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PULSAR TIMING. IV. PHYSICAL MODELS FOR TIMING NOISE PROCESSES

J. M. CORDES
Department of Astronomy, Cornell University

AND

GEORGE GREENSTEIN
Department of Astronomy, Amherst College
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ABSTRACT

This paper is the fourth in a series studying the timing noise process in the isolated pulsars. We quantitatively analyze eight mechanisms that have been proposed to account for the observed phenomenon. Of these we find that five face severe difficulties in meeting the observations. These are the notions that timing noise arises from: (1) starquakes; (2) the random pinning and unpinning of vortex lines ("hard superfluidity"); (3) accretion from the interstellar medium; (4) vortex annihilation at the outer boundary of the superfluid; and (5) pulse-shape changes. We find the remaining three mechanisms to be severely constrained, although not ruled out, by the observations. These are the notions that pulsar timing noise arises from: (6) crust breaking by vortex pinning; (7) the response of the star to a series of heat pulses; and (8) luminosity-related torque fluctuations. We conclude by noting those areas of future research that we feel would be most worthwhile.

Subject heading: pulsars

I. INTRODUCTION

It has long been known that pulsars do not rotate with perfect regularity. Two types of nonsecular behavior have been reported. The first is the glitch, or period jump—a sharp and sudden increase in pulse repetition rate accompanied by an increase in slowing-down rate. The second is a good deal more subtle and initially gave rise to a certain amount of confusion in the literature. It was not until the pioneering work of Boynton *et al.* (1972) that its true nature was correctly identified: it is *noise in the pulsar clock*.

It is with this noise that we are concerned in this paper. We quantitatively analyze eight mechanisms that might be operating in pulsars which could give rise to the observed noise. Of these we find that five face great difficulties in meeting the observational constraints, difficulties sufficiently severe that, in our view at least, they cannot be considered viable candidates. These include the notions that pulsar timing noise arises from: (1) a continuous and erratic quaking of the crust or core of the star; (2) the random pinning and unpinning of vortex lines as they migrate through the crust ("hard superfluidity"); (3) accretion from the interstellar medium; (4) the sudden annihilation of vortex lines at the outer boundary of the superfluid; or (5) pulse-shape changes. We find the remaining three mechanisms to be severely constrained, although not ruled out, by the observations. These are the possibilities that timing noise arises from: (6) the response of the superfluid interior to a continuous and erratic series of heat pulses; (7) the unpinning of vortex lines via crust breaking; or (8) luminosity-related torque fluctuations.

In this paper we are concerned with the theoretical implications of the data, with an eye toward understanding the physical nature of the underlying mechanism responsible for timing noise. We are hampered in this analysis by the fact that *no observation has unequivocably isolated an instance of the underlying event.* We emphasize that the observed structure in the timing residuals, even that structure which can be seen over short time scales, arises from the combined effect of a large number of unresolved events. Pulsar timing noise is precisely analogous in this regard to Brownian motion. It is well known that a particle undergoing Brownian motion occasionally exhibits sharp and apparently discrete jumps in position. But such a jump does not arise from a single collision with an unusually energetic atom. Rather, in any resolvable time interval, the particle suffers a very large number of collisions to the right and a similarly large number to the left, and the observed motion arises from small differences between the two. In the same way, structure in pulsar timing residuals reveals, not the underlying mechanism, but statistical fluctuations in this mechanism.

It is an appealing possibility, although by no means required by the observations, that in certain pulsars we have in fact succeeded in time-resolving the underlying events: they are the glitches. It is obviously significant in this regard that glitches are frequency steps and that the Crab pulsar, which exhibits glitches, also exhibits frequency noise. We can imagine a spectrum of frequency steps, with the many small steps giving rise to the noise process and the occasional large step visible as a glitch. This possibility is discussed in § II below, along with a summary of the observational constraints and a comparison of timing noise with length-of-day variations of the Earth.

In § III we discuss the subtleties of our methodology, which is a comparison of the measured variance of the rotational phase with that predicted by various models.

In §§ IV–VII we turn to a consideration of a number of physical models for the noise process. We establish in each case an estimate of the theoretically predicted magnitude of the noise strength and use the data to test the model. Finally, § VIII summarizes our results and discuss further observational and theoretical studies which would be helpful in future research.

II. DISCUSSION OF OBSERVATIONAL RESULTS

a) *Summary of Pulsar Timing Noise*

The data upon which we based our study have been gathered in an eight-year effort by the University of Massachusetts pulsar observing group. These data are presented in Paper I of this series (Helfand *et al.* 1980) and a method of analysis is described in Paper II (Cordes 1980). Paper III (Cordes and Helfand 1980) applies this method to a sample of fifty pulsars. The central results of Paper III, which we take as input to our study, are as follows:

1. Timing noise is widespread in the pulsars. Indeed, there is every reason to believe that it is present to some degree in every pulsar.
2. Timing noise is correlated with period derivative $\dot{P}$, but weakly correlated with period P. Therefore, timing noise is correlated with $P/\dot{P}$, the spin-down age of the pulsar, and with $(P\dot{P})^{1/2}$, proportional to the pulsar magnetic field.
3. Timing noise is not correlated with height above the galactic plane, radio luminosity, or with pulse-shape changes.

For 11 out of the 50 pulsars studied, the data re sufficiently good to allow a detailed analysis of the statistics of the noise process. In this group, by far the greatest number show noise whose statistical properties are those of a random walk in the rotation rate (as does the Crab pulsar [Boynton *et al.* 1972; Groth 1975]). We will refer to this as frequency noise (FN). However, not all pulsars exhibit this type of noise: two are undergoing *a random walk in the slowing-down rate* (slowing-down noise, SN) and two *a random walk in the phase* (phase noise, PN). The noise process is idealized as consisting of a series of steps in the phase ϕ, frequency ν, or frequency derivative $\dot{\nu}$. The step sizes have mean-square values $\langle\Delta\phi^2\rangle$, $\langle\Delta\nu^2\rangle$, or $\langle\Delta\dot{\nu}^2\rangle$ and are Poisson distributed in time with an average rate R. For the three types of noise, the *strength* is defined as

$$S_{\rm PN} \equiv R\langle\Delta\phi^2\rangle\ ,$$
$$S_{\rm FN} \equiv R\langle\Delta\nu^2\rangle\ ,$$
$$S_{\rm SN} \equiv R\langle\Delta\dot{\nu}^2\rangle\ . \qquad (1)$$

Measured values of the strength parameters are listed in Table 1.

Table 1 also lists upper and lower limits to the magnitude $\Delta\phi$, $\Delta\nu$, or $\Delta\dot{\nu}$ of the underlying events. The upper limits were obtained by combining the measured strengths with a lower limit on the rate R, in turn obtained by noting that the rate must exceed one event per minimum block length analyzed in Paper III. The lower limits were obtained as follows. If the underlying frequency steps in the FN pulsars all have the same sign (i.e., if the noise process has nonzero mean amplitudes), then the noise contributes to the slowing-down rate a term $\dot{\nu}_{\rm noise} = R\langle\Delta\nu\rangle$. We require that this be small compared to the observed slowing-down rate of the pulsar $\dot{\nu}$:

TABLE 1

RANDOM WALK ANALYSIS OF 11 PULSARS

Pulsar	Strength	Magnitude Limits[a]	
	Phase Noise Type		
1133+16	$1.5 \pm 0.9 \times 10^{-14}$ s^{-1}	$\Delta\phi$ lies between 1.8×10^{-14} and	7.5×10^{-4} cycles
2217+47	$1.6 \pm 0.9 \times 10^{-13}$	8.6×10^{-14}	2.5×10^{-3}
	Frequency Noise Type		
0329+54	$7.0 \pm 4.0 \times 10^{-27}$ Hz^2 s^{-1}	$\Delta\nu$ lies between 1.7×10^{-12} and	4.9×10^{-10} Hz
0531+21	$6.6 \pm 3.0 \times 10^{-23}$	1.8×10^{-13}	2.3×10^{-9}
1508+55	$1.0 \pm 0.6 \times 10^{-26}$	1.1×10^{-12}	5.9×10^{-10}
1915+13	$1.1 \pm 0.7 \times 10^{-25}$	5.6×10^{-13}	1.9×10^{-9}
2002+31	$1.0 \pm 0.7 \times 10^{-27}$	5.9×10^{-14}	1.9×10^{-10}
2016+28	$2.0 \pm 1.2 \times 10^{-28}$	4.1×10^{-13}	8.3×10^{-11}
2020+28	$2.0 \pm 1.6 \times 10^{-27}$	1.2×10^{-13}	2.6×10^{-10}
	Slowing-down Noise Type		
0611+22	$1.3 \pm 0.9 \times 10^{-37}$ Hz^2 s^{-3}	$\Delta\nu$ less than 1.8×10^{-15} Hz s^{-1}	
0823+26	$2.0 \pm 1.3 \times 10^{-40}$	8.3×10^{-17}	

NOTE.—Equality holds if all steps have the same sign.

[a] We emphasize that the lower limits only apply if the noise process has a nonzero mean.

$R\langle\Delta\nu\rangle = S/\langle\Delta\nu\rangle \ll \dot{\nu}$, implying $\langle\Delta\nu\rangle \gg S/\dot{\nu}$. Similarly, for PN we require that the contribution $\nu_{\rm noise} = R\langle\Delta\phi\rangle$ of the noise to the rotation frequency be small compared to this frequency. For the SN pulsars, the analogous procedure has been carried out in § IV*c* of Paper III, and we find lower limits on $\Delta\dot{\nu}$ almost precisely equal to the upper limits. Thus, if the noise process in the SN pulsars has nonzero mean, the step size is *determined.*

b) Distribution of Step Amplitudes for the Crab Pulsar

Little is known about the distribution of step amplitudes for the random walks other than values of (or limits on) various moments (such as the random walk strengths and limits on random walk contributions to ν, $\dot{\nu}$, etc.). In Paper II we discussed the extent to which the third moment, which would characterize the asymmetry of the distribution, could be determined. Here we ask whether glitches from the Crab pulsar represent infrequent, large-amplitude events from the tail of the distribution that describes the individual steps of the random walk. The answer, as we shall see, is no and conforms to a similar conclusion made by Groth (1975) on different grounds. Groth determined that the mathematical form of the glitches (a constant plus an exponential; see Table 3) must be different from the frequency perturbations that form the random walk. We now demonstrate that glitches are different from random walk steps because the distribution of frequency steps must necessarily be bimodal.

We consider the constraints that can be put on the distribution of steps from the observables. From the Princeton optical timing data (Groth 1975), we have the quantities $S_{\rm FN} = R_{\rm FN}\langle\Delta\nu^2\rangle = 0.53 \times 10^{-22}$ Hz^2 s^{-1}, $R_{\rm FN} \gtrsim 1d^{-1}$, $\Delta\nu_{1969} = 3 \times 10^{-7}$ Hz, and

$$\Delta\nu_{1975} = 1.2 \times 10^{-6} \text{ Hz},$$

which are, respectively, the random walk strength, the rate, and the magnitudes of the glitches of 1969 September and 1975 June. We also know that no individual frequency steps were observed that are smaller than the observed glitches yet bigger than a "threshold" frequency $\Delta\nu_t \approx 2 \times 10^{-8}$ Hz. As shown in Paper III, a random walk in frequency exhibits apparent (i.e., nonphysical) frequency steps that have a Gaussian probability density function with standard deviation $\sigma_{\Delta\nu} = (S_{\rm FN}\,\Delta t)^{1/2}$, where Δt is a typical sample interval of the measurements. In order to measure confidently a real step in frequency, its amplitude must be greater than $\Delta\nu_t \equiv 3\sigma_{\Delta\nu}$ (i.e., a three-sigma measurement). The quoted value for $\Delta\nu_t$ above is obtained for $\Delta t = 10d$.

We have considered several distributions of frequency steps that are monotonic in $\Delta\nu$. Let $f(\Delta\nu)$ be the distribution of frequency steps $\Delta\nu$, defined such that $\int d(\Delta\nu)f(\Delta\nu) = 1$. The rate at which steps with amplitudes between $\Delta\nu$ and $\Delta\nu + d(\Delta\nu)$ occur is then $R'f(\Delta\nu)d(\Delta\nu)$, where R' is the rate at which all events occur. In terms of the different frequency ranges defined above, the random walk strength and the three rates for random walk steps, the unobserved intermediate-size steps, and the glitches are

$$S_{\rm FN} = R' \int_{-\infty}^{\Delta\nu_t} d(\Delta\nu)f(\Delta\nu)(\Delta\nu)^2\,,$$

$$R_{\rm FN} = R' \int_{-\infty}^{\Delta\nu_t} d(\Delta\nu)f(\Delta\nu)\,,$$

$$R_{\rm int} = R' \int_{\Delta\nu_t}^{\Delta\nu_{1969}} d(\Delta\nu)f(\Delta\nu)\,,$$

$$R_{\rm glitch} = R' \int_{\Delta\nu_{1969}}^{\infty} d(\Delta\nu)f(\Delta\nu)\,. \tag{2}$$

One distribution to consider is a power law of the form:

$$f(\Delta\nu) \propto \begin{cases} \Delta\nu^{-\alpha} & (\Delta\nu_0 \le \Delta\nu \le \Delta\nu_1) \\ 0 & (\text{all other } \Delta\nu) \end{cases}, \tag{3}$$

which has three unknown parameters (α, $\Delta\nu_0$, and $\Delta\nu_1$). Therefore, a total of four parameters must be determined from the four measurements or limits. For $1 < \alpha < 3$ we find, under the constraints $\Delta\nu_0 \ll \Delta\nu_t \ll \Delta\nu_{1969}$, that

$$S_{\rm FN} \approx R'\left(\frac{\alpha-1}{3-\alpha}\right)\left(\frac{\Delta\nu_0}{\Delta\nu_t}\right)^{\alpha-1}\Delta\nu_t^{\,2}\,,$$

$$R_{\rm FN} \approx R'\,,$$

$$R_{\rm int} \approx R'\left(\frac{\Delta\nu_0}{\Delta\nu_{1969}}\right)^{\alpha-1},$$

$$R_{\rm glitch} \approx R'\left(\frac{\Delta\nu_0}{\Delta\nu_t}\right)^{\alpha-1}. \tag{4}$$

A good fit to $S_{\rm FN}$ and $R_{\rm glitch}$ is obtained for $\alpha = 2$ over a range of values of $\Delta\nu_0$ and R' (these are not constrained other than $R' \ge (1d)^{-1}$ because $S_{\rm FN}$ and $R_{\rm glitch}$ depend on R' and $\Delta\nu_0$ in the same way). However, $\alpha = 2$ implies that approximately 30 ($\pm 30^{1/2}$ if steps occur with a Poisson distribution in time) intermediate-sized steps, $\Delta\nu_t \le \Delta\nu \le \Delta\nu_{1969}$, should have been observed in the last 10 years that the Crab pulsar has been observed. Of course, no steps have been observed in this amplitude range, suggesting that glitches and random walk steps are, in fact, part of a bimodal distribution of steps. A distribution steeper than $\alpha = 2$ would allow a fit to the random walk steps such that intermediate-sized steps would occur with a rate $R \lesssim (10 \text{ yr})^{-1}$, but then glitches would be too frequent. Distributions with $\alpha < 1$ and $\alpha > 3$ also do not fit the data. Finally, we have considered an exponential distribution of steps,

$$f(\Delta\nu) = \begin{cases} 1/\langle\Delta\nu\rangle \exp\left(-\Delta\nu/\langle\Delta\nu\rangle\right) & (\Delta\nu \ge 0) \\ 0 & (\Delta\nu < 0) \end{cases}, \tag{5}$$

which also predicts, given a fit for the one parameter $\langle\Delta\nu\rangle$ from $S_{\rm FN}$ and $R_{\rm glitch}$, a large number of intermediate-sized steps.

On the basis of these constraints, we conclude that glitches are phenomenologically different from the frequency perturbations that are responsible for the random walk. Empirically, any such distribution must be bimodal. Combining this result with the conclusion of Groth (1975) and Boynton and Deeter (1980) that random walk steps cannot have the same mathematical form as glitches, we concluded that glitches are caused by a different mechanism than that responsible for random walk steps. The same conclusion holds for the Vela pulsar.

Having settled this we wish to point out a numerical coincidence that appears to lead to the opposite conclusion. Regarding the Crab pulsar glitches as a noise process in its own right, we may define its strength in the usual way, taking R_{glitch} = several per decade and $\delta\nu_{\text{glitch}} = \delta\nu_{1969}$. We obtain a glitch strength the same order of magnitude as the noise strength. We do not understand why this should be so. It is not so for the Vela pulsar. Very likely, it is only a coincidence. But perhaps it is not.

c) *Noise Mixtures*

In Paper III it was found that, given an ample quantity of data, two of the three kinds of random walks we considered could be rejected, whereas one was not inconsistent with the phase residuals of a given pulsar. These results encourage the view that one need not look further for a statistical model for the data. However, it is surprising that "pure" random walks of one kind are sufficient to describe a given pulsar's timing noise, especially since all of the various models we consider below suggest that timing noise should indeed be mixtures of several kinds of random walks. We quantify this point as follows.

In Paper II it was shown that the ensemble-average, mean-square phase calculated over a time span T,

$$\sigma_\phi{}^2(T) = \left\langle \frac{1}{T}\int_0^T dt\phi^2(t) \right\rangle , \tag{6}$$

is

$$\sigma_\phi{}^2(T) = \begin{cases} \frac{1}{2}\ S_{\text{PN}}\, T & (\text{PN}) \\ \frac{1}{12}\ S_{\text{FN}}\, T^3 & (\text{FN}) \\ \frac{1}{120} S_{\text{SN}}\, T^5 & (\text{SN})\,. \end{cases} \tag{7}$$

In paper III strengths for PN, FN, and SN were calculated from estimates of $\sigma_\phi{}^2(T)$ by essentially inverting equation (7), and consistency was demonstrated by determining whether the ratio (e.g., for FN),

$$\begin{aligned} F &= \hat{S}_{\text{PN}}(T_{\max})/\hat{S}_{\text{FN}}(T_{\min}) \\ &= \sigma_\phi{}^2(T_{\max})T_{\max}{}^{-2}/\sigma_\phi{}^2(T_{\min})T_{\min}{}^{-2}, \end{aligned} \tag{8}$$

was consistent with a value of unity.

In order to consider a general mixture of timing noise, we will model the ensemble-average, mean-square phase as

$$\sigma_\phi{}^2(T) = \sum_{j=1}^{M} C_j T^j\,. \tag{9}$$

We do not include a $j = 0$ term because it would correspond to stationary noise in the phase (such as measurement errors due to finite signal-to-noise ratios, etc.), which we assume has already been accounted for. If a pulsar is assumed to have type κ noise (e.g., $\kappa = 3$ for FN), we would estimate a strength using $\hat{S}_\kappa \propto \hat{C}_\kappa = \hat{\sigma}_\phi{}^2(T)T^{-\kappa}$, from which we would obtain a F value

$$F_\kappa = \sum_{j=1}^{M} c_j T_{\max}{}^{j-\kappa} \Big/ \sum_{j=1}^{M} c_j T_{\min}{}^{j-\kappa}\,. \tag{10}$$

If it is found that F_κ is consistent with there being only a type κ random walk, then

$$C_\kappa \gtrsim \sum_{j \neq \kappa} C_j T^{j-\kappa} \tag{11}$$

for both $T = T_{\max}$ and $T_{\min}$. For some models we may suspect there is only one other kind of noise, in which case we have a limit on individual coefficients, $C_j \lesssim C_\kappa T^{\kappa-j}$. Finally, it is clear that whether $T_{\max}$ or $T_{\min}$ should be used depends on the sign of $\kappa - j$. We have, therefore,

$$\begin{aligned} C_j &\lesssim C_\kappa T_{\max}{}^{-|\kappa-j|} \quad (j > \kappa)\,, \\ C_j &\lesssim C_\kappa T_{\min}{}^{|\kappa-j|} \quad (j < \kappa) \end{aligned} \tag{12}$$

as limits on the coefficients.

In Table 2 we give limits on the coefficients for $1 \leq j \leq 5$ for the 11 pulsars analyzed in detail in Paper III.

TABLE 2

LIMITS ON NOISE MIXTURES

Pulsar	Noise Type, k	C_1 (s^{-1})	C_2 (s^{-2})	C_3 (s^{-3})	C_4 (s^{-4})	C_5 (s^{-5})
0329+54	FN, $k = 3$	$10^{-11.0}$	$10^{-18.6}$	...	$10^{-34.6}$	$10^{-43.0}$
0531+21	FN, $k = 3$	$10^{-8.3}$	$10^{-15.2}$	...	$10^{-30.3}$	$10^{-38.5}$
0611+22	SN, $k = 5$	$10^{-7.4}$	$10^{-14.8}$	$10^{-22.1}$	$10^{-29.5}$	...
0823+26	SN, $k = 5$	$10^{-8.1}$	$10^{-16.0}$	$10^{-23.9}$	$10^{-31.8}$	...
1133+16	PN, $k = 1$	...	$10^{-22.2}$	$10^{-30.6}$	$10^{-39.0}$	$10^{-47.4}$
1508+55	PN, $k = 3$	$10^{-10.9}$	$10^{-18.4}$	...	$10^{-34.4}$	$10^{-42.8}$
1915+13	FN, $k = 3$	$10^{-9.7}$	$10^{-17.3}$	...	$10^{-33.1}$	$10^{-41.2}$
2002+31	FN, $k = 3$	$10^{-11.7}$	$10^{-19.4}$	...	$10^{-35.1}$	$10^{-43.3}$
2016+28	FN, $k = 3$	$10^{-12.0}$	$10^{-19.8}$	...	$10^{-36.1}$	$10^{-44.5}$
2020+28	FN, $k = 3$	$10^{-10.8}$	$10^{-18.8}$	...	$10^{-35.0}$	$10^{-43.2}$
2217+42	PN, $k = 1$	...	$10^{-21.2}$	$10^{-29.6}$	$10^{-38.0}$	$10^{-46.4}$

It should be kept in mind that the limits we have derived (eq. [12]) are appropriate for the limit of an infinite number of uniformly spaced samples. However, the errors associated with the application of these formulae to the data are probably no larger than the estimation errors (due to the stochastic nature of the random walks) that would be present even if a large number of uniformly sampled data points were available. Further discussion of noise mixtures can be found in Lamb *et al.* (1978*a*, *b*).

d) *Comparison with Length-of-Day Variations of the Earth*

It is instructive to contrast irregularities in the rotation of the pulsars with those in the only other astronomical object where they have been observed—the Earth. Figure 1 exhibits variations in the length of the day over approximately the past 150 years. The Earth undergoes fluctuations in rotation period of the order $\Delta P/P \approx 10^{-8}$ over a time scale of $\Delta t \sim 10$ yr. Dicke (1966) has argued, on the basis of ancient eclipse observations, that these excursions in period show a strong tendency to return to their starting values; otherwise the derivative of the length of day would not have been as steady as deduced from eclipse records over several millenia. Each event then gives rise to a step in rotational phase of order $\Delta\phi \approx (\Delta\nu)(\Delta t)$. The steps recur at a rate $R \approx 1$ per every 10 yr, implying phase noise in the Earth with a strength $S_{\rm PN} \equiv R\langle\Delta\phi^2\rangle \approx 10^{-17}$ s^{-1}. Among the pulsars, there are two which also exhibit phase noise, although the strength is some three to four orders of magnitude larger in the pulsars than the Earth.

Although the Earth apparently exhibits timing irregularities of the same statistical nature as do the pulsars, the *observational* situation with regard to these irregularities is entirely different. During the time interval shown in Figure 1, the Earth executed some 5×10^4 rotations. A typical pulsar executes this number of rotations in less than a day. If we were to scale arbitrarily the behavior of the Earth to that of the pulsars, then the entire wealth of information contained in Figure 1 would be summarized in a single data point. The converse of this statement, however, is also true: during the 8 years in which pulsar timing noise has been carefully studied, a typical pulsar executes the same number of rotations that the Earth does in approximately 1 million years. That is to say, timing observations of the Earth regularly isolate examples of rotational anomalies but study these anomalies only over relatively short time scales. Pulsar timing observations, on the other hand, have not resolved the underlying anomalies, but are capable of studying their statistical behavior over relatively long time scales. In what follows we will discuss ways in which this latter feature may be able to provide important constraints on the underlying physics.

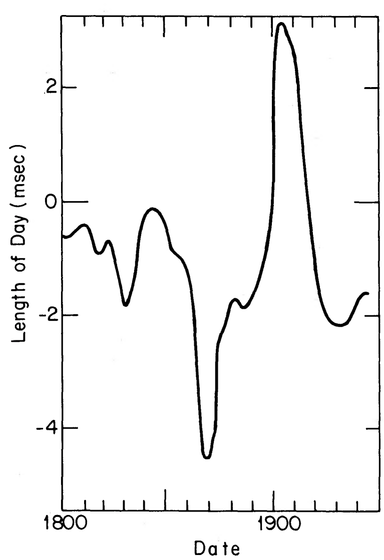

FIG. 1.—Variations in the length of the day as a function of time. From Munk and MacDonald (1960).

III. RANDOM PROCESSES FROM ENSEMBLES OF EVENTS

a) *General Considerations*

The central problem we address in this paper is how different types of idealized processes (e.g., PN, FN, SN) are produced or mimicked by various physical processes. In particular, we consider ensembles of events of the form:

$$\phi(t) = \sum_j a_j \times h(t - t_j) ; \tag{13}$$

and determine the second moment of $\phi(t)$. The statistics of $\phi(t)$ depend on: the shape of $h(t)$ and its time scale W_h (either a pulse width or a rise time) relative to scales that characterize the data; the rate at which events occur, R; whether events occur independently; and the amplitude statistics of a_j. We assume that events occur at independent times, thus allowing ease in computing the second moments (see Appendix). Although it is easy to imagine situations where this assumption is violated (e.g., periodic occurrence of starquakes or other perturbations), our results are unaffected so long as *some* quantity is independent from event to event, such as the amplitude a_j (recall that the textbook approach to a random walk is a strictly periodic sequence of steps with independent amplitudes).

b) *Processes with Overlapping Events*

In calculations that follow, our general procedure is to calculate the rate R necessary to produce the measured strength of the random process and compare R with observational and theoretical constraints on its value. For some physical processes, events can occur independently in, e.g., different regions of the star such that considerable time overlap of events occurs: $RW_h \gg 1$. Examples are localized starquakes, pinning and unpinning of vortices, etc. Other processes, however, may not allow such overlap as a result of there being only one physical entity that can vary. An example of this is the case of radio luminosity variations. On time scales greater than approximately 1 hour, say, the pulse shape is invariable, implying that luminosity variations must occur uniformly over the pulse beam. In the model we consider below, we therefore argue that $RW_h \lesssim 1$ for luminosity variations.

c) *The Time Origin Effect*

In general, the shape of $h(t)$ determines the asymptotic type of random process produced by the ensemble. For example, if $h(t)$ is a pulselike function, then $\phi(t)$ is white noise in the limit $W_h \to 0$. Similarly, random walks in the phase, frequency, or frequency derivative are produced in the limit that $h(t)$ behaves, respectively, as a step function, ramp function, or parabola *over the time span of the data.* If the random process were to commence fortuitously at the onset of data taking ($t = t_d$), then $\phi(t)$ would obviously depend only on events that occur after t_d. Clearly, events have occurred prior to data taking, the consequence being that the type of process determined after t_d depends on events that occur prior, as well as subsequent, to t_d. In the Appendix we discuss this effect in detail. It proves to be difficult to produce a random walk in $\dot{\nu}$ (SN) from perturbations in the frequency that are predicted in most models. SN appears to require generally a random walk in the torque or that t_d (the time that data taking began) is much less than the rise time of frequency perturbations.

To illustrate these comments, consider frequency perturbations of the form

$$\delta\nu(t) = \frac{dh}{dt} = \Delta\nu(1 - e^{-t/\tau})H(t)\,, \qquad (14)$$

where $H(t)$ is the unit step function. If $t_d \ll \tau$ and $T \ll \tau$ (where T is the time span of the data), then $h(t) \propto t^2H(t)$, and the random process will be SN because *all* events are evolving toward their asymptotic form, $h(t) \propto tH(t)$. If $t_d \gg \tau$, however, then events occurring prior to t_d, which have shapes between $t^2H(t)$ and $tH(t)$, will influence the measured statistics; SN will not be produced in the limit $T \ll \tau$.

d) *The Second Moment*

In the Appendix, we show that the second moment of $\phi(t)$ is

$$\langle\phi^2(t)\rangle = \langle\phi(t)\rangle^2 + R\langle a^2\rangle \int_0^t dz h^2(z) \qquad (15)$$

for a random walk commencing at $t = 0$. The net result of the time origin effect is to extend the integration limit from t to $t + W_h$ where, again, W_h is the characteristic width or rise time for $h(z)$. Finally, the quantity we compare with measurements is the integrated variance over an interval $(t, t + T)$:

$$\sigma_\phi^2(t, T) = T^{-1} \int_t^{t+T} dt'[\phi^2(t') - \langle\phi(t')\rangle^2]$$

$$= R\langle a^2\rangle T^{-1} \int_t^{t+T} dt' \int_0^{t'+W_h} dz h^2(z) \qquad (16)$$

IV. PULSE-SHAPE CHANGES

In order to relate arrival-time measurements to rotational phase, it must be assumed that the beam of radio emission is firmly attached to the star. Arrival-time measurements are made (Paper I) by averaging several thousand pulses together and convolving the resultant profile with a reference profile. Such average profiles show no indications of having changed over the last 10 years, suggesting that the radio beam of a pulsar is determined by the strong (10^{12} gauss), frozen-in magnetic field whose diffusion time is expected to be in excess of 10^7 years. Measured profiles deviate slightly from the reference profile because single pulses are generally of different shapes and have an amplitude distribution that may have a long tail. Such deviations decrease with increasing integration time, as has been verified for nine pulsars (Helfand, Manchester, and Taylor 1975). Clearly, the resultant errors in arrival times will be much less than the temporal width of the average profile. Nonrandom, pulse-to-pulse fluctuations will also influence the convergence of a profile to the reference profile. Several pulsars show mode changes wherein the average profile jumps between two distinct shapes, the transition time between modes being less than one pulse period, and the duration of the less frequent mode being several hundred pulse periods. An integration that contains several mode changes will produce errors in arrival times, but there will also be less than the width of the average profile and, moreover, these are expected to have stationary statistics over the 1-year to 8-year spans of available data. Timing noise discussed in Papers I and III has nonstationary statistics (e.g., random walk behavior) and, for some pulsars, implies arrival time variations in excess of the average profile width.

A possible source of nonrandom variation of the average profile concerns the altitude of the emission region. Emission at a given radio frequency may arise from a small radial range well within the velocity-of-light cylinder (see Cordes 1978). If the radius of emission were to vary with time, the rotational phase would vary as $r(t)\nu/c$. It is well known that pulsars exhibit intensity variations on time scales from microseconds to years, and it would not be surprising if the emission radius were also to vary in a correlated way because of, e.g., variations in the growth rate of plasma instabilities that establish coherent radiation. Upper limits on such radial fluctuations can be made because, at a fixed frequency, the average profile width varies as $r^{1/2}$ if it is defined by the open field-line region of a dipolar field (Goldreich and Julian 1969; Cordes 1978). Since pulse shapes have not changed appreciably over the last 10 years, a conservative limit on radial variations is $\Delta r/\bar{r} = 2\Delta W/\bar{W} < 0.1$, where W is the average-profile width, barred quantities are average values, and we place a 5% limit on width variations. Cordes (1978) has shown that $\bar{r} \lesssim 10^8$ cm for a pulsar with a 1-second period. Consequently, an upper limit on such phase variation is $\Delta r/c \approx 0.3$ ms, much smaller than the timing noise observed. We conclude that measurement errors in and of themselves associated with pulse shape changes are not responsible for timing noise.

V. PHYSICAL MODELS INVOLVING CHANGES IN MOMENT OF INERTIA

a) *Crustquakes*

The slowing down of a pulsar results in a steadily decreasing centrifugal bulge at its equator. Eventually the resulting strain becomes sufficient to crack the crust. The bulge falls inward and the rotation rate increases (Ruderman 1969). The subsequent, slow readjustment of the interior superfluid produces the observed post-glitch decay (Baym *et al.* 1969). The characteristic signature of the frequency is given in Table 3, along with those for PN, FN, and SN, and is sketched in Figure 2, superimposed with the spin-down function. We assume that the precise moment in which the crust gives way in a quake is controlled by factors sufficiently numerous and complex (e.g., local fault planes, creep rate, etc.) that the rate of quakes can be considered a random variable. Starquake theory provides a relation between the magnitude $\Delta\nu$ of a frequency step and the time interval t_q between it and the previous step (Baym and Pines 1971):

$$t_q = \nu_q^{\,2}(M)\Delta\nu/\nu^2\dot{\nu} \tag{17}$$

where ν and $\dot{\nu}$ are the pulsar frequency and frequency derivative, and $\nu_q^{\,2}(M)$ is a structure-sensitive parameter whose value depends on the mass of the star.

An ensemble of events of the form in Table 3 will produce FN or PN, depending on the values of τ and Q. The variance of the phase is

$$\sigma_\phi^{\,2}(T) = R(\delta\nu)^2[Q(Q-4)\tau^3/2 + (Q\tau)^2T/2 + Q(1-Q)\tau T^2/3 + (1-Q)^2T^3/12] \tag{18}$$

for a time span $[t_d, t_d + T]$ with $t_d \gg \tau$ ($t = 0$ is the onset of events) and $T \gg \tau$. In the limit $Q = 0$, $\sigma_\phi^{\,2} \propto T^3$ as for FN, and for $Q = 1$, $\sigma_\phi^{\,2} \propto T$ as for PN. On the (τ, Q) plane of Figure 3, we indicate the regions for which PN and FN dominate the variance. The dividing line between these regions is $\tau/T = (1-Q)/Q6^{1/2}$, as indicated. *Note that there is no way of directly producing SN in this model.* We discuss below, however, how SN is *induced* by starquake events and demonstrate that it is negligible compared to the PN or FN produced directly by the events.

TABLE 3

FREQUENCY STEPS FROM PHYSICAL MODELS

Model	Signature of Frequency Step [$\Delta\nu(t)$]
PN	$\delta\phi\delta(t)$
FN	$\delta\nu H(t)$
SN	$\delta\dot{\nu}tH(t)$
Starquake	$\delta\nu(Qe^{-t/\tau} - 1 + Q)H(t)$
Two-component model	
Hard superfluidity:	
case (1)..............	$-\delta\dot{\nu}tH(t)H(\Delta\tau - t)$
case (2)..............	$-\delta\dot{\nu}tH(t)H(\Delta\tau - t) - \delta\dot{\nu}\tau H(t - \Delta\tau)$
Heat pulse.............	$[\delta\nu_1(1 - e^{-t/\Delta t_i}) - \delta\nu_2(1 - e^{-t/\Delta t_f})]H(t)$

NOTE.—$H(t)$ is the unit step function.

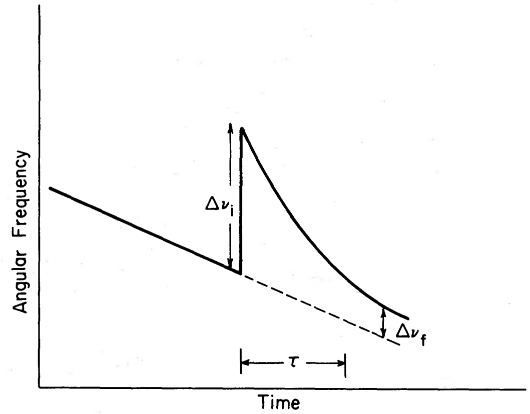

FIG. 2.—The classic signature of a glitch and of a starquake in the two-component model (Baym *et al.* 1969). An initial frequency step of magnitude $\Delta\nu_i$ decays to a final magnitude $\Delta\nu_f$ in time scale τ.

The strength parameter from an ensemble of quakes is

$$S_{\rm FN} = R(\Delta\nu)^2 = (\Delta\nu)^2/t_q = \nu^2\dot{\nu}\Delta\nu/\nu_q^{\,2}(M) \tag{19}$$

in the limit $Q = 0$, and

$$S_{\rm PN} = R(\Delta\nu\tau)^2 = \nu^2\dot{\nu}\Delta\nu\tau^2/\nu_q^{\,2}(M) \tag{20}$$

in the limit $Q = 1$. Note that $Q = 0$ ($Q = 1$) corresponds to the least (most) massive neutron stars.

i) *Frequency Noise*

A free parameter for the strength is $\Delta\nu$, but the measured strengths yield an upper limit on $\Delta\nu$ if we also note that t_q must be smaller than $T_{\rm min}$, where $T_{\rm min}$ is the smallest data block, analyzed in Paper III, in which the random walk is evident. We have

$$\Delta\nu \leq \Delta\nu_{\rm max} = (S_{\rm FN}T_{\rm min})^{1/2} \tag{21}$$

and, therefore, an upper limit on $S_{\rm FN}$ is

$$S_{\rm FN} \leq \nu^2\dot{\nu}\Delta\nu_{\rm max}/\nu_q^{\,2}(M)\,. \tag{22}$$

We consider two different functions for $\nu_q^{\,2}(M)$. Following Baym and Pines (1971) we have

$$\nu_q^{\,2}(M) = A^2/2\pi^2BI\,, \tag{23}$$

if stresses on the crust are only partially relieved by the quakes, and where $A = 3GM^2/25R_*$ and B are the coefficients for gravitational and strain energy, respectively; R_* and I are the stellar radius and moment of inertia. In Figure 4*a* we plot normalized strengths $P^4S_{\rm FN}/\dot{P}$ predicted by equation (22) versus $\Delta\nu$ for the full range of masses (using models of Baym and Pines [1971]), as shown in the cross-hatched region. We also plot measured values of $S_{\rm FN}$ and $\Delta\nu_{\rm max}$ for the seven pulsars that are consistent with FN. We see that *starquake theory cannot be made to yield frequency noise of the observed magnitude for any of the long-period pulsars and will only work (as far as the strength is concerned) for the Crab pulsar if it is of the very lowest mass.* We emphasize in this connection that strong theoretical and observational arguments indicate that low-mass neutron stars do not

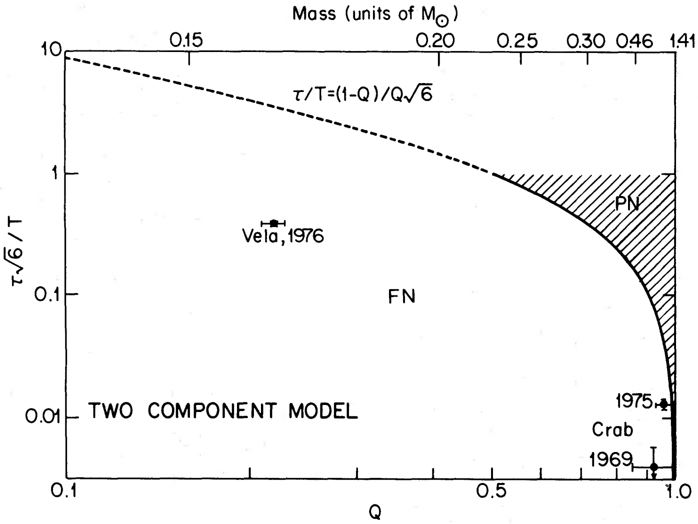

FIG. 3.—Values of τ, the crust-core coupling time, and Q, the fraction of the moment of inertia that is superfluid which produces phase noise, PN (*shaded region*), and frequency noise, FN (the remainder of the plane with $\tau(6^{1/2})/T \leq 1$). Points are plotted for the two Crab pulsar glitches and for one of the four Vela pulsar glitches with values of τ of $\sim$ 5 days and 1 year, respectively ($T = 8$ yr was used in plotting the points). The relationship between Q and M has been taken from Pines and Shaham (1972).

exist (Ruderman 1972). Furthermore, the production of frequency noise (as opposed to phase noise) and the observed value of Q (Fig. 3) for the Crab pulsar require a large-mass neutron star. Consequently, the starquake model appears to be invalidated for all pulsars we have studied.

We note that the solution for the Crab pulsar implies, if starquakes are relevant, that its underlying frequency steps must be almost as large as the present observational upper limits. It seems worthwhile, then, to spend some time in improving this limit in order to search for the frequency steps. If not found, the starquake notion would be disproved for this last FN pulsar as well; if found, the Crab's mass could be inferred, and we would be left with the intriguing question of why the theory works for this one pulsar only.

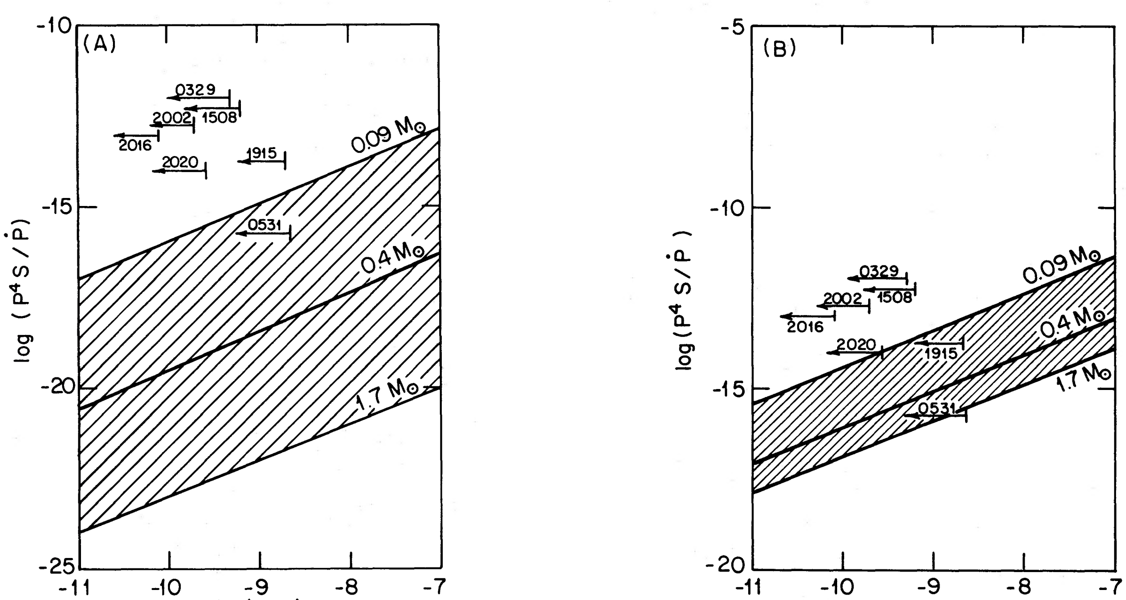

FIG. 4*a* FIG. 4*b*

FIG. 4.—Testing starquake theory. Observed noise strengths and limits on $\Delta\nu$ for the FN pulsars are plotted as arrows. The solid lines give the predictions of starquake theory for the indicated stellar masses. Cross-hatched regions show the full range of allowed values. Case (*a*): v_q^2 from the models discussed by Baym and Pines (1971). Case (*b*): the limiting case, where quakes relieve *all* accumulated stress and the crust shape changes only via quakes. This case produces the maximum possible strengths (see text).

Why does the starquake theory fail for all the long-period pulsars? Formally speaking, the cross-hatched region in Figure 4*a* indicating the predictions of the theory could be moved upward to encompass the observations if v_q^2 could be reduced. Equation (17) indicates the physical meaning of this process: if v_q^2 is reduced, the time interval between quakes will be reduced. It is well known that starquake theory fails to account for glitches in the Vela pulsar in that it cannot be made to yield the observed, relatively small time interval between glitches of approximately 2 years; equation (17), if applied to the Vela glitches, yields t_q greatly in excess of centuries. Reducing v_q^2 would solve this problem as well. Starquake theory cannot be made to account for noise in the long-period pulsars for precisely the same reason that it cannot be made to account for glitches in the Vela pulsar: the rate of buildup of strain in the pulsar is too small.

In this connection we note the somewhat paradoxical fact that the observed strength of the Crab pulsar's noise is far greater than that of any other pulsar. How, then, can starquake theory account for it? From equation (22) we see that the relevant quantity is not $S_{\rm FN}$, but $P^4 S_{\rm FN}/\dot{P}$. This is smaller for the Crab than for any of the long-period pulsars. This is but one instance of an important general point: Within the context of a number of possible models, *timing noise is stronger in the long-period pulsars than in the Crab.*

The above conclusions are not restricted to a particular equation of state. The effect of going to more recent models based on the tensor-interaction, Bethe-Johnson, Reid or mean-field equations of state, discussed by Pandharipande, Pines, and Smith (1976), is to exacerbate the difficulty.

ii) *Phase Noise*

If $\tau \lesssim (1-Q)T/6^{1/2}Q$ is sufficiently small, then the signature of an individual event is a step in phase of order $\Delta\phi \approx \tau \Delta v Q$. Thus starquakes can produce phase noise with a strength of order $S_{\rm PN} \equiv R(\Delta\phi)^2 \approx (\Delta v Q \tau)^2/t_q$ where t_q is the time between events. We have two constraints: both τ and t_q must be less than $T_{\rm min} \approx 1$ year, the length of the minimum block analyzed in Paper III. Combining with equation (17), we find

$$S_{\rm PN} = (Q\tau v)^2 \dot{v} \Delta v / v_q^2(M) < [Q v^2 \dot{v} / v_q^2(M)]^2 T_{\rm min}^3 \qquad (24)$$

as an upper limit to the strength of the phase noise allowed by starquake theory. The limit is largest for lowest $v_q^2(M)$, which in turn occurs for lowest mass. Taking v_q^2 from Baym and Pines (1971) and using $Q = 1$, we find the limiting strength for the PN pulsar PSR 1133+16 to be $S_{\rm PN} \leq 1.8 \times 10^{-19}\ {\rm s}^{-1}$, in contrast to an observed strength some five orders of magnitude larger. For PSR 2217+47, the limit is $S_{\rm PN} = 5.4 \times 10^{-17}\ {\rm s}^{-1}$, in contrast to an observed value a bit more than three orders of magnitude larger. We conclude that *starquake theory cannot account for phase noise for all masses.*

iii) *Slowing-Down Noise*

The signature of the frequency following a starquake (Table 3) does not allow the possibility of SN. That signature, however, does not include an additional effect: a random walk in $\dot{v}$ that arises because the torque is a function of v and the moment of inertia. We show here how SN is induced but that its rms phase is negligible on time scales of interest compared to the FN phase.

If the torque is a function of v and the moment of inertia, $\mathscr{T} = \mathscr{T}(I, v)$, then it can be shown that

$$\frac{\delta \dot{v}}{\dot{v}} = \left(1 - \frac{I}{\mathscr{T}}\frac{\partial \mathscr{T}}{\partial I} + \frac{v}{\mathscr{T}}\frac{\partial \mathscr{T}}{\partial v}\right)\left(\frac{\delta v}{v}\right), \qquad (25)$$

where we have assumed that momentum is conserved: $\delta v/v = -\delta I/I$. Using the torque for magnetic-dipole radiation or for the homopolar generator (Goldreich and Julian 1969),

$$\mathscr{T} \propto B^2 R_*^6 \Omega^3 , \qquad (26)$$

we find that

$$\delta\dot{v}/\dot{v} = \delta v/v = -\delta I/I . \qquad (27)$$

Consequently, starquakes cause both SN and FN (for $Q = 0$), and the ratio of rms phase (cf. equation [7]) over a time interval T is

$$\frac{\sigma_{\phi \rm SN}}{\sigma_{\phi \rm FN}} = \frac{1}{(10)^{1/2}} \left|\frac{\delta\dot{v}/\dot{v}}{\delta v/v}\right| \left(\frac{T}{t_s}\right) . \qquad (28)$$

Since $T \ll t_s = v/\dot{v}$ and $\delta\dot{v}/\dot{v} = \delta v/v$, it is clear that the induced SN is negligible compared to the FN.

b) Corequakes

An alternative to crustquakes involves similar quakes in a possible solid core (Pines, Shaham, and Ruderman 1972) or crustquakes produced by a component of the deceleration torque perpendicular to the spin axis (Pines and Shaham 1972). We are able to put upper limits on these alternatives by considering a second form for $v_q^2(M)$. We imagine that immediately after a quake the star has its fluid shape (i.e., is entirely stress free), that it is infinitely rigid so that this shape does not alter as the star spins down, and that in the next quake the star again relaxes to its new fluid shape. It is clear that these assumptions *overestimate* the magnitude of each corequake, first because real quakes do not entirely relieve the accumulated stress, and second because real neutron star matter smoothly flexes somewhat as it spins down. Furthermore, unless the component of the deceleration torque perpendicular to the spin axis is orders of magnitude larger than its parallel component, the limiting noise strength derived from these assumptions should approximate that derived in this scenario as well.

The oblateness of a fluid star is $\epsilon = I_0 \Omega^2/4A$ and decreases at a rate $\dot{\epsilon} = I_0 \Omega\dot{\Omega}/2A$. In a quake we have, by conservation of momentum, $\Delta v/v = \Delta\epsilon$, and, therefore, the time between quakes, $t_q = \Delta\epsilon/\dot{\epsilon}$, is given by equation (17) with

$$v_q^2(M) = A/2\pi^2 I \approx GM/10\pi^2 R_*^3 , \qquad (29)$$

where the approximate equality holds for a sphere of uniform density. That is, the time between quakes given by this model is a minimum value, t_q now being a factor $B/A \ll 1$ smaller than the Baym-Pines (1971) value obtained by using equation (23). Therefore, the present model yields the maximum possible strength parameters.

Figure 4*b* compares the strength parameters (again normalized as $P^4 S_{FN}/\dot{P}$) with measured values as a function of $\Delta\nu$ using the stellar models described by Baym and Pines (1971). We conclude that the model can be made to work in the case of the Crab pulsar for any mass, but, in the case of the long-period pulsars, it will not work except, possibly, for PSR 1915+13 for a star of low mass. Again, more recent stellar models yield similar conclusions. We conclude that corequakes, and presumably the Pines-Shaham (1972) model as well, cannot account for frequency noise in general.

We conclude by noting that X-ray observations of the Crab pulsar may yield important constraints on the starquake model. In a quake the strain energy released is of order $\Delta E \approx \frac{1}{2} I\Omega^2 (\Delta\nu/\nu)$, most of which is converted to heat within the star. The heating rate is then $L = R(\Delta E) \approx 2\pi^2 I\nu S_{FN}/\Delta\nu$, which, for sufficiently low temperatures, we equate with the blackbody luminosity of the star. Thus,

$$\Delta\nu = \frac{2\pi^2 I\nu S_{FN}}{4\pi R_*{}^2 \sigma T_e{}^4}, \tag{30}$$

and an upper limit on the surface temperature of the star translates into a lower limit on $\Delta\nu$. For the Crab pulsar $T_e < 3 \times 10^6$ K (Toor and Seward 1977), corresponding to $\Delta\nu > 6.7 \times 10^{-10} I_{45}/(R_6)^2$ Hz (where I_{45} is the moment of inertia in units of 10^{45} g cm^2 and R_6 the radius in units of 10 km). Conversely, Table 1 lists a lower limit of $\Delta\nu < 2.3 \times 10^{-9}$ Hz for this object. Already, $\Delta\nu$ is narrowly constrained by the observations. A modest improvement in the upper limit on the Crab pulsar's surface temperature—say, by a factor of 2—would yield significant constraints on the radius and moment of inertia of this object within the context of the starquake model. Unfortunately, this model appears incapable of explaining the observations, so the procedure outlined here is only of academic interest.

VI. PHYSICAL PROCESSES INVOLVING FLUCTUATIONS IN CRUST-SUPERFLUID COUPLING

a) "*Hard*" *Superfluidity*

Anderson and Itoh (1975) have noted a resemblance between pulsar timing noise, on the one hand, and flux migration in "hard" superconductors on the other. The phenomenon occurs in the penetration of a hard superconductor by a magnetic field; bundles of flux tubes pin and unpin in an erratic fashion, resisting the penetration. Anderson and Itoh (1975) proposed that the analogous phenomenon in neutron stars would be the random pinning and unpinning of vortex lines (or bundles of vortex lines) as they migrate outward through the crust and that this effect would cause the pulsar angular velocity to fluctuate in an erratic fashion.

We can develop an order-of-magnitude analysis of this process as follows (see also Lamb *et al.* 1978*a*, *b*). The slowing down of the rotation of a neutron star is accompanied by an outward migration of the vortex lines threading the superfluid. Formally speaking, the lines migrate away from the rotation axis via the Magnus effect in response to the azimuthal friction force on them from the charged-particle component. If the system has a rotation period P and period derivative $\dot{P}$, then the migration velocity v of a line lying a distance r from the spin axis is $v = r\dot{P}/2P = r/2t_s$, where t_s is the spin-down time. For a typical, long-period pulsar, v is of the order of 1 cm per year, about the velocity of continental drift.

As a result of this motion relative to the background superfluid, the line exerts a decelerating torque upon the superfluid, again via the Magnus effect. This is the torque that slows the rotation of the superfluid. It is easy to show that, if v is as given above, then the net decelerating torque, summed over every line in the system, is just sufficient to slow the superfluid at the same rate as the charged-particle system. The star then decelerates uniformly.

From time to time in its outward motion, a given vortex line will encounter a pinning site within the crust. It will hang up there for some length of time. During this time, it no longer migrates outward and no longer exerts a decelerating torque upon the superfluid. The superfluid slows more slowly. To conserve angular momentum, the crust—i.e., the pulsar—slows more rapidly. Thus the observable consequence of the pinning of a line is an increase in slowing-down rate of the pulsar.

The total number N of vortex lines within a neutron star of radius R_* rotating with angular velocity Ω is $N = (\pi R_*{}^2)(2\Omega/\Gamma) \approx 10^{16}/P$, where $\Gamma = h/m_p$ is the quantum of circulation for a Cooper pair with mass m_p = two times the neutron mass. If one line pins, the net decelerating torque $\mathscr{T}$ on the superfluid is reduced by $\Delta\mathscr{T}$, where $\Delta\mathscr{T}/\mathscr{T} \approx 1/N$. If a bundle of n lines as a unit, and all pin and unpin at once, then $\Delta\mathscr{T}/\mathscr{T} = n/N$. If each line is of length L, but only some fraction of the line is prevented by pinning from moving outward, then $\Delta\mathscr{T}/\mathscr{T} = (l/L)(n/N)$ where l is the length of line that does not move. Thus the signature of the onset of pinning is a step in the frequency derivative of the pulsar of magnitude

$$\frac{\Delta\dot{\nu}}{\dot{\nu}} \approx \frac{I_s}{I_c}\frac{l}{L}\frac{n}{N}, \tag{31}$$

where I_s and I_c are the moments of inertia of the superfluid and charged-particle system, respectively.

i) *Phase Noise*

For termination of pinning, we imagine there to be two possibilities. In the first, we assume that the line remains pinned for a time $\Delta\tau$ and, upon unpinning, it jumps back, essentially at the same instant, to the configuration it would have had if it had never pinned at all.

For example, if only part of the line had been prevented by pinning from moving outward, than upon unpinning the "bent" portion would oscillate about this configuration, just as a plucked string vibrates about its ultimate equilibrium state. We assume that the time scale for damping of this transient motion is sufficiently small that it can be neglected. Alternatively, we note that a perfectly uniform vortex array minimizes the free energy of the system and that pinning disturbs this configuration, thus raising the free energy. Without inquiring into the precise nature of the mechanism, we imagine that, upon unpinning, the vortex array rapidly readjusts itself to its lowest energy state.

Under these assumptions, the signature of an individual event has the form of case (1) in Table 3 and is sketched in Figure 5*a* superimposed with the spin-down function. The time between successive pinnings is τ; therefore the event rate is $R = N/\tau$ pinnings per second throughout the star. The variance over a data interval (t, $t + T$) will have the form

$$\sigma_\phi{}^2(T) = \tfrac{1}{8}R(\Delta\dot{\nu})^2(\Delta\tau)^4(T + 2\Delta\tau/5)\,. \qquad (32)$$

For $\Delta\tau \ll T$, this has the form for phase noise with strength parameter

$$S_{\rm PN} = \tfrac{1}{4}R(\Delta\dot{\nu})^2(\Delta\tau)^4 = \frac{N}{4\tau}\left[\frac{\dot{\nu}I_s\,ln\,(\Delta\tau)^2}{I_cLN}\right]^2$$
$$\approx 2.5 \times 10^{-17}\frac{\dot{P}^2}{P^3}\left(\frac{I_s\,ln}{I_cL}\right)^2\frac{(\Delta\tau)^4}{\tau}\,. \qquad (33)$$

We have several constraints on the various parameters in the problem. First, by definition, the length of time $\Delta\tau$ a line remains pinned is less than the time τ between successive pinnings. Second, $\Delta\tau$ must be less than $T_{\rm min} \approx 1$ year, the minimum block length analyzed in Paper III; otherwise the signature of an event is not a pulse in phase. Finally, $l \leq L$. Thus we have an upper limit to the phase noise strength predicted by the model:

$$S_{\rm PN} = 2.5 \times 10^{-17}\frac{\dot{P}^2}{P^3}\left(\frac{I_s}{I_c}n\right)^2 T_{\rm min}{}^3$$
$$= 7.8 \times 10^5\frac{\dot{P}^2}{P^3}\left(\frac{I_s}{I_c}n\right)^2\,. \qquad (34)$$

For the PN Pulsar PSR 1133+16, the predicted limit is

$$S_{\rm PN} = 6.1 \times 10^{-24}\left(\frac{I_s}{I_c}n\right)^2 \,{\rm s}^{-1}\,, \qquad (35)$$

in contrast to an observed noise strength of 1.5×10^{-14} $\rm s^{-1}$. It is immediately obvious that the choice $n = 1$ fails to reproduce the observations, by approximately 10 orders of magnitude. *If each vortex line acts alone, pinning and unpinning independently of all the others, the noise produced by the Anderson-Itoh mechanism is orders of magnitude too small.* It is easy to see why this should be so. From equation (33) the predicted noise strength varies inversely with N, the total number of lines in the star. Because N is so very large, the effects of the erratic motion of a single line are very small.

The observed noise strength for PSR 1133+16 can be reproduced by the theory if it is assumed that bundles containing $n \approx 10^5$ lines act as a unit. It is very difficult to see how the bundles could be so large. For example, the diameter of such a bundle is approximately 5 cm, and it is required by the solution to pin, unpin, *and then pin again* in approximately 1 year. However, its outward migration

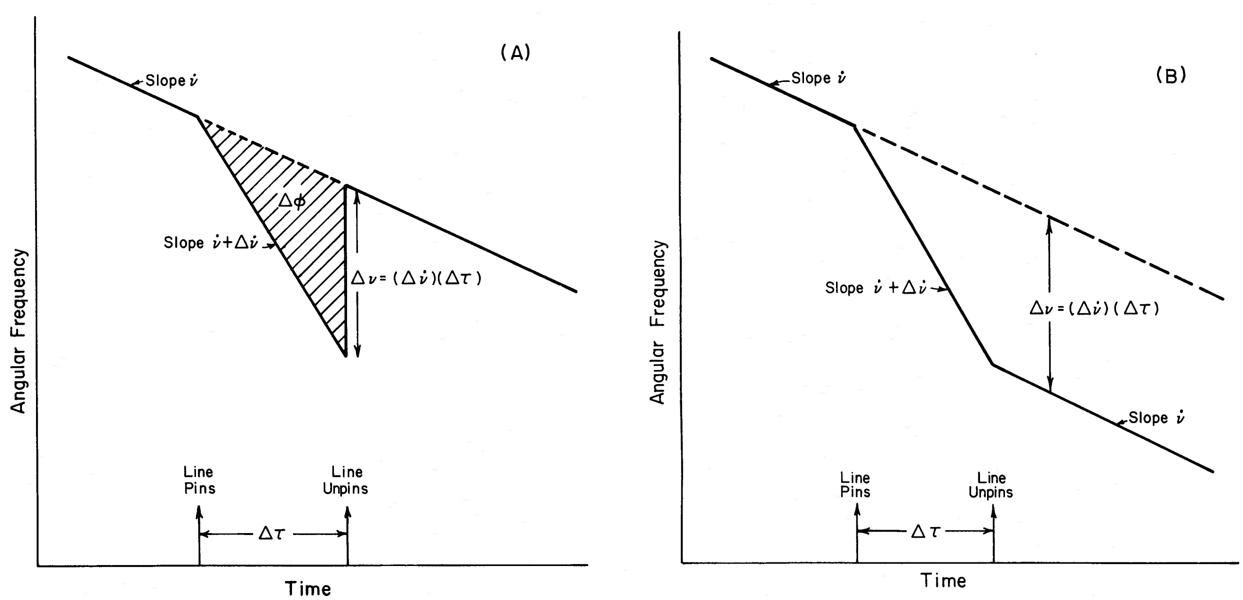

FIG. 5.—Illustrating the characteristic signature of a pinning event. If, upon unpinning, the line returns to the position it would have had if it never pinned at all, (*a*) applies; (*b*) applies if it does not.

velocity carries it, in this year, by only 1% of its thickness. If we go to solutions in which the bundle repins only after moving by significantly more than its thickness, we are forced to still larger bundles. For example, a solution in which a bundle containing $n \approx 10^6$ lines remains pinned for $\Delta\tau \sim 1$ year and then repins approximately 10^3 years later reproduces the observed strength. However, with this choice the difficulties in understanding how the lines could act coherently over such great distances (approximately 20 cm in this case) are exacerbated.

ii) *Frequency Noise*

We now consider a second possibility for the behavior of a vortex line when it unpins. Namely, we suppose that, rather than flipping out to its undisturbed position, the line simply resumes its outward migration. The signature of such an event is given as case (2) in Table 3 and is sketched in Figure 5*b*. The variance now becomes

$$\sigma_\phi{}^2(T) = \tfrac{1}{12}R(\Delta\dot{\nu}\Delta\tau)^2T^3[1 + 3(\Delta\tau/T)^3/5]\,, \quad (36)$$

which implies FN behavior for $\Delta\tau \ll T$ with a strength parameter

$$S_{\rm FN} = R(\Delta\dot{\nu}\Delta\tau)^2 = \frac{N}{\tau}\left(\frac{\dot{\nu}I_s\, ln\, \Delta\tau}{I_c LN}\right)^2$$
$$\approx 10^{-16}\frac{\dot{P}^2}{P^3}\left(\frac{I_s\, ln}{I_c L}\right)^2\frac{(\Delta\tau)^2}{\tau}\,. \quad (37)$$

Since $l \leq L$, $\Delta\tau \leq \tau$, and $I_s \approx I_c$ to the order of magnitude, we have an upper limit

$$S_{\rm FN} \leq 3 \times 10^{-9}\dot{P}^2 n^2 \Delta\tau_{\rm yr}/P^3\,, \quad (38)$$

where $\Delta\tau_{\rm yr}$ is $\Delta\tau$ in years. Combined with the measured strengths, we obtain the lower limits on $n^2(\Delta\tau)_{\rm yr}$ given in Table 4.

The choice $n = 1$ (each line acting alone) forces us to values of $\Delta\tau$, the time a line remains pinned, that are unphysically large; in every case these values are larger than the spin-down age of the pulsar. In one spin-down age, each line migrates outward by approximately the stellar radius, and we would find it difficult to believe any physical model of pinning in which this "migration distance" exceeds microscopic dimensions—the distance between pinning sites, for example. Alternatively, if we require $\Delta\tau$ to be sufficiently small to meet this requirement, the number of lines per bundle is forced to unacceptably large values.

TABLE 4

LOWER LIMITS ON $n^2(\Delta\tau)_{\rm yr}$[a] IN THE ANDERSON-ITOH PICTURES[b]

FN Pulsar	Limit
0329+54	2×10^{11}
0531+21	5×10^6
1508+55	5×10^{10}
1915+13	5×10^9
2002+31	6×10^8
2016+28	6×10^{11}
2020+28	7×10^9

[a] The quantity $(\Delta\tau)_{\rm yr}$ is the length of time (in years) a bundle of lines remains pinned; n is the number of lines in a bundle.

[b] Anderson and Itoh 1975.

b) *Vortex Loss at a Boundary*

Campbell (1979) has conducted detailed numerical studies of the deceleration of a vortex array and noted striking glitchlike behavior as the vortices are lost at the outer boundary of the system. We may estimate the FN strength that could arise from this process as follows.

In uniform rotation the circumferential velocity V of a superfluid is proportional to the number N of vortex lines it contains. If one line is removed from the array, V drops by $\Delta V/V = 1/N$. The pulsar suffers an accompanying frequency increase $\Delta\nu = \nu I_s/I_c N$. Vortex lines encounter the outer boundary of the system at an average rate $R \approx N/t_s$ where t_s is the spin-down age. The strength, then, $S_{\rm FN} \approx (\dot{P}/P^3N)(I_s/I_c)^2 \approx 10^{-16}(\dot{P}/P^2)(I_s/I_c)^2$. Adopting $I_s \approx I_c$ we obtain strengths in the range from 10^{-29} to 10^{-31} Hz^2 s^{-1} for the FN pulsars, four orders of magnitude too small. Alternatively, if a bundle containing n lines annihilates at the boundary, n must be adjusted to be of the order 10^2 to 10^3 to reproduce the observed noise strengths. We conclude, in agreement with Campbell (1979), that some mechanism must be invoked to cause the abrupt loss at the outer boundary of a very large number of lines acting coherently in order to reproduce the observations. Finally, we emphasize the important constraint played by the spherical geometry of the star. In a cylindrical geometry, vortex lines annihilate suddenly upon reaching the boundary. In a spherical geometry, they shorten continuously as they migrate outward, ultimately disappearing "not with a bang but a whimper."

c) *Response to a Heat Pulse*

Greenstein (1979) has studied the dynamical response of a neutron star to a heat pulse and noted that timing irregularities can result. He proposed that occasional large perturbations to pulsar temperatures yield glitches; more frequent (and random) smaller perturbations yield pulsar timing noise.

If the interaction between the superfluid and crust of a neutron star is purely frictional in nature, the superfluid will lag behind the crust in its steady deceleration. The superfluid, then, rotates more rapidly than the crust. The frictional coupling between the two is a strongly increasing function of temperature. A heat pulse then makes the star tend to "seize": the superfluid slows down and the crust—the pulsar—speeds up. We will refer to this process as a "spin-up."

The most important theoretical uncertainty in this analysis concerns the question of vortex pinning. It is clear that *in certain density regimes vortices pin strongly to the crust* (Alpar 1977). Within these regimes none of the above ideas apply. However, it is equally clear that in other regimes the pinning force is weak or absent, and the dominant interaction is in fact frictional in nature.

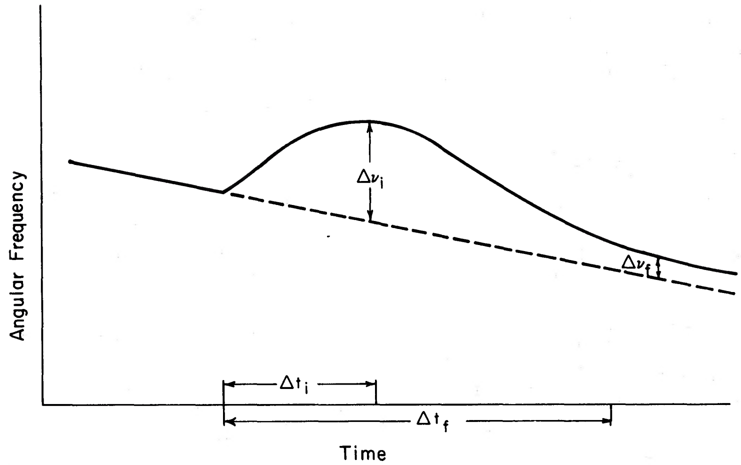

FIG. 6.—Illustrating the characteristic signature of a spin-up.

However, no satisfactory theory of the junction between these regimes exists, and it is unclear to what extent the one affects the other. We will assume that the vortex tangle generated at this junction does not migrate significantly into the frictional regime, in which case the above ideas will be applicable.

Figure 6 illustrates the characteristic signature of a spin-up. The pulsar rotation frequency rises in a time Δt_i to an initial offset $\Delta \nu_i$ and then decays more slowly to a final offset $\Delta \nu_f$ in time Δt_f. Depending on the initial temperature, the magnitude of the perturbation, and, to a lesser extent, the mass of the star, the two time scales can range from minutes to decades, and the final offset can range from close to the initial offset to very much less than it. An approximate analytical expression for the perturbation is given in Table 3 where $\Delta \nu_i \approx \delta \nu_1$, and $\Delta \nu_f \approx \delta \nu_1 - \delta \nu_2$ for $\Delta t_f \gg \Delta t_i$.

We can easily obtain important constraints on this picture with the help of the two-component model, in which the superfluid is supposed to be rotating uniformly (Baym *et al.* 1969). Within this model the torque equations for the superfluid and crust are:

$$I_s \dot{\omega} = -\frac{I_s I_c}{I\tau}(\omega - \Omega)\,,$$

$$I_c \dot{\Omega} = \frac{I_s I_c}{I\tau}(\omega - \Omega) - \mathscr{T}\ ; \tag{39}$$

where ω is the angular velocity of the superfluid and Ω that of the crust; I_s is the moment of inertia of the superfluid, I_c of the crust, and I of the star as a whole; $\mathscr{T}$ is the radiation torque (which equals $-I\dot{\Omega}$ in the steady state); and τ is the crust-core coupling time. From these equations it is easy to show that, if τ changes from τ_i to τ_f essentially in an instant (as the result of a heat pulse), then the *immediate* change in slowing-down rate is an increase,

$$\frac{\Delta \dot{\nu}}{\dot{\nu}} = \frac{I_s}{I_c}\left(1 - \frac{\tau_i}{\tau_f}\right)\,, \tag{40}$$

(valid for times much less than Δt_i). The ultimate offset $\Delta \nu_f$, obtained from equation (25) and by conserving angular momentum, will be

$$\frac{\Delta \nu_f}{\nu} = \left(\frac{I_s}{I_c}\right)\frac{\tau_i - \tau_f}{t_s}\,, \tag{41}$$

where $t_s = \Omega/\dot{\Omega}$ is the spin-down age. For the coupling time, we adopt Feibelman's (1971) result $\tau = (A/T)\exp(T_a/T)$; where A is a complicated function of density only; T is the internal temperature of the star; and $T_a = \Delta^2/\epsilon_f k$, where Δ is the superfluid energy gap, ϵ_f the neutron Fermi energy, and k the Boltzmann constant. If, in the perturbation, the temperature changes by $\epsilon \equiv \Delta T/T \ll 1$, we obtain to the first order in ϵ,

$$\frac{\Delta \dot{\nu}}{\dot{\nu}} = -\frac{I_s}{I_c}\frac{T_a}{T}\epsilon \tag{42}$$

for $t \ll \Delta t_i$, and

$$\frac{\Delta \nu_f}{\nu} = \frac{I_s}{I_c}\frac{\tau}{t_s}\frac{T_a}{T}\epsilon \tag{43}$$

for $t \gg \Delta t_f$.

For application to realistic neutron stars, we need to choose values of A and T_a representative of that region of the star in which the moment of inertia peaks. For a 1 $M_\odot$ star, this occurs close to or below the base of the crust, in which regime

$$\tau \approx (1.3 \times 10^4/T)\exp(10^7/T)\,. \tag{44}$$

If the event that triggers the spin-up liberates an energy ΔE as heat, then $\Delta T = \Delta E/C$, where C is the specific heat of the star as a whole. The expression $C = 10^{30}T(M/M_\odot)$ ergs K^{-1} closely approximates Tsuruta's (1979) detailed computations of the specific heat. Then the perturbation $\epsilon \equiv \Delta T/T = \Delta E/[10^{30}T^2(M/M_\odot)]$. X-ray observations provide us with an upper limit to ΔE. If the triggering events occur at a rate R, then, for sufficiently low temperatures, an upper limit on pulsar temperatures sets an upper limit on the blackbody luminosity and hence the heating rate:

$$\Delta E \leq 4\pi R_*{}^2 \sigma T_e 4/R$$
$$\leq 4\pi R_*{}^2 \sigma T_e{}^4 T_{\min} \lesssim 2 \times 10^{40} \text{ ergs} , \quad (45)$$

where $T_{\min} \approx 1$ year, and we have adopted an upper limit $T_e \leq 10^6$ K (Helfand, Chanan, and Novick 1979). Conversely, for any assumed rate of events in excess of $1/T_{\min}$, we may find the corresponding upper limit on ΔE.

i) *Frequency Noise*

We have calculated $\sigma_\phi{}^2(T)$ for heat-pulse perturbations, yielding a result of the form of equation (18) for the starquake model, but not one sufficiently illuminating to reproduce here. When $T \gg \Delta t_f > \Delta t_i$ and

$$\Delta \nu_i = \Delta \nu_f (1 - \Delta t_i / \Delta t_f)^{-1} ,$$

the response to a random series of heat pulses will be frequency noise. According to the above results, the magnitude of the steps $\Delta \nu$ depends on the temperature of the star and the energy supplied by each triggering event. Because the time scale Δt_f is of the order of the coupling time, τ must not exceed $T_{\min} \approx 1$ year. In Figure 7 we plot the magnitude $\Delta \nu$, the noise strength $S_{\rm FN}$, and the coupling time τ for various choices of the temperature and the rate of events.

If one triggering event occurs per year, the maximum allowable energy released by the event is fairly large: $\Delta E = 2 \times 10^{40}$ ergs. This is sufficient to produce large spin-ups and to yield noise strengths as large as observed. The constraints on $\Delta \nu$ and τ force us to go to temperatures $T \geq 2 \times 10^6$ K. The predicted noise strength decreases with increasing temperature: the limit on T translates into $S_{\rm FN} \leq 2 \times 10^{-26}$ Hz^2 s^{-1}, comfortably within the observed range for the long-period pulsars so long as the pulsar temperature is not too high. However, if we go to one triggering event per week, the energetics forces us to much lower energies $\Delta E = 4 \times 10^{38}$ ergs. This is *not* sufficient to produce the observed noise: if T is adjusted so as not to violate the constraints on $\Delta \nu$ and τ, the predicted noise strength is $S_{\rm FN} \leq 6.3 \times 10^{-28}$ Hz^2 s^{-1}, which is too close for comfort to the bottom of the observed range.

We conclude that *spin-up theory is capable of yielding frequency noise of the observed amplitude.* Within the theory a number of specific predictions can be made. For example, with our particular choices for A, T_a and M, the interior temperature of the star is constrained to lie between 2 and 4 million degrees, and the individual events cannot recur very much more frequently than once a year. A modest improvement in the timing observations should be able to test this latter prediction. The former translates into predicted surface temperatures lying between 3×10^4 K and 4×10^5 K, depending on the mass of the star. The upper limit of this range lies close to the best presently available limits on pulsar temperatures (Helfand, Chanan, and Novick 1979); the lower limit is hopelessly far below it. We emphasize, however, that the particular numerical values we obtain depend quite sensitively on A, T_a, and M, not to mention the equation

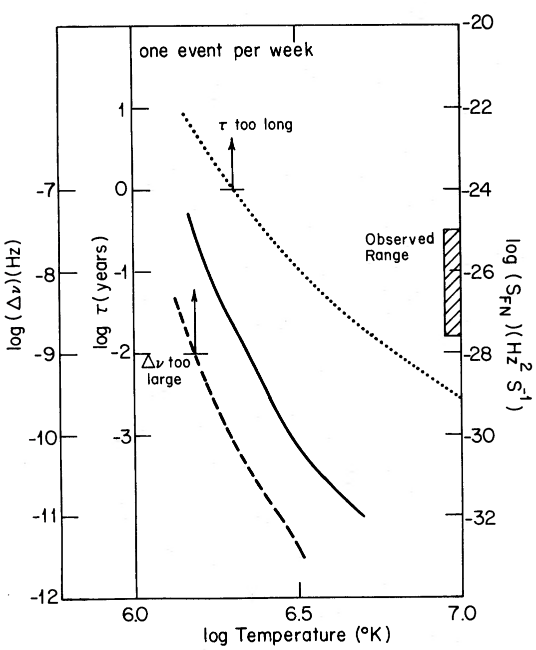

FIG. 7*a*

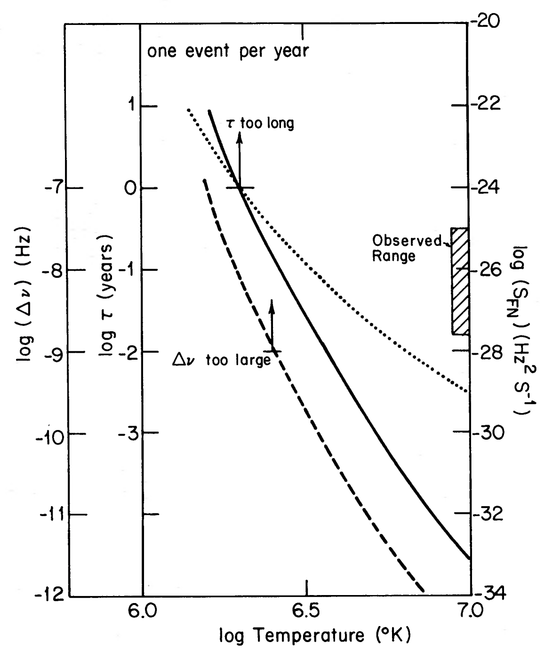

FIG. 7*b*

FIG. 7.—Testing spin-up theory. *Solid lines:* the predicted noise strength. *Dashed lines:* the magnitude $\Delta \nu$ of the frequency step. *Dotted lines:* the crust-core coupling time τ (of the order of the time scale of the spin-up). We have specialized to a one-second pulsar with spin-down age 5×10^6 yr, a one-solar mass star, and we have assumed $I_s \approx I_c$. ΔE has been taken to be the maximum allowed by the observations. One event per year reproduces the observations. One event per week does not.

of state of the star. Realistic predictions will only become possible when detailed models are constructed which include some realistic treatment of the vortex tangle generated at the boundary of the pinning regime.

We are not able to account for the observed noise if we go to choices for ΔE, the thermal energy involved in each triggering event, significantly below the upper limits set by X-ray observations. This places important constraints on the physical nature of the triggering mechanism. For example, Greenstein (1979) argued that strain energy release from starquakes could provide the necessary perturbation to pulsar temperatures. This is indeed sufficient, when amplified by the resulting spin-up, to account for the giant Vela pulsar glitches. Paradoxically, it is not enough to account for pulsar timing noise. The reason is that the energy released in a quake is proportional to the initial rotational energy of the star; this is so low in the slower pulsars that we are driven to impossibly large quakes. We had reached an analogous conclusion earlier in this paper in our discussion of starquake theory: in an important theoretical sense *timing irregularities are stronger in the slow pulsars than in the fast ones.*

We emphasize that starquakes are ruled out as possible triggers only if their released energy is *spread uniformly over the entire star.* If so the thermal energy is so diluted that it cannot produce much of a perturbation to the temperature. If a significant fraction of this energy release is concentrated at the fault plane, however, we reach an entirely different conclusion. The thermal conductivity within a neutron star (Ray 1979; Flowers and Itoh 1976) is sufficiently low that a local " hot spot " can persist for time scales approximately on the order of years. In such a circumstance, the same ΔE produces a significantly larger ΔT and a correspondingly larger spin-up. To go further in this direction would require an analysis of the way in which the released strain energy is converted to heat and of the full three-dimensional, hydrodynamical spin-up problem.

According to equation (43), the magnitude of a spin-up is proportional to the period derivative of the star (since $t_s = P/\dot{P}$). Thus, even if the triggering mechanism itself were to be independent of $\dot{P}$, the observed correlation of timing noise with $\dot{P}$ can be understood.

ii) *Phase Noise*

If the characteristic signature of an event as sketched in Figure 5 involves a considerable " overshoot," $\Delta\nu_f \ll \Delta\nu_i$, then the random process will have a phase noise component. In the limit of $T \gg \Delta t_f > \Delta t_i$ and $\Delta\nu_f = 0$, pure phase noise will result, with a strength parameter

$$S_{\rm PN} = R(\Delta\phi)^2 \approx R(\Delta\nu_i \Delta t_f)^2 \,. \qquad (46)$$

Applying the observational constraints, $R^{-1} < T_{\min}$ and $\Delta t_f < T_{\min} \sim 1$ yr, we require $\Delta\nu_i \gtrsim 2$–6×10^{-11} Hz for the two PN pulsars, PSR 1133+16 and PSR 2217+47. *Such frequency steps are easily accommodated in the heat-pulse model.*

Finally, we note that whether a spin-up will or will not exhibit an appreciable overshoot, and hence PN, depends within this model on the detailed form of the rotation curve within the star. There is no simple formula, analogous to that which obtains in the starquake picture (Baym *et al.* 1969), from which the mass or temperature of the star can be inferred from the ratio of $\Delta\nu_i$ to $\Delta\nu_f$, that is, whether it is an FN or PN pulsar.

iii) *Slowing-Down Noise*

Only if the " build-up " time scale Δt_i in Figure 5 is long compared to the time elapsed since the noise process *commenced* will the consequence of a heat pulse be an apparently permanent step in the frequency derivative and the noise process be SN. Because Δt_i is of order τ, and because the noise process has presumably been operating for a significant portion of the entire history of a pulsar, this implies values of τ in excess of 10^5–10^7 years. Although we cannot rigorously exclude this possibility we find it unnerving, to say the least, and we will not consider it further.

d) Crust-Breaking by Vortex Pinning

Ruderman (1976) has noted that, when vortex lines pin to the crust, the are pinning to something rotating at a different angular velocity than the background superfluid. Under these circumstances, a shearing Magnus force is transferred to the crust. He argues that, in certain density regimes, the pinning force is sufficiently strong that the lines never unpin but rather the crust ultimately fractures. Upon breaking free in this manner, the lines suddenly migrate outward, yielding a pulsar frequency jump.

If, immediately following a fracture, the crust and fluid essentially corotate, then the mismatch $\delta\Omega$ in angular velocity between them builds up at a rate $\delta\Omega = \dot{\Omega}t$ within this model. When $\delta\Omega$ reaches

$$\delta\Omega_{\rm crit} \approx \theta_m \mu GM/2pR_*{}^3\Omega \,, \qquad (47)$$

fracture will occur (Ruderman 1976; note we have revised his eq. [19]), where θ_m is the maximum strain angle before fracture, $\mu \approx 10^{30}$ dynes cm^{-2} is the shear modulus, and p is the pressure at the base of the crust. Upon fracture, the angular momentum in the pinned vortex lines is suddenly transferred to the crust, resulting in an increase of crust rotation frequency,

$$\Delta\nu/\nu = (I_s/I)(\delta\Omega_{\rm crit}/\Omega) \,. \qquad (48)$$

An ensemble of fractures will yield a frequency noise strength

$$S_{\rm FN} = (\Delta\nu)^2/\Delta t \,,$$

or

$$S_{\rm FN} = (\dot{P}/P^2)(I_s/I)^2(\delta\Omega_{\rm crit}/2\pi) = t_s{}^{-1}(I_s/I)^2(\theta_m \mu GM/8\pi^2 pR_*{}^3) \,, \qquad (49)$$

where I_s is the moment of inertia of the superfluid that penetrates the crust, and t_s is the spin-down time scale.

In Ruderman's (1976) first scenario (where normal neutron fluid exists interior to superfluid neutrons), $p \approx 7 \times 10^{32}$ dynes cm^{-2} and $I_s/I \approx 10^{-2}$; in the second (where vortices extend throughout the star or terminate at a stellar core), both p and I_s/I are approximately 10 times larger. Adopting $M = M_{\odot}$, $R = 10$ km and the canonical value $\theta_m = 10^{-5}$, we obtain predicted noise strengths six to seven orders of magnitude *larger* than observed. *The problem within this model is not why pulsar timing noise is so large: it is why it is so small.*

We can reproduce the observed strengths by going to values of the maximum strain angle θ_m of the order 10^{-11}–10^{-13}, depending on the pulsar and the scenario. Such values are very much smaller than expected (Smoluchowski 1970). However, these values of θ_m are also required in order to make Δt be as short as one year for the longer-period pulsars. Alternatively, Ruderman (1976) notes that, within the long-period pulsars, the crust may not be expected to crack before the vortex lines unpin or pull individual pinning nuclei from their lattice positions. It remains to be seen whether an intermediate situation, capable of yielding strengths neither too large nor too small, is possible. Ruderman's estimates assume that the moment of inertia of the crust is unaltered in the fracture. It seems likely, however, that the crust will be deformed when it is broken, although whether an increase in the moment of inertia is produced is not clear. An increase would cause a *decrease* in $\Delta\nu$, which would lessen the increase in $\Delta\nu$ due to the angular momentum imparted to the crust.

The model outlined above is expected to yield phase noise or frequency noise according to the same conditions as were discussed for the starquake model. Slowing-down noise is again not possible for the physics we have described above. One possibility, however, is that $\dot{\nu}$ will vary in time because the crust lattice will creep (Smoluchowski and Welch 1970; Ruderman 1976), thereby allowing vortex lines to move outward at a slow rate. The resultant $\dot{\nu}(t)$ is proportional to temperature (eq. [31] in Ruderman 1976) so temperature fluctuations may cause the requisite variations in $\dot{\nu}$ to produce SN.

VII. PHYSICAL PROCESSES INVOLVING TORQUE FLUCTUATIONS

a) *Accretion From the Interstellar Medium*

In the binary X-ray sources, accretion from the companion produces period fluctuations. We ask whether accretion from the interstellar medium could produce analogous effects in the isolated pulsars. The classical expression for the accretion rate, $\rho\pi R_a{}^2 v$ g s^{-1} (where ρ is the density of the interstellar medium, v the pulsar space velocity, and $R_a = 2GM/v^2$, the accretion radius), works out to approximately 10^9 g s^{-1} for the plausible numbers. A rough estimate indicates that such a rate may in fact be sufficient to produce timing irregularities of the observed magnitude.

We emphasize, however, that most of this "accreted" matter never reaches the magnetosphere of the star. Rather, it is blown away by the pulsar radiation. Blandford *et al.* (1973) have argued that, if the rotational energy loss of a pulsar is eventually converted to relativistic particles, then the pulsar will "blow a bubble" in the interstellar medium of an approximate 1 pc radius. The fact that this radius is very much larger than the accretion radius implies that the isolated pulsars never accrete anything. However, it has not been demonstrated that the radiation flux is isotropic; if the interstellar medium could find "holes" down into which to fall, the effect could be important.

A stronger argument against this possibility is based on the fact that, in a sample of 48 pulsars, no correlation is found between timing noise and altitude z above the galactic plane (Paper III). For example, PSR 1508+55 has $z = 730$ pc but nevertheless has a perfectly reasonable noise strength. Among the pulsars listed in Table 1, the mean z is approximately 190 pc. The pulsar distribution as a whole has a scale height of 230 pc (Manchester and Taylor 1977). In contrast, molecular clouds are confined to within roughly 50 pc of the galactic plane (mid-point to half-power of the distribution: Sanders, Solomon, and Scoville 1979) and H I to within 120 pc (Burton 1976). Furthermore, this model predicts an inverse correlation of timing noise with space velocity. In the sample of 25 pulsars for which interferometer velocities are available, this correlation is not observed. We strongly doubt, therefore, that the effect could be significant.

b) *Torque Fluctuations*

Pulsar spin-downs occur as a consequence of an electromagnetic torque which derives from two sources: magnetic dipole radiation and a stellar wind. Angular momentum is carried away mostly by electromagnetic fields, but a small amount is in particles which convert some of their energy to coherent radio emission. Radio emission is variable on time scales from microseconds to years, and it is plausible that the associated particle flow may vary correspondingly, thus causing timing noise. We consider here the details of such fluctuations and the prospects for observationally linking luminosity variations with timing noise (see also Lamb *et al.* 1978*a*, *b*).

i) *Current and Mass Flow in Pulsar Magnetospheres*

Cheng and Ruderman (1980) have argued that particle flow from the magnetic polar cap has two components: (1) a positive current composed of ions and positrons; (2) a secondary electron-positron plasma generated from γ-rays emitted by primary positrons.

The current and plasma interact as a beam-plasma system that is unstable to charge-bunching via a two-stream instability. Hence both components are required for producing coherent radio emission. In this model, the *current* flow is essentially constant; it is fixed by the rotation rate and magnetic field of the star to the extent that those quantities are constant. The current can fluctuate only on time scales smaller than approximately 10 μs, the time necessary for the polar cap region to become unstable to pair production if the ion flow were interrupted. The *plasma* flow can vary on any time scale, however, because it originates from the primary positrons, whose number may change so long as the ion number varies such that the current is constant. Ions are produced by thermionic emission and, therefore, the quantity that controls fluctuations is the polar cap temperature. Cheng and Ruderman (1980) demonstrate that the temperature is self-regulated by back heating of electrons accelerated into the polar cap.

In this model (and in *any* model that is similar to that of Goldreich and Julian 1969), most of the work done against the star is by the current (viewed as a $\boldsymbol{J} \times \boldsymbol{B}$ force). The neutral plasma is less important in the spin-down, but the energy it carries can vary, as does the radio emission for which it is responsible.

ii) *Timing Noise from Plasma Variations*

The energy loss rate in particles is

$$\dot{E}_p = \gamma mc^2 \dot{N} , \tag{50}$$

where $\dot{N}$ is the number of particles exiting the magnetosphere per second. The radio luminosity is at the expense of particle energy, so

$$L = \Delta\gamma mc^2 \dot{N} . \tag{51}$$

Compared to the spin-down loss rate, $\dot{E} = I\Omega\dot{\Omega}$, $\dot{E}_p$ is negligible according to theory while L is observationally miniscule. Indeed $L/\dot{E}$ ranges from $10^{-9.5}$ for the Crab pulsar to 10^{-2} for PSR 1819 − 22. Remarkably, however, $L/\dot{E}_p$ is of order unity, implying that the radiation efficienciy $\epsilon = \Delta\gamma/\gamma$ is large. The level of fluctuation of $\dot{E}$ which can appear as timing noise is, therefore, $\delta\dot{E}_p = \delta L/\epsilon$.

Frequency noise will be produced if luminosity variations of amplitude δL occur in pulses of width W such that $t \gg W$. These pulses produce steps in frequency with amplitude

$$\delta\nu = W\delta L/4\pi^2\epsilon I\nu . \tag{52}$$

Therefore, the strength parameter is

$$S_{\rm FN} = R\langle(\delta\nu)^2\rangle = R\langle(W\delta L)^2\rangle/(4\pi^2\epsilon I\nu)^2 . \tag{53}$$

Assuming nonoverlapping events ($RW \sim 1$) and adopting the values $\delta L \sim L \sim 10^{28} L_{28}$ ergs s^{-1}, $I = 10^{45} I_{45}$ g cm^2, and $W = W_d$ days, we have

$$S_{\rm FN} = 10^{-32.3}(\delta L_{28}/I_{45}\nu)^2 W_d \epsilon^{-2} . \tag{54}$$

Comparison with measured values shows that $S_{{\rm FN},\delta L}$ is too small by four to six orders of magnitude. The deficit can be made up by assuming that fluctuations with $W_d \sim 10^2$ occur and appear as timing noise in the least massive neutron stars, with $I_{45} = 0.1$ and $\delta L_{28}/\epsilon = 10$.

It is interesting to note that L is not measurably correlated with period derivative. In the Goldreich-Julian (1969) model, however,

$$\dot{P} = 4\pi^2 c^{-3} P^{-1} B^2 R^6/I . \tag{55}$$

Therefore, all other things being equal, the least massive stars will show the largest values of $S_{\rm FN}$ and $\dot{P}$, a correlation that is established in Paper III. Whether this model is viable or not depends on whether the luminosity does, in fact, vary by 100%. The radio luminosity has only been inferred from narrow-band intensity measurements by assuming that the shape of the radio spectrum is time invariant. The narrow-band intensity, indeed, varies by large fractions on time scales from microseconds to years, but there is evidence that such variations are not correlated over the whole radio spectrum (Cole, Hesse, and Page 1970; Backer and Fisher 1974; Rankin, Payne, and Campbell 1974; Helfand, Fowler, and Kuhlman 1977). Suppose that intensity variations with characteristic time τ are correlated over a radio frequency range, $\Delta f_{\rm cor}(\tau) < \Delta f$, where Δf is a characteristic bandwidth of the spectrum. We have

$$\delta L(\tau)/\langle L\rangle \approx m_I(\tau)[\Delta f_{\rm cor}(\tau)/\Delta f]^{1/2} , \tag{56}$$

where $m_I(\tau)$ is the modulation index (ratio of rms to mean) of narrow-band measurements and may be conservatively estimated to be unity. At present there are no good estimates of $\Delta f_{\rm cor}$.

iii) *Timing Noise from Current Fluctuations*

The current is steady as long as the rotation rate and magnetospheric structure are constant. Average pulse shapes, which are probably determined by the magnetic field, are measureably constant on time scales of years (Helfand, Manchester, and Taylor 1975). Observations do not preclude pulse-shape fluctuations of 10% on short time scales, however. In the Cheng-Ruderman (1980) model (and polar cap models in general), the size of the polar cap, and hence the current, is governed by the global structure of the magnetosphere, particularly that near the Alfvén radius. We therefore consider torque fluctuations caused by variations of the polar cap size. This involves, of course, closed field lines temporarily opening. Observationally, this suggests average profile fluctuations at some level.

Variations in current flow imply $\delta\dot{\nu}/\dot{\nu} = f \leq 1$. Imagining pulselike variations with width W, we obtain a random walk in frequency with strength

$$S_{\rm FN} = R\langle(\delta\dot{\nu}W)^2\rangle = (RW)f^2\dot{\nu}^2 W . \tag{57}$$

Adopting the values $\dot{\nu} = 10^{-15}\dot{\nu}_{15}$ and $RW = 1$ (nonoverlapping events), we have

$$S_{\rm FN} = 10^{-30}\dot{\nu}_{15}{}^2 f^2 W \ {\rm Hz}^2\ {\rm s}^{-1} . \tag{58}$$

Note that the strength is a factor $\dot{E}f\epsilon/L$ larger than that for luminosity fluctuations.

To achieve random walk strengths equal to those measured requires $f^2 W = 10^{-3.4}$ s for the Crab pulsar, and values ranging from 3.0 s for PSR 1915+13 to approximately 10^3 s for PSR 2016+28. Therefore, except for the Crab, the torque varies by 100% on time scales of seconds to minutes, or it varies by smaller fractional amounts on larger time scales. We note that pulse-timing measurements do not preclude such variations. In order to measure $\dot{\nu}$, a time scale $\delta t \approx 0.05\dot{\nu}^{-1/2} \approx 10^6 \dot{\nu}_{15}{}^{-1/2}$ s is necessary (we assume arrival time measurements are accurate to 1 ms). Consequently, even 100% variations of $\dot{\nu}$ on the required time scales would not be directly detectable because several thousand pulses are required to form an average profile stable to approximately 5%. The required values of W are smaller than the time for achieving a stationary average profile. Therefore,

$RW > 1$ if different parts of the open field-line region fluctuate independently. The result is that W is *decreased* from values mentioned above.

For the Crab pulsar, there is the interesting possibility that giant radio pulses are associated with timing noise. Giant pulses have $W \sim 0.1$ ms and $R = 1/30$ s (Hankins 1980). From equation (58) and the measured strength, we find $f \approx 360$. Whether torque variations with $f = \delta\dot{\nu}/\dot{\nu} = 360$ can occur will not be discussed here. Considerably smaller values of f may suffice, however, depending on the nature of giant pulses. If such pulses represent temporal fluctuations that would be observed in the corotating frame of reference, then $W = 0.1$ ms, as above, but R may be greater than observed if some giant pulses occur when the pulsar beam is not pointed at the Earth. Alternatively, a giant pulse may be caused by a rotating beam with a lifetime in the range of 0.1 ms to $P = 33$ ms. Therefore the giant pulse may persist for a time 0.1 ms $\leq W \leq$ 33 ms. If $W = 33$ ms, then $f \approx 13$ is required to produce the observed timing noise.

iv) *Phase Noise*

We cannot imagine how torque variations could produce phase noise because torque variations that appear asymptotically as the derivative of a Dirac delta function would be required.

v) *Slowing-Down Noise*

Torque variations will produce SN if they are of the form of step functions. The strength parameter becomes

$$S_{SN} = R\langle(\delta\dot{\nu})^2\rangle \approx R(\dot{\nu}f)^2 , \qquad (59)$$

where $f = \delta L/\epsilon\dot{E}$ for the case of plasma fluctuations, and f is arbitrary for current fluctuations. Assuming the former case, the strengths are equal to those measured if $R\epsilon^{-2} \approx 10\ s^{-1}$ for both pulsars PSR 0611+22 and PSR 0823+26. Physically, we see no reason why torque fluctuations should appear as SN in these two pulsars and FN in others.

vi) *Pulse Nulling and Timing Noise*

In some pulsars a dramatic form of luminosity fluctuation occurs: the radiation at all frequencies shuts off for one to several hundred pulse periods (Backer 1970; Ritchings 1976). According to the model outlined above, such nulls can only be interpreted as a termination (or major perturbation) of the plasma flow. In that context, the time scale for nulls is too small to produce measurable timing noise (cf. eq. [54]). It is nonetheless of interest to ask whether pulsars that display pulse nulling are also those with considerable timing noise. Ritchings (1976) determined the fraction of null pulses for 27 pulsars for which timing data were also available. We find that the activity parameter (a logarithmic measure of the strength parameter; see Paper III) for pulsars with 5 to 55% null pulses is not statistically different from pulsars with a strong upper limit ($\leq 5\%$) on the fraction of null pulses.

VIII. PROSPECTS FOR THE FUTURE

We close by outlining those areas of further work which we believe would be the most helpful.

Two improvements in the timing observations—finer time resolution and observations extending over longer baselines—are called for. With regard to the first, it is obviously a serious problem that no timing observations have ever succeeded in isolating an example of the underlying event responsible for the noise process. Nevertheless there are a variety of reasons for believing that these events may lie not far below the present threshold of detectability. For example (§ II), the amplitude of the step size for the SN pulsars can be determined if the noise process has a nonzero mean. As indicated in Table 1, the step size so obtained is fairly large. Secondly, as emphasized in § V*a*, if crustquakes are relevant, the underlying steps in the Crab pulsar must be almost as large as the present observational limit. Finally (§ VI*c*), it is easier to understand the noise process with the heat-pulse model if the process is of a moderately slow rate: one event per year as opposed to one per week.

The presently available data are sufficient in only 11 out of 50 pulsars to allow a classification of the noise process into PN, FN, or SN. More pulsars could be added to the data set if timing observations of them were to be extended over longer baselines. We also note that it might be possible to obtain a handle on the signature of the underlying events in this way. As indicated by equations (18), (32), and (36), as the time span of the observations T is varied, $\sigma_\phi{}^2(T)$ changes its T-dependence. Thus a given pulsar is expected to change the character of its noise process from, for example, PN to FN as the observations are carried out over successively longer and longer periods of time. We emphasize that, in most of the theoretical models that we have considered, we are already in the large-T limit. However, we do recommend this consideration to anyone wishing to propose new mechanisms for timing noise.

X-ray observations of blackbody radiation from the pulsars can also be used to place important constraints on the underlying mechanism. For example, we have been able to relate the surface temperature of the pulsar to the amplitude of its frequency steps in the starquake picture in equation (30) and to various parameters appearing in the heat-pulse model in § VI*c*.

None of the models we have considered are able to account for all three types of noise as the relevant parameters are varied over their respective ranges. The crust-breaking model of Ruderman (1976) appears to be capable of yielding PN and FN strengths far greater than observed, at least in the young pulsars; it remains to be seen, however, whether this mechanism can be made to operate at the observed level in the long-period pulsars. The heat-pulse model of Greenstein (1979) also appears to be capable of accounting for the PN and FN pulsars. Within this model the most important unresolved problems are that of the hydrodynamics of the boundary between the vortex pinning and the frictional regions of the star, and that of the degree to which the vortex tangle generated at this boundary migrates into the frictional region.

We believe that SN can be understood only in terms of variations in the torque acting on the star. For this reason, and because they have not been explored as fully as models involving processes internal to the star, we urge a more complete study of such models.

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APPENDIX

MOMENTS OF AN ENSEMBLE OF EVENTS WITH UNKNOWN TIME ORIGIN

The moments of an ensemble of events,

$$\phi(t) = \sum_j a_j h(t - t_j)\,, \tag{A1}$$

can easily be calculated by using the procedures for shot noise if the t_j are independent and Poisson distributed. In Paper II it was shown that the first and second moments of ϕ are

$$\langle \phi(t) \rangle = R\langle a \rangle \int_0^t dt' h(t - t')\,, \tag{A2}$$

$$\langle \phi^2(t) \rangle - \langle \phi(t) \rangle^2 = R\langle a^2 \rangle \int_0^t dt' h^2(t - t')\,, \tag{A3}$$

where R is the rate of events, and it is assumed that the events commence at $t = 0$ (i.e., $\phi(t) = 0$ for $t \leq 0$).

Measurements of a moment involve an integral over some interval $[t_d, t_d + T]$. If we consider such an integral to be an estimate of a moment, then the ensemble-average of that estimate is, for the second moment,

$$\sigma_\phi{}^2(t_d, T) = \left\langle \frac{1}{T} \int_{t_d}^{t_d+T} dt' [\phi^2(t') - \langle \phi(t') \rangle^2] \right\rangle . \tag{A4}$$

For nonstationary random processes (those for which $h(t) \neq 0$ as $t \to \infty$), $\sigma_\phi{}^2(t, T)$ will depend on t, the unknown time elapsed since the onset of the random process. In this paper, we do not know the value of t, but we are interested in comparing data in the interval $[t_d, t_d + T]$ with a random process of a particular form. To do so, we will divide the events contributing to the random process into three groups: (1) those events that occurred between 0 and t_d and whose time-dependence $h(t - t_j)$ has reached an asymptotic form [e.g., a power of $(t - t_j)$]; (2) those events that occurred in $[t_d - W_h, t_d]$ such that $h(t - t_j)$ has not reached an asymptotic form; and (3) those events that occur in $[t_d, t_d + T]$. It can be shown that events in the first group will contribute only low-order polynomial components to the phase. Events in the third group constitute a random process commencing at the time t_d. Events in the second group occur prior to t but have shapes that evolve in $[t_d, t_d + T]$ toward the asymptotic forms of those of the first group. Events in the second group will, therefore, influence the form of the random process.

In the following, we ignore events in the first group mentioned above because, for event signatures that we are interested in, they contribute only low-order polynomial components to the phase. These components are indistinguishable from those associated with the steady spin-down, which are removed from the data before analyzing the timing noise. To demonstrate this, suppose that event signatures assume the following form for large t:

$$\lim_{t \gg W_h} h(t) = \sum_{n=0}^{m} C_n t^n\,, \tag{A5}$$

where m is a small integer. For events produced by starquakes (Table 3), for example, C_n is 0 for $n \geq 2$. The phase produced by events in the limit $t \gg W_h$ can be written as

$$\phi_1(t) = \sum_{k=0}^{m} t^k \sum_{n=k}^{m} C_n (-1)^{n-k} \binom{n}{k} \sum_j a_j t_j{}^{n-k}\,, \tag{A6}$$

which is clearly a low-order polynomial in time. All the sums over j, which are over events $t_j \leq t_d - W_h$, are constant over the data interval. Consequently, $\phi(t)$ is a mth-order polynomial that is removed along with the spin-down polynomial. Therefore, the events contributing to ϕ_1 do not contribute to the phase fluctuations in the data interval.

The analysis in Papers II and III essentially compared the T dependence of the phase variance of the data with $\sigma_\phi{}^2(t_d, T)$ for the case that $t_d = 0$; i.e., that the random process commenced at the beginning of the data acquisition. If this were the case, then the type of random process would be determined by the shape of $h(t)$ for those events that occur only in the interval of the data. If the random process commences at the time of onset of data taking, then

$$\mu_2(t) = \langle \phi^2(t) \rangle - \langle \phi(t) \rangle^2 = R\langle a^2 \rangle \int_0^{t-t_d} dz h^2(z)\,, \tag{A7}$$

from which it is clear that the t dependence of $\mu_2(t)$ will depend on the form of $h^2(z)$.

More realistically, however, we will assume that events with a characteristic time W_h have occurred such that $t_d \gg W_h$. It is then clear that, as far as the form of the random process is concerned, we must consider those events that occur with $t_j \in [t_d - W_h, t_d]$ as well as those that occur with $t_j \in [t_d, t_d + T]$. We can write the phase as

$$\phi(t) = \phi_A(t) + \phi_B(t)\,, \tag{A8}$$

where

$$\phi_A(t) = \sum_{t_j \in [t_d - W_h, t_d]} a_j h(t - t_j) \quad \text{(A9)}$$

$$\phi_B(t) = \sum_{t_j \in [t_d, t_d + T]} a_j h(t - t_j) \, . \quad \text{(A10)}$$

The events contributing to ϕ_A are in a time interval that does not overlap the interval containing events of ϕ_B. Therefore, the second moments of ϕ_A and ϕ_B add,

$$\langle \phi^2(t) \rangle = \langle {\phi_A}^2(t) \rangle + \langle {\phi_B}^2(t) \rangle \, , \quad \text{(A11)}$$

and it can be shown that

$$\mu_2(t) = R\langle a^2 \rangle \int_0^{t + W_h - t_d} dz h^2(z) \, . \quad \text{(A12)}$$

Comparison with equation (A7) indicates that the effect of events prior to t_d is to extend the integration interval from $[0, t - t_d]$ to $[0, t + W_h - t_d]$. This extension strongly limits the kinds of random processes that can be expected from ensembles of events. For example, suppose perturbations in the frequency of the form

$$\Delta\nu(t) = a(1 - e^{-t/\tau})H(t) \quad \text{(A13)}$$

occurred. For $t \ll \tau$, $\Delta\nu(t) \approx (a/\tau)tH(t)$, and if equation (A7) were the valid equation (i.e., events commence when data sampling begins), then the random process would appear as a random walk in rotation-frequency derivative for which $\mu_2(t) \propto t^5$. Using the realistic equation (A12), however, yields $\mu_2(t) \propto \tau^3$, i.e., the dominant term is independent of t if $t \ll \tau$.

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JAMES M. CORDES: Department of Astronomy, Space Sciences Building, Cornell University, Ithaca, NY 14853

GEORGE GREENSTEIN: Astronomy Department, Amherst College, Amherst, MA 01002