* occurs as the hydrogen shell again takes over as the main energy producer in the star, while the helium burning continues its exponential decay for $\sim 40\%$ of the flash cycle.

b) *Variation in Flash Behavior with Total Mass*

One of the main aims of the present work was to investigate the variation with total mass M of the behavior of the surface luminosity during the helium-shell flash, since TSB require an increase in the height of the surface luminosity pulse as total mass decreases. There are two ways in which, for a given core mass M_c, a variation in the total stellar mass (or envelope mass) could affect the behavior of the surface luminosity: (1) by absorbing or releasing different amounts of internal-gravitational energy during the hydrostatic readjustment occurring during the shell flash and (2) by altering the effective outer boundary conditions on the core, these conditions being the result of integration from the stellar surface through the envelope to the core boundary.

If the envelope is to absorb a significant amount of the energy output of the core, the thermal time scale for the envelope must be longer than the time scale on which the structure of the envelope is changing at the particular phase of the flash in question. For the envelopes considered here ($M_{\rm env} \lesssim 2.4\ M_\odot$), the thermal time scales are of the order of a few years, and the only part of the flash cycle rapid enough for energy absorption in the envelope to be important is around the minimum at *B* in Figure 1. Calculations carried out on stars of 0.8, 1.0, 2.0, and 3.0 $M_\odot$ show that this minimum is less deep in stars of increasing envelope mass which have longer thermal time scales. However, the height of the surface luminosity pulse and its shape at luminosities above the quiescent luminosity maximum is unaltered by envelope mass in the stars studied with $M < 3\ M_\odot$ and $M_c \lesssim 0.9\ M_\odot$. Thus, except for the minimum around *B*, the envelope is close to thermal and hydrostatic equilibrium and normal instability mechanisms (pulsation, dynamical instability) can be expected to apply (although the luminosity-core mass relation will be different from that pertaining to the quiescent evolutionary phase). Significant envelope energy absorption may occur in stars with more massive envelopes (longer envelope thermal time scales) or with larger core masses (shorter flash time scales).

The second way in which the envelope can affect the flash (and also the quiescent evolution) is by altering the temperature and pressure at the core boundary, say, at the top of the hydrogen-burning shell. From the point of view of the core, the envelope provides two boundary conditions, and these will, in general, be a function of envelope mass. However, these two boundary conditions turn out to be independent of envelope mass in the low-mass stars considered here. The important feature of these stars is the existence of an extensive radiative zone between the bottom of the convective envelope and the hydrogen-depleted region, across which temperature increases by at least an order of magnitude and pressure by four orders of magnitude. In this radiative region, the opacity is very nearly constant (since it is predominantly due to electron scattering) and, in addition, the region contains very little mass so that $M_r = M_{\rm core} = {\rm constant}$ is a very good approximation. Integrating the equations of hydrostatic equilibrium and radiative transfer under these conditions with an equation of state for a perfect gas plus radiation gives the two relations (Schwarzschild 1958; Eggleton 1967; Paczyński 1970*b*):

$$P = 16\pi\, \sigma G M_c T^4 / 3\kappa L\,, \tag{1}$$

and

$$T = \beta G M_c \mu m_{\rm H} / 4kr\,, \tag{2}$$

where the symbols have their usual meanings. The effect of the outer part of the envelope on these relations is contained in constants of integration which are negligible provided T and P change significantly through the radiative zone. From the last requirement, it follows that the existence of a significant radiative zone between the hydrogen-depleted region and the convective envelope is a sufficient condition for the core evolution to be independent of the envelope mass.

The width of the radiative zone above the core depends on total mass, luminosity, and mixing length, l. Examples of dependence of the temperature T_b at the base of the convective envelope on total mass and luminosity are given by Scalo, Despain, and Ulrich (1975) and Paczyński (1977), although agreement between the two sets of calculations is poor (the present calculations agree very well with those of Paczyński). There is an increase in T_b (decrease in the width of the radiative zone) with M in each of the sets of calculations so that in heavy stars, the core mass-luminosity relation for quiescent evolution and the behavior of shell flashes may be different from that in stars of smaller total mass but similar core mass. In fact, Iben (1977) and Iben and Truran (1978) give a core mass-luminosity relation derived from stars of total mass $\lesssim 7\ M_\odot$ which does involve total mass. Finally, it should be noted that there is some uncertainty in the value of T_b owing to its dependence on l. Test calculations with l in the range 0.5–2.0 pressure scale heights showed that T_b increases rapidly with l initially but then levels off at $T_b \sim 4 \times 10^6$ K for the range of total and core masses studied. Thus the present models lie in the region where the core boundary conditions (1) and (2) are valid and core evolution is independent of envelope mass.

c) *Flash Calculations for a 2 $M_\odot$ Star*

In view of the above results, variation with core mass of all quantities associated with a flash cycle except depth of

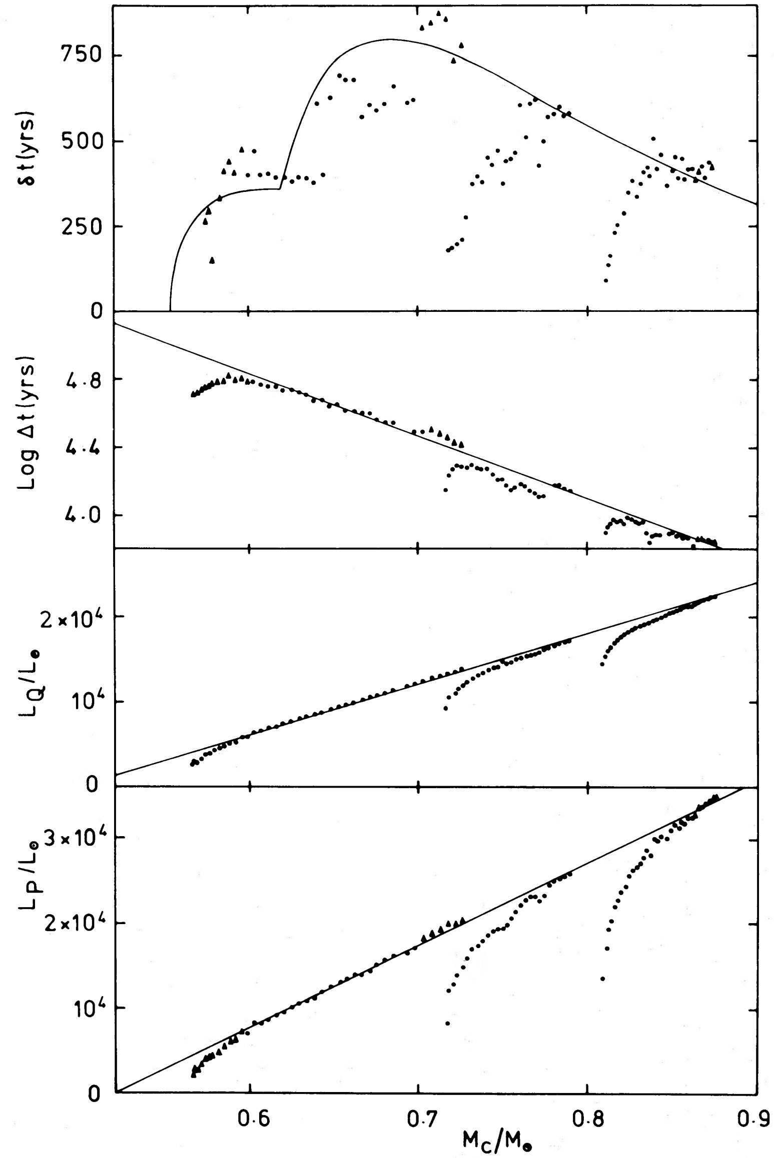

FIG. 2.—Some properties of individual shell flashes plotted against core mass M_c at the flash. L_P is the peak surface luminosity at the shell flash; L_Q is the quiescent luminosity maximum prior to the flash; Δt is the time since the preceding flash; and δt is the time interval during which the surface luminosity at the flash exceeds the preceding quiescent luminosity maximum. The continuous lines are derived from fitting formulae given in the text. The nuclear reactions $^{14}N(\alpha, \gamma)\,^{18}F(\beta^+\nu)\,^{18}O(\alpha, \gamma)\,^{22}Ne$ were included only in flashes represented by a triangular symbol.

the minimum at B can be found from computations on a star of given total mass. A large number of the present calculations were performed on a 2 $M_\odot$ star, and variation of selected properties with core mass for these calculations is shown in Figure 2. The properties of each flash were obtained from three series of models, with initial core masses of 0.53, 0.7, and 0.8 $M_\odot$. For each series, the flash amplitude grew rapidly at first, but, by about 15 flashes, full amplitude for the current core mass had been reached and the flash amplitude increased steadily with core mass thereafter. (For the flashes with $M_c < 0.775\ M_\odot$ in the series with initial core mass of 0.7 $M_\odot$, and $M_c < 0.862$ in the series with initial core mass of 0.8 $M_\odot$, the nuclear reaction rates were taken from a preliminary version of the formulae in Fowler, Caughlan, and Zimmerman [1975]. The flash characteristics obtained with these rates differ slightly from those of the later flashes obtained with the published reaction rates.)

From Figure 2 it is clear that there exists a linear relation between core mass and quiescent luminosity maximum L_Q, this relation being

$$\frac{L_Q}{L_\odot} = 59{,}250\left(\frac{M_c}{M_\odot} - 0.495\right). \qquad (3)$$

The coefficient for dependence of L_Q on M_c is the same as that given by Paczyński (1970*a*) and Uus (1970). However, because of the different zero point, luminosities given by equation (3) are larger than the Paczyński-Uus values which were obtained from flash-suppressed models. The present results are also consistent with a linear relation between log Δt (where Δt is the time interval between flashes) and M_c, as previously found by Paczyński (1975) and Becker and Iben (1980). A formula fitting the present results is

$$\log \Delta t = 3.68\left(1.914 - \frac{M_c}{M_\odot}\right). \qquad (4)$$

The slope for the dependence of Δt on M_c differs from the slope obtained by the above authors who collectively considered a wider range in M_c, but generally their values of Δt were derived from the first few flashes of a series or from estimates of Δt. As can be seen from Figure 2 the first flashes in each sequence of models tend to have Δt values considerably smaller than the value corresponding to full amplitude flashes at the same core mass. The present formula runs through the upper envelope of the compilation of interflash time scales plotted by Becker and Iben (1980) and seems appropriate for fully developed flashes. During the latter 75% of the period of recovery to quiescent maximum, the time dependence of the luminosity is approximately represented by

$$L(t) = L_Q\left[1 - \exp\left(6.5 - 9.5\frac{M_c}{M_\odot} - 7\frac{t}{\Delta t}\right)\right],$$

where t is measured from the time of the preceding flash.

Two quantities of particular interest in relation to studies of envelope ejection are the height L_P of the surface luminosity peak at the flash and the time interval δt during which the luminosity at the flash peak exceeds the quiescent luminosity maximum. The value of L_P for fully developed flashes increases linearly with core mass and is described by the relation

$$\frac{L_P}{L_\odot} = 97{,}000\left(\frac{M_c}{M_\odot} - 0.52\right). \qquad (5)$$

From Figure 2, it appears that for fully developed flashes δt lies predominantly in the range 400–800 yr over the range of core masses studied. A formula for the dependence of δt on M_c is given below in the light of an examination of the shape of the surface luminosity peak.

The behavior of the surface luminosity with time is displayed in Figure 3 for fully developed flashes at a number of core masses. Each flash peak is made up of two components: the first component is a pulse that includes the rise to maximum, the peak, and the initial part of the decline and is due to the escape from the core of accumulated energy released by the powerful first flash of the helium shell; the second component consists of a slower exponential decay which occurs when the energy output from the star is close to equilibrium with the output of the helium-burning shell. At the lower core masses, a small bump occurs on the exponential decay as a result of the second smaller flash of the helium shell. If $t = 0$ is assumed to correspond to the time of surface luminosity maximum, then the surface luminosity when greater than the quiescent maximum is given approximately by

$$L(t) = L_P \exp\left(-t^2/\tau_P{}^2\right), \quad t < t_1, \qquad (6a)$$

$$= L_E \exp\left(-t/\tau_E\right), \quad t > t_1, \qquad (6b)$$

where the pulse time scale τ_P (in years) is given by

$$\log \tau_P = 3.61 - 1.75\frac{M_c}{M_\odot},$$

and the exponential decay parameters are given by

$$\frac{L_E}{L_\odot} = 97{,}000\left(\frac{M_c}{M_\odot} - 0.54\right),$$

and

$$\log \tau_E = 5.525 - 3.0\frac{M_c}{M_\odot}.$$

The time t_1 is defined as the value of $t > 0$ at which the values of $L(t)$ given by equations (6a) and (6b) are equal. The time interval δt during which the surface luminosity exceeds the quiescent maximum can be obtained as a function of M_c from equations (3) and (6) and is shown in Figure 2 together with values obtained from model calculations. For low core masses ($M_c < 0.62\ M_\odot$), only the pulse peak (eq. [6a]) extends above the quiescent maximum, and δt increases rapidly with core mass at first but then levels off as the time scale of the pulse decreases. As the core mass increases beyond 0.62 $M_\odot$, the contribution from the decaying helium shell (eq. [6b]) becomes large enough to affect the surface luminosity at values above the quiescent maximum, causing a second rapid increase in δt. The decrease in time scales for still larger core masses eventually produces a decline in δt.

It should be noted here that at least some of the above formulae will be composition dependent. For example, the quiescent luminosity maximum is a function of envelope composition, and the surface luminosity during the flash phase may also depend on envelope composition. Calculations in progress are designed to investigate composition dependence of flash and quiescent evolution and results will be reported in a subsequent paper.

d) Mixing

In the present calculations, there was no mixing of helium-burning products to the surface via the dredge-up mechanism (Iben 1975; Sugimoto and Nomoto 1975), and there was no significant contact between the intershell convection zone and hydrogen-containing material. Both of these results are in agreement with other flash calculations for low-mass ($M \lesssim 3\ M_\odot$) stars having core masses less than 0.9 $M_\odot$ (Sweigert 1974; Gingold 1974, 1975; Paczyński 1977; Schonberner 1979). In the models with $M_c > 0.66\ M_\odot$, envelope convection extended in a continuous zone down to the outer part of the extinct hydrogen-burning shell during the phase of the surface luminosity peak. There were also one or two small

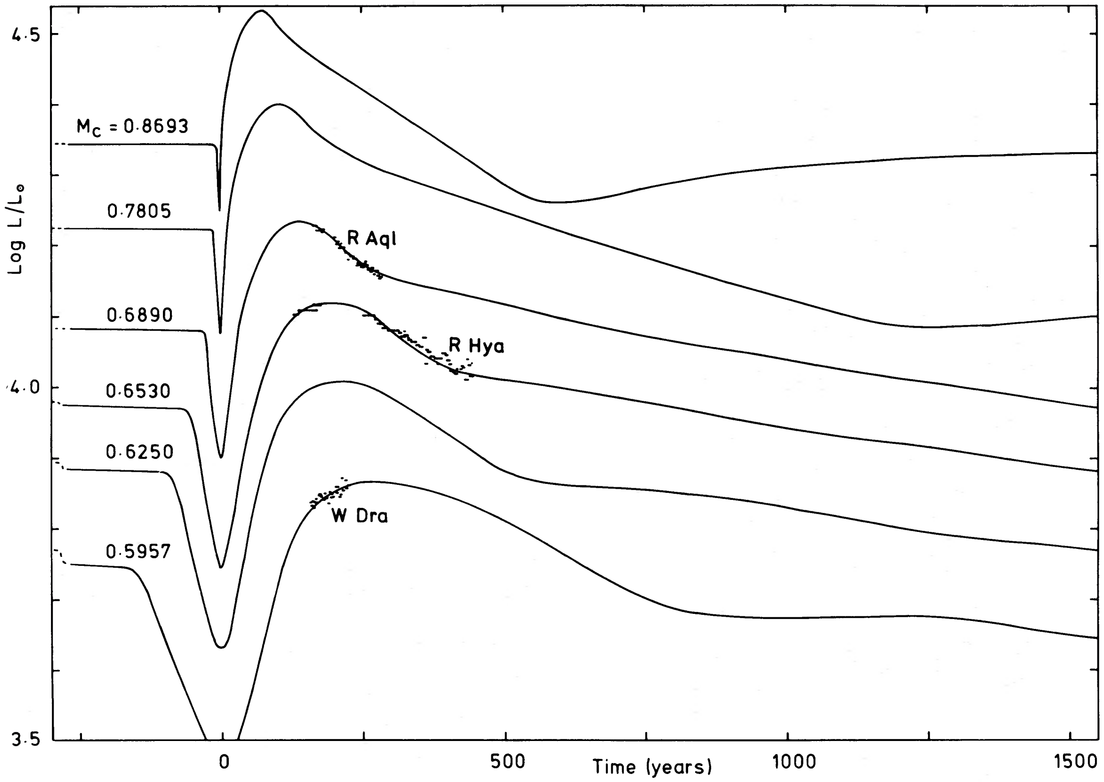

FIG. 3.—Behavior of surface luminosity L as a function of time for a number of different values of core mass M_c. The variation of the surface luminosity of the Mira variables R Hya, R Aql, and W Dra as deduced from changes in period is also shown (see text for details).

isolated convection zones below the main envelope convection zone. Contact between the envelope convection zone and the extinct hydrogen-shell region resulted in a fractional enrichment in envelope ^{14}N at the rate of $\sim 2 \times 10^{-4}\ M_\odot/M_{\rm env}$ per flash. This rate of enrichment remained constant to within the cycle-to-cycle accuracy of the flash calculations for envelope masses from 0.07 $M_\odot$ to 2.3 $M_\odot$ but occurred only for core masses $M_c > 0.66\ M_\odot$. The above very small amount of mixing could cause a significant increase in envelope nitrogen abundance only for stars with small envelope masses or large core masses. As an example, a 1 $M_\odot$ star which ejects its envelope with a core mass of 0.8 $M_\odot$ has a ratio of final to initial nitrogen abundance of 1.04.

IV. PERIOD CHANGES IN MIRA VARIABLES AND THE HELIUM-SHELL FLASH

The Mira variables lie on the giant branch at the large luminosities characteristic of the AGB phase so that helium-shell flashes would be expected to influence the pulsation of these variables via surface luminosity and radius changes. It has been known for many years that the Mira variables R Hya and R Aql are undergoing continuous period changes (Plakidis 1932; Sterne and Campbell 1937), and Wood (1975*b*) has suggested that these period changes may result from the luminosity pulse produced by a helium-shell flash. Unfortunately, most Mira variables do not have regular periods but show random cycle-to-cycle variations in period of a few percent as well as abrupt period changes of $\sim 2\%$ at intervals of 10–50 yr (Plakidis 1932; Vasiljanovskaja, Kiselev, and Kiseleva 1970; Nadjenko 1974; Wood 1975*b*). Both of these effects make the search for the continuous period changes which should result from helium-shell flashing difficult, even though dates of maximum have been accumulated for over 100 yr for many Mira variables.

In Figure 4, $O - C$ curves of dates of maximum for a selected number of Mira variables are shown to illustrate the typical characteristics of these curves. Data were obtained from the tabulations of Cannon and Pickering (1909), Campbell (1926), and Campbell (1955) together with more recent data kindly supplied by J. A. Mattei of the AAVSO and F. M. Bateson of the Variable Star Section, RASNZ. In total, the $O - C$ curves of 48 well-observed Mira variables were examined, with 23 of these variables having over 100 yr of recorded data. The decreasing periods of R Aql and R Hya are very obvious in Figure 4, but an interesting new candidate for a continuously changing period is W Dra, whose period seems to be increasing. However, continuing observations of W Dra are desirable to confirm the changing period, since a glance at the $O - C$ curve of T Cep shows how an apparent increase in period over 70 yr can suddenly be reversed by an abrupt period change. S Her is a good example of a Mira showing abrupt period changes; with the present data set, these abrupt changes would mask any continuous period change on a time scale longer than ~ 3000 yr. T Her also shows abrupt period changes, but these are small and period change on a time scale shorter than 8×10^4 yr can be ruled out (this star has the most stable period of any examined).

In order to relate period changes in R Hya, R Aql, and W Dra to helium-shell flash calculations, it is necessary to convert the rate of period change into a rate of change in luminosity. For this purpose, the three relations,

$$C = P\left(\frac{M}{M_\odot}\right)^a\left(\frac{R}{R_\odot}\right)^{-b} = \text{constant}\,, \qquad (7)$$

$$L = 4\pi\,\sigma R^2 T_{\text{eff}}^{\;4}\,, \qquad (8)$$

and

$$\log\frac{L}{L_\odot} = \alpha - \beta\log T_{\text{eff}}\,, \qquad (9)$$

are required. Equation (7) is a period-mass-radius relation for pulsation; equation (8) is the definition of effective temperature; and equation (9) represents the position of giant branch stars in the H-R diagram. Combining equations (7), (8), and (9) gives the desired relation,

$$\log\frac{L}{L_\odot} = \frac{2}{b(1+4/\beta)}\log P(\text{days}) + \left(\frac{4}{4+\beta}\right) \times \left\{\alpha + \beta\log\left[\left(\frac{4\pi\sigma R_\odot^{\,2}}{L_\odot}\right)^{1/4}\left(\frac{M}{M_\odot}\right)^{a/2b} C^{-1/2b}\right]\right\}. \qquad (10)$$

From this equation it is clear that the rate of change of luminosity for a given rate of change of period depends only on β and b and is independent of stellar mass M, generalized pulsation constant C, and the parameters a and α. Since the rate of change of luminosity during a shell flash is a unique function of the core mass (or surface luminosity), a determination of this rate of change should, by comparison with flash calculations, give an accurate estimate of the core mass and absolute bolometric luminosity of the star involved.

Unfortunately, the values of β and b are not well known. The value of β represents the slope of the line in the H-R diagram along which a star moves during a helium-shell flash. Since the deviation of helium-shell flashing stars from the quiescent giant branch is small (in the present calculations, the deviation was to the cool side of the quiescent giant branch for luminosities above quiescent maximum, and of magnitude $\delta T_{\text{eff}} \lesssim 50$ K), β may be obtained from the slope of the quiescent giant branch in the H-R diagram. However, this slope seems to vary significantly: converting Eggen's (1975) old disk giant branch in the (M_{bol}, $R_K - J_K$)-plane to the (M_{bol}, log T_{eff})-plane using the temperature calibration of Ridgway *et al.* (1980) gives $\beta = 16.67$, while using Johnson's (1966) temperature calibration gives $\beta = 10.66$; values of β in the range $10 < \beta < 20$ are obtained for the old galactic cluster M67 and the globular clusters 47 Tuc, M71, M13, and M3 using the observationally determined giant branches of Cohen, Frogel, and Persson (1978) and Frogel, Persson, and Cohen (1979, 1980*a*, *b*). The old disk giant branch of Eggen will be adopted here, giving $\beta = 16.67$ and $\alpha = 62.5$. The value of b was determined from the linear, nonadiabatic pulsation calculations of Fox and Wood (1981) for luminous giant stars. For the fundamental mode, values of b in the range $2.0 < b < 2.5$ are indicated for a wide range of conditions; for the first overtone $1.5 < b < 2.5$, where the low values of b occur for AGB stars with $M \gtrsim 2\ M_\odot$ and the larger values of b

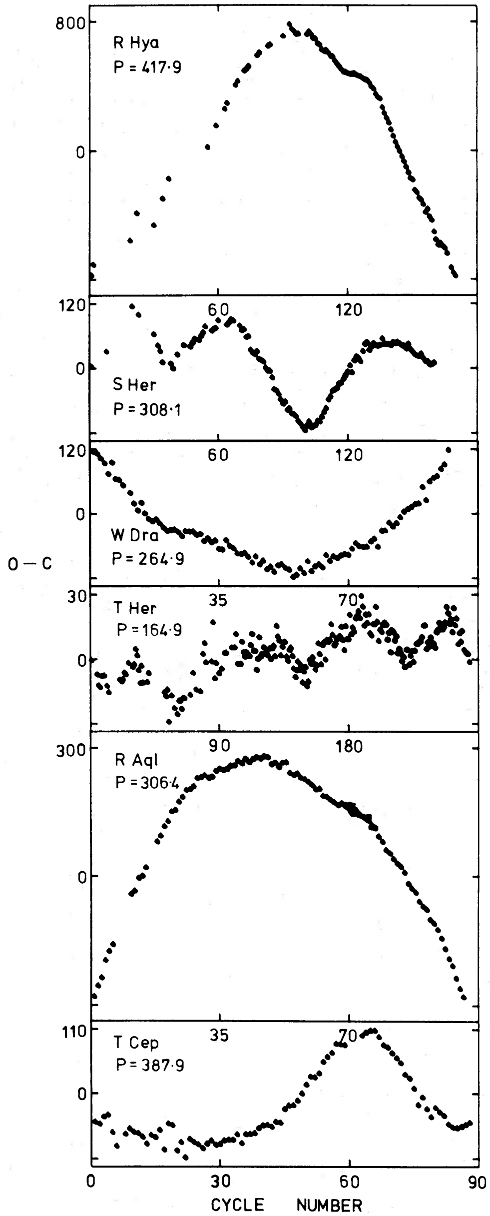

FIG. 4.—Observed dates (*O*) minus computed dates (*C*) of maximum vs. cycle number for a number of Mira variables. Computed dates of maximum are derived by assuming the constant value of the period shown in the figure. Periods and dates are in days. Some early observations of R Hya and R Aql were omitted because of the difficulty of determining the cycle number unambiguously.

occur for stars with $M \lesssim M_\odot$ and small envelope mass. The values $a = 0.74$, $b = 2.0$, and $C = 5.75 \times 10^{-3}$ (days) are adopted here for the fundamental mode. For the first overtone, the values $a = 1.0$, $b = 2.0$, and $C = 2.8 \times 10^{-3}$ (days) are adopted, although it must be stressed that these are mean values only and that there is a significant variation with mass and input parameters in the values of a, b, and C for the first overtone.

By fitting only the observed *time scale* for luminosity change to the computed time scales in Figure 3, it should be possible to derive accurate core masses and present luminosities for the three Miras without assuming values for α, a, M, or C. In the remainder of this section, such a comparison of observational and theoretical time scales is used to constrain the possible values of the parameters of R Hya, R Aql, and W Dra, and particularly the mode of pulsation which is at present in dispute (Wood 1975*a* favors the first overtone while Hill and Willson 1979 prefer the fundamental).

Using equation (10) and the above values of b and β, $\log L/L_\odot$ was plotted against time in Figure 3 for the Mira variables R Hya, R Aql, and W Dra. In plotting the Mira data, mean periods over three cycle intervals (or longer if successive known dates of maximum are more than three cycles apart) are calculated and lines are drawn at the corresponding $\log L/L_\odot$ over the time interval covered. The zero of time for each Mira is adjusted to give a best fit to the theoretical curves; the zero point in $\log L/L_\odot$ for each variable is discussed below.

For R Hya, four very old (1662–1708) and valuable dates of maximum are known which show that the period was increasing up to ~ 250 yr ago but that it has been declining since then. The long series of data available means that the time of luminosity maximum is well defined and, as a result, a fit between theoretical and observational time scales is possible for only a relatively small range in M_c and $L/L_\odot$. For the given values of b and β, the R Hya time scale is consistent with core masses in the range $0.61 \lesssim M_c/M_\odot \lesssim 0.66$. The core mass ($0.653\ M_\odot$) corresponding to the plotted curve lies near the upper limit of acceptability and was chosen to give a present bolometric luminosity for R Hya close to the value -5.3 to -5.5 mag deduced for R Hya from *VRIJHKL* observations and a distance modulus of 6.1 obtained from a common proper motion companion and membership of the Hyades group (Eggen 1975; Robertson and Feast 1980).

In the case of R Aql, the maximum luminosity does not occur in the interval covered by the observational data (the observations coincide with part of the luminosity decline). The deduced rate of period change is consistent with a wide range in core mass $0.64 \lesssim M_c/M_\odot \lesssim 0.88$, where in the upper part of this range the observed time scale fits the exponential decay portion of the surface luminosity peak rather than the decay of the pulse as at lower core masses. In Figure 3, the data for R Aql are shown arbitrarily plotted at a core mass of $0.689\ M_\odot$.

For W Dra, no useful constraints can be put on the values of M_c and $\log L/L_\odot$ from time scale considerations because of the short length of time spanned by the observations. The existing data are shown plotted in Figure 3 at an arbitrary core mass of $0.5957\ M_\odot$.

A notable feature of the period changes in R Hya and R Aql is that the time scale for period decrease in R Hya (950 yr) is longer than that for R Aql (650 yr) even though R Hya has the longer period. In the theoretical calculations, all time scales decrease with core mass so that if all Miras had identical values of α, β, M, a, b, and C, a longer period would be associated with a shorter time scale of period decrease. A difference between R Hya and R Aql in one or more of the parameters α, β, M, a, b, or C therefore seems likely.

Unfortunately, the uncertainties discussed above in the parameters α, β, a, b, and C mean that a determination of the mode of pulsation of R Hya and R Aql from the present results is difficult (the mass M is a further uncertainty, with plausible values lying in the range $0.8 \lesssim M/M_\odot \lesssim 2.5$, e.g., Feast 1963). With the values of α and β adopted above and with values of a, b, C, and M in the ranges described, the luminosities derived directly from pulsation theory via equation (10) can be made consistent with the luminosities derived above from time scales of period change when either fundamental or first overtone pulsation is assumed. However, the luminosities given by equation (10) for the fundamental mode lie at the low end of the ranges permissible from time scale fitting; this means that if the effective temperatures of the Mira variables are ~ 2600 K, as suggested by lunar occultations measured at infrared wavelengths (Robertson and Feast 1980), rather than ~ 3200 K as given by equation (9) with the adopted values of α and β, then the fundamental mode will cease to be consistent with the luminosities required from time scale considerations.

V. THE EFFECT OF HELIUM-SHELL FLASHING ON AGB EVOLUTIONARY SCENARIOS

In the models of Wood and Cahn (1977) and Tuchman, Sack, and Barkat (1979) a typical low-mass star evolving up the AGB eventually becomes luminous enough to pulsate as a Mira variable, after which it continues to evolve up the AGB until it becomes so luminous that it ejects its envelope (however, the lowest-mass stars may actually lose all their envelope mass via stellar winds before or during the Mira phase). The luminosities for onset of Mira pulsation and envelope ejection are functions of mass as well as luminosity, so that a region of intrinsic Mira instability is defined in the ($\log L/L_\odot$, M)-plane.

The nonlinear pulsation calculations of TSB were designed to define the region of Mira pulsation in the ($\log L/L_\odot$, M)-plane. Having found a pulsation zone, TSB then attempted to reproduce the observed (period, number density)-relation for Miras by considering evolution of AGB stars through this zone. In order to obtain a reasonable fit between observed and predicted periods and number densities, TSB were forced to assume that all Miras have mass $M \gtrsim 2\ M_\odot$, and the reason they offered for this was that the helium-shell flash in stars with $M \lesssim 2\ M_\odot$ could propel a star below the Mira region in the ($\log L/L_\odot$, M)-plane during quiescent evolution right through to the ejection region. In this way, AGB stars with $M \leq 2\ M_\odot$ would never be seen as Miras except in the very short time interval while they were traversing the Mira region during shell flashes. Since the Mira pulsation region found by TSB is of approximately constant width in log $L/L_\odot$ for all masses, the only way in which stars with $M < 2\ M_\odot$ could be selectively removed from the Mira population as described above is for the pulse height (in units of log L_P/L_Q) at the helium flash to increase with *decreasing* mass. The present calculations show that this is not the case. In fact since the Mira strip found by TSB in the (log $L/L_\odot$, M)-plane has a larger onset luminosity (or core mass) for larger total mass, stars of increasing mass entering the Mira region during quiescent evolution will have larger core masses and therefore larger values of log L_P/L_Q from equations (3) and (5). Therefore, the requirement of TSB that stars with $M < 2\ M_\odot$ are not Miras cannot be justified on theoretical grounds. It should also be mentioned here that on observational grounds, Feast (1963) estimated that Miras have masses $M \lesssim 2\ M_\odot$. Finally we note that the present results do not assist in resolving the main problem with the raw theoretical Mira period distribution of TSB (i.e., too many short-period Miras), since the effect of shell flashing is to selectively remove the *longer*-period Miras from the intrinsic instability region.

In the models of WC, a strip of Mira pulsation in the (log $L/L_\odot$, M)-plane was sought empirically, and the effect of helium-shell flash luminosity peaks on termination of AGB evolution was neglected. A region in the (log $L/L_\odot$, M)-plane was obtained which reproduced the observed period distribution and number density of Miras, and this region was identified as the region of intrinsic Mira instability. However, in the light of the present results this identification must be modified to allow for the effect of the helium-shell flash. Now, the empirically deduced region of pulsation must correspond to the low-luminosity part only of the intrinsic pulsation region such that stars occupying the empirical region in their quiescent state do not reach the intrinsic envelope ejection line during the helium-shell flash. From the present results, the intrinsic ejection line in the (log $L/L_\odot$, M)-plane lies approximately 0.15 in log $L/L_\odot$ above the ejection line given in WC. It has been implicitly assumed here that the surface luminosity peak at the shell flash is of sufficient duration for envelope ejection to occur.

VI. DISCUSSION AND CONCLUSIONS

One of the main aims of this study was to investigate the variation with M and M_c of the height above quiescent luminosity maximum of the surface luminosity peak during the helium-shell flash cycle. It has been shown that for the low-mass ($M \leq 3\ M_\odot$) stars considered, the core evolution and surface luminosity are independent of total mass (except for a very short interval corresponding to the first luminosity minimum at the onset of the flash) because of the existence of a substantial radiative zone above the region where nuclear burning occurs. Both the peak surface luminosity L_P at the flash and the quiescent luminosity maximum L_Q are found to increase linearly with core mass. For the envelope composition (X, Z) = (0.68, 0.02) used in these calculations, the flash peak exceeds the quiescent maximum for $M_c > 0.559\ M_\odot$ and log $L_P/L_Q \rightarrow 0.21$ as the core mass increases. Finally, comparison of the extensive period changes in the Mira variables R Hya and R Aql with period changes predicted to occur as a result of the helium-shell flash shows good agreement, and constraints are placed on the luminosity of these stars independent of any direct observational determination. Consistency between observed and predicted rates of period change in R Hya and R Aql can be obtained within the large existing uncertainties in pulsation theory for either first overtone or fundamental mode pulsation.

Finally, the present results are compared with other calculations in order to explain why TSB found an apparent increase in log L_P/L_Q as total mass decreased. Part of the reason for this finding is that existing low-mass ($M \leq M_\odot$) flash calculations (Schwarzschild and Harm 1967; Sweigart 1971, 1973; Gingold 1974) were done with an envelope metal abundance $Z = 0.001$. Reducing Z decreases the quiescent luminosity for a given core mass and calculations in progress indicate that the flash surface luminosity peak increases as Z is lowered. Both these effects act to increase log L_P/L_Q as Z decreases. Thus the large values of log L_P/L_Q found in the above-mentioned calculations should be attributed to the low value of Z, not the low total mass. (In addition, the calculations of Schwarzschild and Harm [1967] and Sweigart [1971] were done without including radiation pressure in the gas physics. Since radiation pressure supplies a large fraction of the total pressure near the core boundary, its neglect could have a significant effect on the computational results.) Of the remaining low-mass flash calculations for which published values of log L_P/L_Q are available, the values log L_P/L_Q obtained by Paczyński (1977) for a star with $M = 3\ M_\odot$ and (X, Z) = (0.7, 0.03) and by Schonberner (1979) for stars of initial mass 1.0 $M_\odot$ and 1.45 $M_\odot$ and (X, Z) = (0.74, 0.02) agree well with the values predicted on the basis of the present results. However, the values of log L_P/L_Q given by Gingold (1975) for $M = 2\ M_\odot$ and (X, Z) = (0.68, 0.02) have smaller values of log L_P/L_Q than those obtained here (this result may be due to the shell shifting used by Gingold), while the values given by Christy-Sackmann and Paczyński (1975) for $M = 3\ M_\odot$ and (X, Z) = (0.70, 0.03) have larger values of log L_P/L_Q, at the corresponding core masses. Existing flash calculations (Weigert 1966; Iben 1975) involving heavier stars ($M \geq 5\ M_\odot$) do not show a surface luminosity pulse which exceeds the quiescent luminosity maximum. As noted earlier, the surface luminosity in these heavier stars may not be independent of envelope mass since the core is not isolated from the convective envelope by a substantial radiative zone and, furthermore, the greater envelope mass may absorb more of the luminosity pulse than will be the case in a star of low mass. Good agreement between the surface luminosity variations in the present calculations and those of Weigert (1966) and Iben (1975) is therefore not necessarily expected, and log L_P/L_Q

indeed seems to be smaller in stars with $M > 5\ M_{\odot}$ than in lower-mass stars having the same core mass.

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P. R. Wood and D. M. Zarro: Mount Stromlo and Siding Spring Observatories, Private Bag, Woden, P.O. ACT 2606, Australia