THE ASTROPHYSICAL JOURNAL, 237:216–226, 1980 April 1

PULSAR TIMING. II. ANALYSIS OF RANDOM WALK TIMING NOISE: APPLICATION TO THE CRAB PULSAR

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Received 1979 May 21; accepted 1979 October 8

ABSTRACT

Random walk irregularities in the rotations of pulsars are analyzed by use of the variance of rotational phase, through which consistency with the data can be demonstrated and a strength parameter determined. Application to Princeton optical timing data of the Crab pulsar yields a strength $R\langle\delta\nu^2\rangle = 0.66 \pm 0.31 \times 10^{-22}$ Hz^2 s^{-1}, in agreement with Groth. Higher-order moments of the phase, which would characterize the *distribution* of step amplitudes, can be determined only if a small number of steps occur in the analyzed time interval. The required intervals ($\lesssim 1$ day) for the Crab pulsar unfortunately have phases dominated by random measurement errors. Step amplitudes with a nonzero mean will bias the observed braking index, $n = \nu\ddot{\nu}/\dot{\nu}^2$, from the index that describes the torque mechanism that spins down the star. For the Crab pulsar, a 20% bias requires rates larger than 1 s^{-1} if the random walk is damped on a time scale larger than 100 years.

Subject heading: pulsars

I. PULSAR TIMING NOISE

Pulsars spin down on time scales of 10^3 to 10^9 years according to what is apparently a deterministic torque law. Although non-dipolar magnetic field components and field decay may contribute, measurements on the Crab pulsar (PSR 0531+21) indicate that dipole radiation would produce spin-down that is similar to that observed (Boynton *et al.* 1972). However, a random component of the rotation—which is in the form of a random walk in the rotation frequency for the Crab pulsar (Groth 1975*c*)—represents a departure from the deterministic spin-down. Such timing noise is common to many pulsars (Manchester and Taylor 1974; Helfand *et al.* 1980 [Paper I]; Gullahorn and Rankin 1980), but it is not known whether it is due to Crab style random walks or to less frequent, resolvable events in which momentum is transferred between different components of the star or between the star and its environment.

This paper presents an analysis of random walks and develops the means for testing whether a random walk is consistent with the observed timing noise. The variance of rotational phase is used to show consistency with the data and to determine a strength parameter for the random walk. The analysis parallels that of Groth (1975*b*) except that the integrated variance is dealt with, rather than a decomposition of the variance into polynomial components. The analysis is applied to timing data on 10 pulsars in the next paper of this series (Cordes and Helfand 1980, Paper III).

The next two sections consider analytical models for the deterministic and the random components of the rotation; the extent to which these components can be separated is discussed. Section IV presents the analysis of the variance and the accompanying measurement errors, which is then applied to the Princeton optical timing data of the Crab pulsar, the results for which agree with those of Groth (1975*c*). Finally, § V discusses the case where the step amplitudes of the random walk have a distribution with a nonzero mean. Higher-order moments of the phase can in principle yield information about the distribution of step amplitudes, but accurate estimates of the moments are not yet available. The effect of nonzero mean amplitudes on the braking index is also considered for the Crab pulsar.

II. MODELS FOR ROTATIONAL PHASE

a) *Arrival Time Measurements*

The arrival times of pulses at an observatory can be expressed in terms of the rotational phase

$$\phi'(t) = \int_0^t dt_1 \nu'(t_1)\,, \tag{1}$$

where $\nu'(t)$ is the rotation frequency in the nonintertial frame of the observatory. As discussed in Paper I, topocentric (Groth 1975*a*) arrival times are measured by averaging many single pulses together to form an average pulse profile; comparison with a reference profile then yields the arrival time. Barycentric arrival times are obtained through knowledge of propagation delays and of the observatory's motion in the solar system. Errors in barycentric arrival times derive from errors in the ephemeris and in the position (and proper motion) of the pulsar. In addition to these systematic errors, there are random errors due to finite signal-to-noise ratios and to intrinsic flux variations. We expect that a number of processes intrinsic to the pulsar contribute to the rotational phase, and therefore we model the barycentric phase as

$$\phi(t) = \phi_S(t) + \phi_R(t) + \phi_A(t) + \phi_M(t)\,. \tag{2}$$

Here ϕ_S describes the smooth spin-down of the star on very long time scales (10^3–10^9) years; ϕ_R is a random component that may operate on a variety of time scales; ϕ_A is an astrometric component that includes position errors, proper motion errors, and ephemeris errors; ϕ_M is measurement error.

Proper motion was discussed in Paper I, and for the remainder of this paper we will ignore it as a source of error because systematic errors appear to be dominated (over the 10 years' duration over which pulsars have been observed) by random measurement errors. Measurement errors are independent from one observation to the next and therefore, since they are also zero mean, we can write their second moment as

$$\langle \phi_M(t_k)\phi_M(t_l)\rangle = \sigma_M{}^2(t_k)\delta_{kl} . \quad (3)$$

Here—and elsewhere in the paper—angular brackets denote average over an ensemble, and δ_{kl} is the Kronecker delta.

b) Neutron Star Rotation

The ϕ_S and ϕ_R terms in equation (2) are written separately because it is presumed that they arise, respectively, from a deterministic spin-down torque that is applied to the star via its magnetosphere and from the non–rigid-body nature of the neutron star and its magnetosphere.

The rotation of a neutron star crust has been analyzed (Baym *et al.* 1969; Lamb, Pines, and Shaham 1978; Greenstein 1979) in terms of a differential equation for the crust angular frequency. The large-amplitude glitches—and the subsequent time behavior in particular—observed from the Crab and Vela pulsars (see Manchester and Taylor 1977, and references therein) and PSR 1641−45 (Manchester *et al.* 1978) have supported the view that neutron-star interiors contain neutron superfluid. However, it is not clear what quantities are involved in determining the amplitudes of the glitches. Nor is it known what underlies the more common timing-noise phenomenon evident in the Crab pulsar and the majority of other pulsars. Mathematically, any of a number of quantities may lead to the observed randomness in the rotational phase. The external torque itself may possess a random component, as for the case of X-ray binaries (Lamb, Pines, and Shaham 1978). Alternatively, the crust moment of inertia may vary due to starquakes (Pines and Shaham 1972); or the temperature of the star may vary, thus effecting changes in the frictional coupling of the crust with interior components (Greenstein 1979). Given the latitude of these possibilities, the approach that lends itself best to comparison with observation is to write fluctuations in the rotation frequency as

$$\delta\nu(t) = \sum_j a_j h_j(t - t_j) , \quad (4)$$

where the amplitude a_j and the occurrence time t_j of the jth event vary according to the appropriate probability distributions. The function $h_j(t)$ is the response of a star to an impulsive perturbation. In the next section we will model $h_j(t)$ in an idealized way. Meanwhile it is instructive to consider the rotation of a rigid object under the action of a random torque.

i) *Rigid Neutron Stars*

The rotation of a rigid neutron star is described by the equation

$$I\dot{\Omega} = N(t) , \quad (5)$$

where I is the moment of inertia, N is the torque, and $\Omega = 2\pi\nu$. The statistical properties of Ω are determinable from those of $N(t)$, an analysis that is equivalent to studies of one-dimensional Brownian motion with the Langevin equation (see Chandrasekhar 1943, eq. [132]). Suppose $N(t)$ can be written as the sum of a stochastic term δN and a deterministic term N_0:

$$N(t) = N_0(t) + \delta N(t) , \quad (6)$$

where N_0 is in general a function of Ω:

$$N_0(t) = \alpha\Omega^n ; \quad (7)$$

n is the so-called braking index. Then Ω is

$$\Omega(t) = \Omega_S(t) + \Omega_R(t) , \quad (8)$$

where

$$\Omega_S(t) = \Omega_0[1 - (n - 1)\dot{\Omega}_0(t - t_0)/\Omega_0]^{1/(n-1)} \quad (9)$$

is the spin-down function, with Ω_0 and $\dot{\Omega}_0$ being the appropriate values at the reference time t_0. The characteristic time scale of Ω_S is $T_S \sim |\Omega_0/\dot{\Omega}_0|$, and for radio pulsars it ranges from $10^{3.3}$ years (Crab pulsar) to $10^{9.8}$ years (PSR 1952+29). If $\Omega_R \ll \Omega_S$, as appears to be the case for all pulsars that have been carefully observed, then to first order in Ω_R/Ω_S,

$$\Omega_R(t) = I^{-1} \int^t dt'\delta N(t') + nI^{-1} \int^t dt' \frac{\dot{\Omega}_S(t')}{\Omega_S(t')} \int^{t'} dt''\delta N(t'') . \quad (10)$$

For times $t - t_0 \ll T_S$, the second term of equation (10) is negligible compared to the first. In general, however, noise in the torque of a particular kind will produce noise Ω_R that is a mixture of two kinds of noise. If the torque is a sequence of delta functions with random amplitudes, for example, then the first term of equation (10) will be a random walk in Ω_R and the second term will represent a random walk in $\dot{\Omega}_R$. We have assumed that torque variations are responsible for timing noise. Variations in the moment of inertia as an alternative source of noise will essentially reproduce the results for torque variations.

ii) *Nonrigid Stars*

Nonrigid stars differ from rigid ones in that their response to an impulsive torque will not be instantaneous. The two-component model of Baym *et al.* (1969) has a characteristic response time τ. Lamb, Pines, and Shaham (1978) have thoroughly analyzed the general case of a three-component star, although their treatment has assumed that N_0 is not a function of Ω. As far as the purposes of this paper are concerned, the question is whether the time interval T over which observations are made is large or small compared to the response time. For $T \gg \tau$, the star may as well be considered rigid; however, if $T \ll \tau$, then the crust is effectively decoupled from the superfluid. The analysis of rotational phase described herein cannot directly determine τ if individual events are undetectable. However, the two cases $T \gg \tau$ and $T \ll \tau$ will be manifested as two different *kinds* of timing noise, as defined in § III.

c) *On the Separability of ϕ_R and ϕ_S*

Implicit in our model of the rotational phase (eq. [2]) is that the deterministic spin-down is mathematically separable from random variations that occur on much shorter time scales. In practice we can adequately describe the spin-down function by expanding it in a Taylor series:

$$\phi_S(t) = \phi_S(t_0) + \sum_{k=1} \frac{d^k\phi_S(t_0)}{dt^k}(t - t_0)^k/k!\,, \quad (11)$$

and evaluating the rotation frequency, frequency derivative, etc., at the epoch t_0:

$$\nu_S(t_0) = d\phi_S(t_0)/dt\,, \quad (12)$$

$$\dot{\nu}_S(t_0) = d^2\phi_S(t_0)/dt^2\,. \quad (13)$$

Except for the Crab pulsar, only ν_S and $\dot{\nu}_S$ appear to contribute significantly to the phase over the available time spans of the data, and the corresponding terms in the phase dominate the random component. Of course, one can measure nonzero values of $\ddot{\nu}$ for a number of pulsars (Gullahorn and Rankin 1978, 1980), but these values are much larger than those predicted by spin-down mechanisms that are thought acceptable. At present, we will attribute such values of $\ddot{\nu}$ to changes in curvature in $\phi = \phi_S + \phi_R$ produced solely by the random component, ϕ_R. For the Crab pulsar, $\ddot{\nu}_S$ has also been measured (e.g., Groth 1975*c*), but $\dddot{\nu}_S$ has not been because random variations are comparable to the $\dddot{\nu}_S(t - t_0)^4/24$ term of the phase.

Our assumption that a deterministic spin-down phase $\phi_S(t)$ is separable from the random phase suggests that we should treat ϕ_R and its derivatives as zero-mean random processes:

$$\langle d\phi_R{}^l(t)/dt^l\rangle = 0\,, \quad l = 0, 1, 2, \ldots\,. \quad (14)$$

Physically there is no reason why ϕ_R should necessarily have zero-mean derivatives. As far as measurements are concerned, however, there is no direct way to separate non–zero-mean derivatives of ϕ_R from ν_S, $\dot{\nu}_S$, etc., unless one has a model that relates rms values of $\dot{\phi}_R$, $\ddot{\phi}_R$, $\dddot{\phi}_R, \ldots$, to their mean values. The observable consequences of ϕ_R being a non–zero-mean random process will be discussed in § V. Meanwhile we will attribute zero-mean derivatives to ϕ_R. It should be noted that although the *ensemble-average* derivatives of ϕ_R are assumed to be zero, the derivatives for any particular realization will be distributed over a finite range centered on zero.

III. MODELS FOR THE RANDOM ROTATIONAL COMPONENT

a) *Superposition of Events*

The random component of the rotational phase (which would be the integral of eq. [4]) can be represented as a superposition of contributions

$$\phi_R(t) = \sum_j a_j g_j(t - t_j)\,, \quad (15)$$

where the jth term occurs at a time t_j with amplitude a_j and with a response function $g_j(t)$. In general we need not make any assumption about the rate of occurrence of events. If we assume that events occur independently, however, then the t_j are Poisson distributed and ϕ_R takes on the mathematical form of shot noise. We are most interested in the moments of ϕ_R, and the formalism of shot noise (Rice 1954) easily provides them. As shown in the Appendix, the first and second moments of ϕ_R are

$$\langle\phi_R(t)\rangle = \langle a\rangle \int dt' R(t')g(t - t')\,, \quad (16)$$

$$\langle\phi_R(t_1)\phi_R(t_2)\rangle = \langle\phi_R(t_1)\rangle\langle\phi_R(t_2)\rangle + \langle a^2\rangle \int dt' R(t')g(t_1 - t')g(t_2 - t')\,, \quad (17)$$

where we have further assumed that the moments of a are independent of time and that all events have the same time behavior. The rate at which events occur, $R(t)$, has not been assumed to be time independent. If ϕ_R is a zero-mean random process, then $\langle a\rangle = 0$. As shown in later sections, our primary diagnostic for studying the random process will be the second moment.

b) *Random Walk Processes*

Boynton *et al.* (1972) and Groth (1975*b*) hypothesized that rotational irregularities from the Crab pulsar might be produced by a random walk in either the phase, the frequency, or the frequency derivative. In general, a random walk $r(t)$ is the integral of a sequence, $q(t)$, of pulses, $\Delta(t)$:

$$r(t) = \int_0^t dt' q(t')\,, \quad (18)$$

$$q(t) = \sum_j a_j\Delta(t - t_j) = \Delta(t) * \sum_j a_j\delta(t - t_j)\,, \quad (19)$$

where $\Delta(t)$ is a pulselike function and $q(t)$ can be represented as the convolution ($*$) of $\Delta(t)$ with a sequence of delta functions. If the rate at which impulses occur is independent of time and if the amplitudes have stationary statistics, then $q(t)$ is a stationary random process. However, the random walk $r(t)$ is nonstationary regardless of whether or not $q(t)$ is stationary.

We apply the formal definitions of a random walk in equation (18) to the model for phase irregularities in equation (15) by defining random walks in the kth derivative of the phase (following Lamb, Pines, and Shaham 1978) as

$$d^k\phi_R(t)/dt^k = r(t)\,. \tag{20}$$

The duration of $\Delta(t)$ is an important diagnostic for the structure of neutron stars if it is large (or small) enough to be measurable. We will assume that $\Delta(t)$ is not resolved by observations, in which case $\Delta(t)$ is effectively a delta function and $r(t)$ is a sequence of unit step functions, $H(t)$. Under this assumption, the random walks for $K = 0, 1, 2$ correspond to phase noise (PN), frequency noise (FN), and slowing-down noise (SN) as defined by Groth (1975*b*). We have

$$\phi_R(t) = \sum_j \delta\phi_j H(t - t_j) \quad (\text{PN}, k = 0)\,, \tag{21}$$

$$\dot\phi_R(t) = \sum_j \delta\nu_j H(t - t_j) \quad (\text{FN}, k = 1)\,, \tag{22}$$

$$\ddot\phi_R(t) = \sum_j \delta\dot\nu_j H(t - t_j) \quad (\text{SN}, k = 2)\,, \tag{23}$$

from which the quantity $g(t)$ in equation (15) can be identified and the moments determined from equations (16) and (17). Table 1 gives $\langle\phi_R(t)\rangle$ and

$$\langle\phi_R(t_1)\phi_R(t_2)\rangle - \langle\phi_R(t_1)\rangle\langle\phi_R(t_2)\rangle$$

assuming that random walks commence at $t = 0$. Section IV will make frequent use of the second moments of the random walks in the analysis of pulse arrival times.

c) The Time Origin of Random Walks

The moments expressed in Table 1 assume that random walks begin at $t = 0$. It is clear that the beginning of a set of measurements of pulse arrival times has nothing to do with the onset of the random walk. The rms phase for PN, FN, and SN increases, respectively, as $t^{1/2}$, $t^{3/2}$, and $t^{5/2}$, and therefore the observed statistics will depend on the time origin of measurements. It can be shown that if measurements begin at $t = t_0$, then the random walk can be written as a random walk commencing at $t = t_0$, ϕ_R', plus terms describing the random walk prior to t_0:

$$\begin{aligned}\phi_R(t) &= \phi_R(t_0) + \phi_R'(t - t_0) \quad (\text{PN})\,,\\ &= \phi_R(t_0) + \dot\phi_R(t_0)(t - t_0) + \phi_R'(t - t_0) \quad (\text{FN})\,,\\ &= \phi_R(t_0) + \dot\phi_R(t_0)(t - t_0) + \tfrac{1}{2}\ddot\phi_R(t_0)(t - t_0)^2\\ &\quad + \phi_R'(t - t_0) \quad (\text{SN})\,.\end{aligned} \tag{24}$$

The net effect of the random walk prior to t_0 is to adulterate the random walk beginning at t_0 by a net phase for PN, a net phase and frequency for FN, and a net phase, frequency, and frequency derivative for SN. That is, if random walks are assumed to commence at $t = t_0$, then the estimates for ν_S and $\dot\nu_S$ at $t = t_0$ will generally have absorbed contributions from the random walks. The absorbed contributions will have zero mean values over an ensemble (if the random walk steps have zero mean amplitudes), but for any given set of observations they will be distributed with standard deviations

$$\delta\nu_S \sim \langle[\dot\phi_R(t_0)]^2\rangle \quad (\text{FN and SN})\,, \tag{25}$$

$$\delta\dot\nu_S \sim \langle[\ddot\phi_R(t_0)]^2\rangle^{1/2} \quad (\text{SN})\,. \tag{26}$$

If SN is operative in any pulsar, then the measured braking index, $n = \nu\ddot\nu/\dot\nu^2$, will be in error according to the error of the frequency derivative. The most important consequence is that the ensemble-average second moment, $\langle\phi_R^2(t)\rangle$, at the start time of observations is larger for a random walk starting at $t = 0 < t_0$ than for a walk starting at $t = t_0$. Fortunately, and this is the main point of this discussion, this effect is obviated by fitting a polynomial to the observed phase because the coefficients of the polynomial absorb the undesired terms.

IV. ESTIMATION OF RANDOM WALK PARAMETERS

a) Analysis of the Second Moment

In this section we outline a procedure for estimating the random walk strength when only the composite quantity

$$\phi = \phi_S + \phi_R + \phi_M \tag{27}$$

TABLE 1

MOMENTS FOR THREE RANDOM WALK PROCESSES

Quantity	PN	FN	SN
$\langle\phi_R(t)\rangle$[a]	$R\langle\delta\phi\rangle t$	$\frac{1}{2}R\langle\delta\nu\rangle t^2$	$\frac{1}{6}R\langle\delta\dot\nu\rangle t^3$
$\langle\phi_R(t_1)\phi_R(t_2) - \langle\phi_R(t_1)\rangle\langle\phi_R(t_2)\rangle$[a]	$R\langle\delta\phi^2\rangle t_<$	$\frac{1}{6}R\langle\delta\nu^2\rangle t_<^2(3t_> - t_<)$	$\frac{1}{120}R\langle\delta\dot\nu^2\rangle t_<^3(10t_>^2 - 5t_<t_> + t_<^2)$
$\langle\nu_R(t)\rangle$	$R\langle\delta\phi\rangle$	$\int_0^t dt' R(t')\langle\delta\nu(t')\rangle$	$\int_0^t dt' R(t')(t - t')\langle\delta\dot\nu(t')\rangle$
$\langle\dot\nu_R(t)\rangle$	$\frac{d}{dt}[R\langle\delta\phi\rangle]$	$R\langle\delta\nu\rangle$	$\int dt' R(t')\langle\delta\dot\nu(t')\rangle$
$\langle\ddot\nu_R(t)\rangle$	$\frac{d^2}{dt^2}[R\langle\delta\phi\rangle]$	$\frac{d}{dt}[R\langle\delta\nu\rangle]$	$R\langle\delta\dot\nu\rangle$

NOTE.—$t_<(t_>)$ is the smaller (larger) of t_1 and t_2.

[a] These quantities were derived assuming R and the step amplitude moments are independent of time.

is measured. The difficulty arises because ϕ_S is a low-order polynomial and ϕ_R also has components with similar polynomial behavior. Groth (1975*c*) tackled the problem by expanding ϕ into a series of orthogonal polynomials and comparing the polynomial coefficients with those predicted for the random walk models. The procedure followed here is to estimate ϕ_S by a polynomial of order m, ϕ_P, and then compare the variance of the residuals that is in excess of measurement-error variance with the variance expected for a random walk.

Let the polynomial fit to ϕ be

$$\phi_P = \phi_S + \phi_{PR} + \phi_{PM} , \qquad (28)$$

where ϕ_{PR} and ϕ_{PM} are the low-order polynomial components of ϕ_R and ϕ_M, respectively. The residuals $\mathscr{R} = \phi - \phi_P$ have an ensemble-average second moment

$$\sigma_{\mathscr{R}}^2 = \langle(\phi - \phi_P)^2\rangle = \langle(\phi_R - \phi_{PR})^2\rangle + \langle(\phi_M - \phi_{PM})^2\rangle , \qquad (29)$$

assuming measurement errors are independent of ϕ_R and that ϕ_R is zero mean.

In practice, ensemble-average quantities are unavailable, and one must deal with integral *estimates* of the moments such as the following:

$$\sigma_R^2(T) \equiv T^{-1} \int_0^T dt \phi_R^2(t) , \qquad (30)$$

as an estimate for the second moment of $\phi_R(t)$. The measurable quantity of interest is

$$T^{-1} \int_0^T dt(\phi_R - \phi_{PR})^2 = \sigma_{\mathscr{R}}^2(T) - \sigma_M^2 , \qquad (31)$$

which is an underestimate of the desired quantity, $\sigma_R^2(t)$. We quantify the underestimation of σ_R^2 by considering the ratio of pre-fit rms phase to post-fit rms phase:

$$C_R(m, T) \equiv \left[\int_0^T dt\phi_R^2(t) \Big/ \int_0^T dt(\phi_R - \phi_{PR})^2\right]^{1/2} , \qquad (32)$$

where m is the order of the polynomial ϕ_P. The quantity $C_R(m, T)$ is a "correction" factor for the rms value of ϕ_R after a polynomial fit. It itself is a random variable because it fluctuates from realization to realization of ϕ_R.

TABLE 2

CORRECTION FACTORS FOR THREE RANDOM-WALK PROCESSES

ORDER OF POLYNOMIAL	CORRECTION FACTORS PN	FN	SN
0.................	1.73	1.58	1.38
1.................	2.90	6.46	3.87
2.................	3.70	15.5	23.7
3.................	4.14	27.3	71.1
4.................	4.70	36.1	151.7
5.................	5.20	49.0	234.5

Table 2 lists the average correction factors obtained from 100 realizations of each kind of random walk. The factors can be worked out analytically (in the limit of a large number of equally spaced samples), but it is tedious except for a zeroth order polynomial, results for which agree within 5% of the values listed in Table 2. Simulations were performed for a range of time spans T and for both uniform and irregular sampling. The correction factors are (1) independent of the time span T as long as $RT \gg 1$ and the number of samples N satisfies $N \gg m$, (2) independent of the sampling as long as the number of missing samples is less than $\sim 0.5N$ and the largest gap is smaller than $\sim 0.2T$, and (3) variable over a factor of 8 range from one realization to the next. Simulations also show that for a given random-walk strength, the variance σ_R^2 can vary by a factor of 1000. The independence of $C(m, T)$ from T is not surprising because it reflects the fact that a random walk has no characteristic time scale (as long as individual steps are unresolved). In other words, apart from a scale factor, the spectrum of a random walk in polynomial space has the same form independent of time span T.

Note that the correction factors for SN vary the most with order of polynomial and those for PN vary the least. This follows because SN has a narrower spectrum in polynomial space than does PN. Therefore, for example, a fifth-order polynomial absorbs a much larger fraction of the variance of SN than of the PN or FN variance.

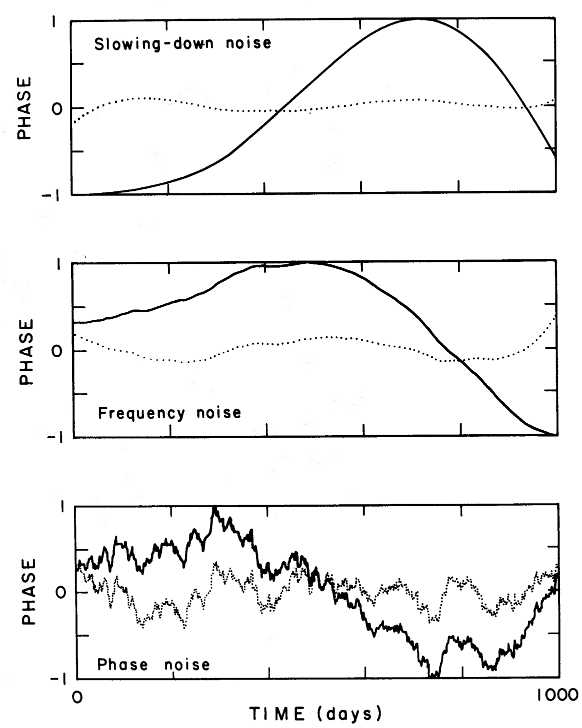

FIG. 1.—Realizations of three kinds of random walks computed with $T = 1000$ days, $R = 1$ day^{-1}. The phase is shown before (*solid line*) and after (*dotted line*) removing a least-squares, third order polynomial fit. The pre-fit phase has been shifted by an arbitrary amount for clarity; at $t = 0$ the pre-fit phase is actually zero.

Figure 1 shows a realization of each kind of random walk and the residuals of each walk from a third-order least-squares polynomial fit. It is apparent that the fit decreases the variance of SN the most and PN the least.

Strengths of the random walks are calculated according to

$$S = \langle C_R(m, T)\rangle^2 \left(\frac{\sigma_{\mathscr{R}}^2(T) - \sigma_M^2}{\langle \sigma_R^2(T)\rangle_u}\right) S_u \,, \quad (33)$$

where $\langle \sigma_R^2(T)\rangle_u$ is the ensemble-average second moment for a unit strength ($S_u = 1$) random walk. For arbitrarily sampled data, we have

$$\langle \sigma_R^2(T)\rangle_u = N^{-1} \sum_{j=1}^{N} \langle \phi_R^2(t_j)\rangle \,, \qquad 0 \le t_1 \le t_N \le T \,, \quad (34)$$

where $\langle \phi_R^2(t_j)\rangle$ is obtained from Table 1 assuming that the random walks are zero mean and that the strengths, $S_{\rm PN} = R\langle \delta\phi^2\rangle$, $S_{\rm FN} = R\langle \delta\nu^2\rangle$, and $S_{\rm SN} = R\langle \delta\dot{\nu}^2\rangle$ are unity with units of s^{-1}, $Hz^2\ s^{-1}$, and $Hz^2\ s^{-3}$, respectively.

b) *Computation of Measurement Errors*

The variance of measurement errors can be calculated empirically by finding the variance of the residuals after removing a running linear trend. A local approximation to the residual phase $\mathscr{R}_j = \phi(t_j) - \phi_P(t_j)$ is

$$\hat{\mathscr{R}}_j = \mathscr{R}_{j-1} + \frac{(\mathscr{R}_{j+1} - \mathscr{R}_{j-1})(t_j - t_{j-1})}{t_{j+1} - t_{j-1}} \,. \quad (35)$$

For equally spaced samples, the measurement error variance can be estimated as

$$\sigma_M^2 \approx \frac{2}{3N} \sum_{j=1}^{N} (\mathscr{R}_j - \hat{\mathscr{R}}_j)^2 \,, \quad (36)$$

which itself will be in error according to the amplitudes of the quadratic and higher-order components of the residuals. If the time differences (e.g., $t_{j+1} - t_{j-1}$) are constrained below some upper limit, then the estimation error of σ_M^2 can be constrained below some desired limit.

c) *Application to the Crab Pulsar*

Groth (1975*c*) determined that timing noise from the Crab pulsar was consistent with a random walk in the frequency with strength $R\langle \delta\nu^2\rangle = 0.53 \times 10^{-22}\ Hz^2\ s^{-1}$. To show that timing noise is consistent with a random walk model, it is necessary to show that the noise and the model have consistent moments over a range of time scales. This determination also requires that the random walk strength not vary considerably with time. In this section we apply the technique of this paper to the optical timing data of the Crab pulsar that was analyzed by Groth.

Barycentric arrival times were calculated from the topocentric arrival times tabulated by Groth (1975*a*) by using the Lincoln Lab solar system ephemeris and by using the identical observatory and pulsar coordinates used by Groth. The 348 arrival times were divided into 12 sub-blocks as defined by Groth (1975*c*), and an *m*th-order polynomial was least-squares fitted to the data of each block. Blocks shorter than a year were fitted by a polynomial of order $m = 3$; longer blocks were fitted with $m = 4$ because the phase contributed by the third derivative $\dddot{\nu}_S$ (which can be estimated from $\dddot{\nu}_S = n(2n - 1)\dot{\nu}_S^3 \nu_S^{-2}$) becomes comparable to the rms phase of the random process for $T \sim 3$ yr. The fits differed from those of Groth in that (1) non-orthogonal polynomials were used and (2) all points were weighted equally whereas Groth weighted points proportional to the inverse square of the measurement errors. The rms residual from the fit was computed, and rms measurement errors were computed according to equation (36). Comparison with measurement errors tabulated by Groth (1975*a*) indicates that equation (36) indeed gives an accurate estimate of the rms measurement error. Such estimates will prove to be essential for the analysis of other pulsars described in Paper III.

Random walk strengths calculated from equation (33) are shown in Table 3. The results agree with Groth's insofar as phase noise and slowing-down noise are inconsistent with the data because the derived strengths for long time spans differ by two or more orders of magnitude from strengths for short time spans. The PN and SN strengths are not the same as Groth's because, as will be demonstrated below, they are functions of the order of the polynomial fit. The strengths for frequency noise differ by less than an order of magnitude on all time scales from 94 to 1628 days, suggesting that frequency noise is indeed consistent with the second moment of the residuals. The average FN strength obtained from blocks 4 through 12 is $R\langle \delta\nu^2\rangle = 0.66 \times 10^{-22}\ Hz\ s^{-1}$, whereas the longer blocks (2 and 3), which are less affected by a possible phase-noise component (Groth 1975*c*), yield $R\langle \delta\nu^2\rangle = 1.22 \times 10^{-22}\ Hz\ s^{-1}$; both values agree (within the errors to be discussed below) with those obtained by Groth. A final point of comparison between our results and Groth's concerns the slowing-down parameters, ν_S, $\dot{\nu}_S$, $\ddot{\nu}_S$. Despite the different weighting procedures used, these parameters and the derived braking indices agree with Groth's within the uncertainty caused by measurement errors.

The derived strengths for PN and SN are—given that the random walk is FN—functions of both the time span of the data and the order of the polynomial that is fitted to the data. It follows from equation (33) that the estimated strengths are, for uniformly sampled data,

$$\hat{S}_{\rm PN} = S_{\rm FN}[C_{\rm PN}(m)/C_{\rm FN}(m)]^2 T^2/6 \,, \quad (37)$$

$$\hat{S}_{\rm SN} = S_{\rm FN}[C_{\rm SN}(m)/C_{\rm FN}(m)]^2 10 T^{-2} \,, \quad (38)$$

where we have dropped the T dependence of the correction factors. It can be verified that the strengths for PN and SN in Table 3 conform to those predicted by these equations.

TABLE 3

RANDOM WALK STRENGTHS FOR THE CRAB PULSAR

Block (1)	Julian Day Limits (JD − 2,440,000) Begin (2)	End (3)	Length (days) (4)	Number of Points (5)	RMS Residual (ms) (6)	RMS Error (ms) (7)	Random Walk Strengths $R\langle\delta\phi^2\rangle \times 10^{11}$ (s^{-1}) (8)	$R\langle\delta\nu^2\rangle \times 10^{22}$ ($Hz^2\,s^{-1}$) (9)	$R\langle\delta\dot{\nu}^2\rangle \times 10^{36}$ ($Hz^2\,s^{-1}$) (10)
1......	535	2164	1628	312	5.01	0.048	802.0	1.54	1.38
2......	535	1420	883	208	2.61	0.055	355.0	1.98	5.57
3......	1591	2164	572	104	0.601	0.028	35.0	0.456	2.91
4......	288	526	189	24	0.175	0.107	5.49	0.355	6.58
5......	535	704	167	54	0.071	0.020	1.06	0.129	4.14
6......	829	926	95	39	0.056	0.017	1.07	0.378	35.0
7......	969	1065	94	46	0.057	0.028	1.13	0.403	39.0
8......	1175	1288	111	28	0.123	0.030	4.32	1.01	67.7
9......	1296	1420	122	41	0.151	0.013	6.49	1.38	76.6
10......	1591	1700	107	34	0.086	0.014	2.63	0.697	46.3
11......	1704	1803	98	31	0.080	0.013	2.39	0.852	78.0
12......	1974	2164	183	37	0.214	0.049	8.35	0.730	17.6

d) *Estimation Errors for the Strength of the Random Process*

The strength of the random process can be determined only as accurately as measurement errors and the stochastic nature of the random process allow. If the rms residual is much larger than measurement errors, then mis-estimation of measurement errors is inconsequential. Long data spans provide the best estimates for the strength, as far as measurement errors are concerned, because the random walk phase increases with data span length T and because the error in the estimate of σ_M decreases with T. However, the nonstationarity of the random process itself ultimately determines the accuracy to which its strength can be determined. Consider the quantity

$$\sigma_R^2(T) = T^{-1}\int_0^T dt\phi_R^2(t)\,. \tag{39}$$

For a *stationary* process, σ_R^2 converges to its ensemble-average value, $\langle\phi_R^2(T)\rangle$, as $T \to \infty$. That is, the fractional estimation error in $\sigma_R^2(T)$,

$$\epsilon \equiv [\langle\sigma_R^4(T)\rangle - \langle\sigma_R^2(T)\rangle^2]^{1/2}/\langle\sigma_R^2(T)\rangle\,, \tag{40}$$

goes to zero as $T \to \infty$. For nonstationary random walks, however, $\sigma_R^2(T)$ does not converge to its ensemble average value. For PN we have

$$\epsilon(T) = \left(\frac{2}{3}\right)^{1/2}\left[1 + \frac{\langle\delta\phi^4\rangle}{\langle\delta\phi^2\rangle^2 RT}\right]^{1/2} \xrightarrow[T\to\infty]{} \left(\frac{2}{3}\right)^{1/2}\cdot \tag{41}$$

If phase steps $\delta\phi$ are of roughly equal amplitude, then for $RT \gg 1$, the estimated second moment will vary from one realization to another with a standard deviation $(2/3)^{1/2}\langle\sigma_R^2(T)\rangle$, *regardless of the value of T.* If phase steps themselves are highly variable, then $\langle\delta\phi^4\rangle/\langle\delta\phi^2\rangle^2 \gg 1$ and ϵ will be correspondingly greater for finite T. For FN and SN we have

$$\lim_{T\to\infty} \epsilon(T) = (33/35)^{1/2} \quad \text{(FN)}$$

$$= (905/924)^{1/2} \quad \text{(SN)}\,; \tag{42}$$

and for finite T, ϵ will deviate from these values by terms of order $\langle\delta\nu^4\rangle/\langle\delta\nu^2\rangle^2 RT$ and $\langle\delta\dot{\nu}^4\rangle/\langle\delta\dot{\nu}^2\rangle^2 RT$, respectively.

To rephrase these results, estimates of the random-walk second moment do not improve as one integrates for longer times (as long as we ignore measurement error). The rms phase $\phi_R(T)$ has a fractional fourth moment that is smaller than that for a Gaussian random variable for which $\epsilon = \sqrt{2}$.

Numerical simulations indicate that the variance of the *residual* phase $\sigma_{\mathscr{R}}^2(T)$ and the correction factors have statistics similar to those of $\sigma_R^2(T)$. Moreover, $C_R(m)$ and $\sigma_{\mathscr{R}}$ appear to vary independently, and therefore the variance of the strength estimates is the sum of the variances of $C_R(m)$ and $\sigma_{\mathscr{R}}$. For FN, this means that the fractional variance of the strength approaches $\sqrt{2}$, the value for the square of a Gaussian random variable. Consequently, for the purposes of an error analysis of our strength determinations, we can assume that a strength estimate that is the sum of N independent strength estimates is a χ^2 random variable with N degrees of freedom.

The results of this section yield a prescription for assigning errors to the derived strengths of FN for the Crab pulsar. Summing strength estimates for the nine shorter blocks (4–12) yields an estimate with 9 degrees of freedom, and therefore the 68% confidence interval for the strength is $R\langle\delta\nu^2\rangle = 0.66(+0.31, -0.30) \times 10^{-22}$ Hz^2 s^{-1}. Blocks 2 and 3 yield $R\langle\delta\nu^2\rangle = 1.22(+0.92 -1.13) \times 10^{-22}$ Hz^2 s^{-1}. For blocks 4–12, the sample standard deviation is 0.39×10^{-22} Hz^2 s^{-1}, indicating that the strengths vary as expected from our error analysis.

e) *Low versus High Rate Random Walks*

Analysis of the second moment yields direct information on only the product, $R\langle\delta\nu^2\rangle$, for the Crab pulsar. However, the fact that a consistent strength is obtained on time scales down to 10 days (i.e., from the 10–13 order polynomials fit to 100 day spans of data by Groth) and that structure is evident in the residuals with time scales of a few days implies that the rate is larger than one event every few days. In anticipation of the results on other pulsars, it is worthwhile to consider how a second-moment analysis can determine the rate R. The parameter that distinguishes small rates from large ones is

$$\zeta = \langle a^4 \rangle / \langle a^2 \rangle^2 RT , \quad (43)$$

where a is the step amplitude in the appropriate units and T is the time span for the analysis. If $\zeta \ll 1$, then the strength estimates will vary with a ratio of standard deviation to the mean of $\sim\sqrt{2}$, as discussed before (cf. eq. [41]). However, as $\zeta \rightarrow 1$, strength estimates will vary in excess of $\sqrt{2}$. When $\zeta \gg 1$, individual steps will occur at resolvable times, and R and $\langle a^4 \rangle / \langle a^2 \rangle^2$ can be separately estimated. If $\zeta \gg 1$, an alternative method of analysis would be in order because the goal would then be to measure the properties of individual steps after first demonstrating their reality.

V. RANDOM WALKS WITH FINITE MEAN STEPS

As previously stated, the mean values of the steps of a random walk get absorbed into the slowing-down parameters. In this section, we demonstrate that the mean and third moment of the step amplitude can be measured under certain circumstances.

a) *The Second Moment*

The quantity available to us for studying the random walk is the residual phase after subtracting a least-squares polynomial. Define

$$\langle \delta m_2 \rangle = \left\langle T^{-1} \int_0^T dt [\phi(t) - \phi_l(t)]^2 \right\rangle \quad (44)$$

as the second moment of the phase after subtracting a polynomial of order l, $\phi_l(t)$. For FN, we can write

$$\langle \delta m_2 \rangle_{\rm FN} = A_l R^2 \langle \delta\nu \rangle^2 T^4 + B_l R \langle \delta\nu^2 \rangle T^3 . \quad (45)$$

It can be shown that $A_l = 0$ for $l \geq 2$ because the mean value of $\delta\nu$ produces a second-order phase variation, corresponding to a frequency derivative $R\langle \delta\nu \rangle$. Unfortunately, such a frequency derivative cannot be separated from that of the slowing-down function. For the case of slowing-down noise, however, we have

$$\langle \delta m_2 \rangle_{\rm SN} = A_l R^2 \langle \delta\dot{\nu} \rangle^2 T^6 + B_l R \langle \delta\dot{\nu}^2 \rangle T^5 ; \quad (46)$$

and in this case, $R\langle \delta\dot{\nu} \rangle$ corresponds to a frequency second derivative, which may possibly be measurable for slow pulsars because $\ddot{\nu}_s$ is negligible if the braking index is as small as it is for the Crab pulsar. For PN and FN, however, it is clear that the second moment of the phase residuals can yield information on only the quantities $R\langle \delta\phi^2 \rangle$ and $R\langle \delta\nu^2 \rangle$, respectively.

b) *The Third Moment*

Now consider the third moment of the residuals,

$$\langle \delta m_3 \rangle = \left\langle T^{-1} \int_0^T dt [\phi(t - \phi_l(t)]^3 \right\rangle , \quad (47)$$

which can be written for FN as

$$\langle \delta m_3 \rangle_{\rm FN} = \alpha_l R^3 \langle \delta\nu \rangle^3 T^6 + \beta_l R^2 \langle \delta\nu \rangle \langle \delta\nu^2 \rangle T^5 + \gamma_l R \langle \delta\nu^3 \rangle T^4 . \quad (48)$$

Only if the distribution of $\delta\nu$ is symmetric about the origin will $\langle \delta m_3 \rangle_{\rm FN}$ be zero. For $l > 2$, $\alpha_l = \beta_l = 0$, whereas $\gamma_l \neq 0$ in general, and therefore information about $R\langle \delta\nu^3 \rangle$ is obtainable from the third moment. Numerical simulations indicate that $\gamma_2 \approx -1.1 \times 10^{-5}$, $\gamma_3 \approx -3.3 \times 10^{-6}$, $\gamma_4 \approx -8.3 \times 10^{-7}$.

We can estimate the third moment for the Crab pulsar by assuming that $\langle \delta\nu^3 \rangle = \pm \langle \delta\nu^2 \rangle^{3/2}$ and therefore $R\langle \delta\nu^3 \rangle = \pm R^{-1/2} S_{\rm FN}{}^{3/2}$, where $S_{\rm FN} = R\langle \delta\nu^2 \rangle \approx 0.53 \times 10^{-22}$ Hz^2 s^{-1}. Expressing R in inverse days, we have (after a fourth-order fit)

$$\langle \delta m_3 \rangle = \gamma_4 R \langle \delta\nu^3 \rangle T^4 \approx \pm 5.25 \times 10^{-6} R_{\rm day}{}^{-1/2} T_{1000}{}^4 \text{ cycles}^3 , \quad (49)$$

where T_{1000} is the span of time in units of 1000 days. Measurement errors have a zero third moment over an ensemble, but an estimate of the third moment will have a standard deviation of

$$\sigma_3 \approx \sigma_M{}^3 N^{-1/2} , \quad (50)$$

which is smaller than the predicted $\langle \delta m_3 \rangle$ for most of the 12 blocks of Crab data defined in Table 3. Again the dominant error in δm_3 is due to the stochastic nature of the random walk. The standard derivation of δm_3 is

$$\langle \delta m_3{}^2 \rangle^{1/2} \approx K_l (R \langle \delta\nu^2 \rangle T^3)^{3/2} \quad (51)$$

if we assume that $\langle \delta m_3 \rangle = 0$ (i.e., $\delta\nu$ has a symmetric distribution), and simulations indicate that $K_2 = 4.3 \times 10^{-6}$, $K_3 = 1.0 \times 10^{-6}$, $K_4 = 2.3 \times 10^{-7}$. For the Crab, we have

$$\langle \delta m_3{}^2 \rangle^{1/2} = K_4 (R \langle \delta\nu^2 \rangle T^3)^{3/2} \approx 4.6 \times 10^{-5} T_{1000}{}^{4.5} \text{ cycles}^3 , \quad (52)$$

which, upon comparison with equation (49), indicates that δm_3 will not be measurably distinct from stochastic fluctuations unless $R_{\rm day} T_{1000} \ll 1$, or $\langle \delta\nu^3 \rangle \gg \langle \delta\nu^2 \rangle^{3/2}$. It is known that $R_{\rm day} \gtrsim 1$. Furthermore, estimates of the third moment for the Crab pulsar are within the bounds of equation (52). Therefore, the primary conclusion is that the frequency steps do not have a *highly* skewed distribution. The main hope for measuring δm_3 would be to make intraday measurements, but then measurement errors (eq. [50]) would dominate the third-moment estimate.

c) *Finite Mean Steps and the Braking Index of the Crab Pulsar*

If the random walk steps have non-zero-mean amplitudes, then the random walk contributes to the measured *systematic* frequency and frequency derivatives. The FN process that operates for the Crab pulsar has frequency derivatives

$$\left\langle \frac{d^k \nu_{\rm FN}(t)}{dt^k} \right\rangle = \frac{d^{k-1}}{dt^{k-1}} (R\langle \delta\nu \rangle) . \quad (53)$$

Consequently, the measured braking index may or may not reflect that of the deterministic torque process. In particular, we show here how the measured braking index for the Crab pulsar, $n_m = 2.51$, may have been biased away from the torque index of 3, as predicted by the electromagnetic torque model (Pacini 1968; Gunn and Ostriker 1970) and by the relativistic-wind model (Goldreich and Julian 1969).

Let n_S be the index of the spin-down torque model. That is, given the rotation frequency, $\nu = \nu_S + \langle \nu_{\rm FN} \rangle$, we have the interrelationship

$$n_S = \nu \ddot{\nu}_S \dot{\nu}_S^{-2} . \quad (54)$$

The *observed* braking index is

$$n_m = n_S (1 + \langle \ddot{\nu}_{\rm FN} \rangle / \ddot{\nu}_S)(1 + \langle \dot{\nu}_{\rm FN} \rangle / \dot{\nu}_S)^{-2} , \quad (55)$$

and can be written as

$$n_m = n_S \ddot{\nu}_m (\dot{\nu}_m - x)^2 / (\ddot{\nu}_m - y) , \quad (56)$$

where $x = R\langle \delta\nu \rangle$, $y = dx/dt$, and $\dot{\nu}_m$ and $\ddot{\nu}_m$ are the *measured* derivatives of the frequency.

A set of values of $(x, y) = (R\langle \delta\nu \rangle, dR\langle \delta\nu \rangle/dt)$ can cause the measured braking index to deviate from the torque index, n_S; however, only a subset of values is physically plausible. If the random walk is a consequence of rotation (e.g., the starquake model), then it is plausible that $|R\langle \delta\nu \rangle|$ should decrease with time as the oblateness of the star decreases. Suppose the random walk is a series of spin-ups: $R\langle \delta\nu \rangle > 0$. We therefore require $dR\langle \delta\nu \rangle/dt \leq 0$. Moreover, we should expect that the dissipation time for the random walk,

$$t_d \equiv \left| R\langle \delta\nu \rangle \Big/ \frac{d}{dt} R\langle \delta\nu \rangle \right| , \quad (57)$$

be reasonable, i.e., much greater than the 10 years over which the Crab pulsar has been observed and probably not much different from the spin-down time scale, $T_S = \nu_m/\dot{\nu}_m = 2600$ years.

In Figure 2 we show those values of x and y that satisfy equation (56) for various values of the true braking index, n_S. We have taken $n_m = 2.51$, $\dot{\nu}_m = -3.8559 \times 10^{-10}$ Hz s^{-1}, and $\ddot{\nu}_m = 1.2341 \times 10^{-20}$ Hz s^{-2}. If the steps are spin-ups ($\delta\nu > 0$), then reasonable dissipation times ($t_d \gtrsim 100$ years, say) are obtained only for very large rates: $R \gtrsim 10^5$ steps per day for $n_S = 3$. We estimate the rate by assuming all steps have the same amplitude. Then a *lower limit* to the rate is

$$R \gtrsim (R\langle \delta\nu \rangle)^2 / R\langle \delta\nu^2 \rangle = x^2 S_{\rm FN}^{-1/2} . \quad (58)$$

As $n_S - n_m \to 0^+$, $t_d \geq 100$ years for smaller rates; indeed, $R_{\rm day} = 1$ is possible only if $|n_S - n_m| \lesssim 10^{-3}$. If the random walk comprises spin-downs, then similarly large rates are required if n_S deviates significantly from the measured index.

In summary, if the step rate of the random walk can be shown to be approximately 1 step per day, then the measured braking index is essentially equal to the true braking index of the spin-down torque. At present, observations indicate only $R_{\rm day} \gtrsim 1$, and it is likely that only theoretical considerations will provide an upper limit to the rate. Consequently, it is possible that the observed braking index may deviate significantly from the spin-down index.

VI. CONCLUSIONS

In this paper, randomness in the rotations of pulsars has been modeled and a technique has been presented for estimating the strength parameter of the random process. In application to the Crab pulsar it has been demonstrated that a random walk in the rotation frequency, characterized by a strength parameter $S = R\langle \delta\nu^2 \rangle$, is consistent with the data. In principle, more information is contained in the

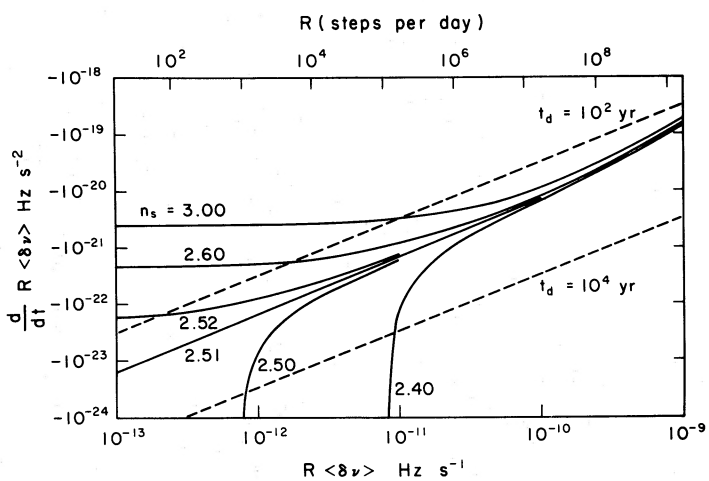

FIG. 2.—Plot of those values of $R\langle \delta\nu \rangle$ and its time derivative that yield the observed braking index, $n_m = 2.51$, for the Crab pulsar assuming different values for the actual spin-down index, n_s. The dashed lines represent damping times for the random walk of 10^2 and 10^4 years. The rate on the upper horizontal scale was derived assuming that all steps of the random walk are of equal amplitude and by using the known strength for the random walk, $R\langle \delta\nu^2 \rangle$.

data. For example, one would like to determine the rate R and the third moment of the step amplitudes, $\langle \delta\nu^3 \rangle$. This is not possible with available data because one must have $RT \lesssim 1$, where T is the time interval of the data span. For the case of the Crab pulsar, measurement errors are too large to analyze data sets with $T \lesssim 1$ day that would be required.

Showing consistency of a random walk is a necessary but not a sufficient condition for demonstrating that a random process is indeed occurring in a pulsar's rotation. In general one can only compare hypotheses and show which is the best. For the case of the Crab, it was found that no deterministic process could account for the observed timing residuals, thus suggesting a random process. One's confidence in the random walk interpretation increases when it is proved to be consistent over a variety of time intervals as was done by Groth (1975*c*) and in the present paper. Over the 5 years spanned by Princeton timing data, the random process appears to have stationary statistics. It is to be expected, however, that the random walk will eventually prove to be nonstationary on longer time scales because the random walk will be "mixed" with another kind of random walk (cf. § II and the equation of motion) and because the strength parameter is likely to decrease as a function of time.

The next paper in this series will apply the techniques of this paper to radio timing data of several pulsars. Activity in the rotations of these pulsars is certain, but it is not known whether the responsible random process is composed of a large or a small number of events.

I thank G. Greenstein, D. J. Helfand, and J. H. Taylor for innumerable discussions during the course of this work, which was supported by NSF grants MPS 75-03377 and ATS 75-23581. This is contribution 309 of the Five College Astronomy Department.

APPENDIX A

MOMENTS OF A SUPERPOSITION OF EVENTS

Here we derive moments of random processes

$$\phi(t) = \sum a_j g(t - t_j) \tag{A1}$$

under the assumption that events occur at times t_j that are Poisson distributed. Following Rice (1954), we compare the fourth-order characteristic function

$$\psi(\omega_1, \omega_2, \omega_3, \omega_4) \equiv \left\langle \exp\left[i \sum_{k=1}^{4} \omega_k \phi(t_k)\right]\right\rangle \tag{A2}$$

with the expression

$$\psi = \exp\left(\sum_{k=1}^{4} i^k \gamma_k\right), \tag{A3}$$

$$\gamma_k \equiv \frac{\langle a^k \rangle}{k!} \int dt' R(t') \left[\sum_{j=1}^{4} \omega_j g(t - t_j)\right]^k \tag{A4}$$

to find the moments

$$\begin{aligned} m_1(t) &= \langle \phi(t) \rangle , \\ m_2(t_1, t_2) &= \langle \phi(t_1)\phi(t_2) \rangle , \\ m_3(t_1, t_2, t_3) &= \langle \phi(t_1)\phi(t_2)\phi(t_3) \rangle , \\ m_4(t_1, t_2, t_3, t_4) &= \langle \phi(t_1)\phi(t_2)\phi(t_3)\phi(t_4) \rangle . \end{aligned} \tag{A5}$$

In general we need not assume that a and R are independent of time. For our purposes, full generality is not necessary, and we therefore obtain

$$m_1(t) = R\langle a \rangle \int dt' g(t - t') , \tag{A6}$$

$$m_2(t_1, t_2) = m_1(t_1)m_1(t_2) + R\langle a^2 \rangle \int dt' g(t_1 - t')g(t_2 - t') , \tag{A7}$$

$$\begin{aligned} m_3(t_1, t_2, t_3) = &-2m_1(t_1)m_1(t_2)m_1(t_3) + m_1(t_1)m_2(t_2, t_3) + m_1(t_2)m_2(t_1, t_3) + m_1(t_3)m_2(t_1, t_2) \\ &+ R\langle a^3 \rangle \int dt' g(t_1 - t')g(t_2 - t')g(t_3 - t') . \end{aligned} \tag{A8}$$

The fourth-order moment is given only for the case that $\langle a \rangle = \langle a^3 \rangle = 0$:

$$m_4(t_1, t_2, t_3, t_4) = m_2(t_1, t_2)m_2(t_3, t_4) + m_2(t_1, t_3)m_2(t_2, t_4) + m_2(t_1, t_4)m_2(t_2, t_3) + R\langle a^4 \rangle \int dt' g(t_1 - t')g(t_2 - t')g(t_3 - t')g(t_4 - t') \,. \quad \text{(A9)}$$

If R and a are time dependent, they must be included in the above integrals.

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