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On the mass of nonflat components of galaxies

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The requirement that a galaxy be stable against two main types of perturbation—barlike disturbances in the plane of the galaxy and bending of that plane—imposes strong constraints on the admissible mass of the halo, which is confined to the narrow range $1.0\text{–}2.5 \le M_{\rm halo}/M_{\rm disk} \le 3\text{–}4$. Some possible explanations for the noncoplanar structure in the central regions of galaxies are mentioned.

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1. INTRODUCTION

If the velocity distribution of the stars in a system is sufficiently anisotropic, a hose-pipe instability can occur, causing the plane of symmetry of the stellar system to bend.[1,2] We have pointed out in a previous letter[3] that the presence of an extended halo should increase the velocity dispersion of the stars in the plane and thereby exert a destabilizing influence on the system. The question arises, then, of determining an upper limit $M_{h,\max}$ on the mass of the halo such that the system will still be stable. A lower limit on the halo mass is set by the condition that the system be stable against barlike perturbations.[4–8] Ostriker and Peebles[7] estimate this limit as $M_{h,\min} \approx (1.0\text{–}2.5) M_d$, where M_d is the mass of the disk component. Thus a stable stellar system may have a halo whose mass is confined to the range $M_{h,\min} \lessdot M_h \lessdot M_{h,\max}$.

Our primary aim in this paper is to estimate $M_{h,\max}$, but the results we shall obtain may have other applications as well. Let us briefly mention some of them.

At the present time the chief problem concerning the spiral structure of galaxies is generally conceded to be posed by the source that generates the density waves.[2,9] One promising mechanism for exciting these waves is thought to be a standing barlike wave at the center of the galaxy. However, excitation of the barlike mode is itself impeded by the presence of a large spherical subsystem acting like a massive halo around the flat component. The departures from circular rotational velocity observed in the central parts of galaxies are probably associated with bending of the flat subsystem.

As another application, one is able to estimate the maximum flattening of elliptical (weakly rotating) galaxies from the criterion of stability against bending and barlike distortion: $c/a \gtrsim 0.3$.

In Sec. 2 we shall analyze the stability of bending oscillations for a model homogeneous stellar ellipsoid with a spherical halo. The degree to which our stability criteria are of universal application will be considered in Sec. 3, where we also generalize the Ostriker–Peebles criterion[7] for stability of a disk against barlike perturbations to the case of systems of finite thickness. The results obtained are discussed in Sec. 4. In an Appendix we derive equations for the membrane oscillations of a collisionless disk.

2. STABILITY OF BENDING OSCILLATIONS FOR ELLIPSOID WITH HALO

Let us model a galaxy as a spheroid with semiaxes a and c, homogeneous in density, composed of stars, and embedded in an extended halo, also homogeneous, and spherical in shape. The gravitational potential within a spheroid is expressed by $\Phi_0 = {}^1/_2\Omega^2(x^2+y^2) + {}^1/_2\omega_0^2 z^2$. It consists of the potential of the halo and of the spheroid itself: $\Omega^2 = GM_h/a^3 + \Omega_d^2$, $\omega_0^2 = GM_h/a^3 + \omega_d^2$, where G is the gravitational constant; Ω_d^2, ω_d^2 depend on the density of the ellipsoid and the lengths of its semiaxes (explicit equations for them may be found, for example, in the recent book by Fridman and one of us[2]).

The moments of the distribution function – quantities that we are hopeful of evaluating – take the form

$$\overline{v_r^2} = {}^1/_2(1-\gamma^2)(1-r^2/a^2-z^2/c^2), \quad \overline{(v_\varphi-\bar{v}_\varphi)^2} = \overline{v_r^2}, \; \bar{v}_\varphi = \Omega\gamma r, \quad \overline{v_z^2} = {}^1/_2\omega_0^2 c^2(1-r^2/a^2-z^2/c^2). \tag{1}$$

The parameter γ here denotes the ratio of the angular velocity of the centroid to the circular velocity ($\gamma^2 \le 1$); r, φ, z are cylindrical coordinates; and a bar represents an average over the distribution function.

We shall confine attention to three modes (Φ_1 denotes the perturbation in the potential):

a. The dome mode m = 0:

$$\Phi_1 = z(az^2+br^2+d). \tag{2}$$

b. The mode m = 1:

$$\Phi_1 = z(x+iy)(ar^2+bz^2). \tag{3}$$

c. The saddle mode m = 2:

$$\Phi_1 = az(x+iy)^2. \tag{4}$$

The names "dome" and "saddle" allude to the shape of the perturbation in the plane of symmetry of the ellipsoid. The coefficients a, b, d depend on time alone, as $e^{-i\omega t}$.

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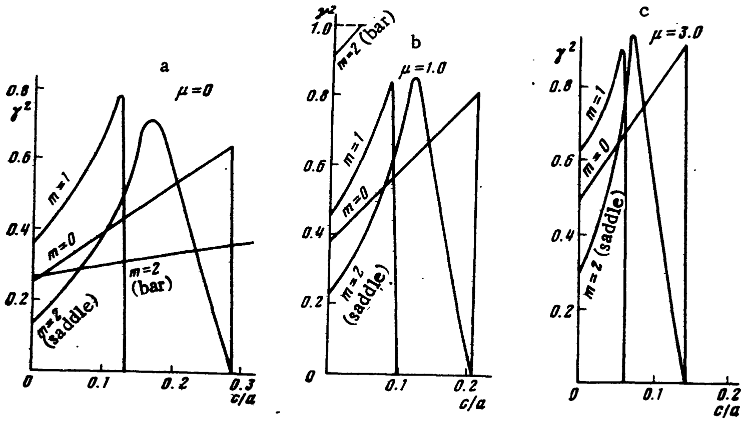

FIG. 1. Regions of instability for homogeneous spheroids with a halo. a) Halo mass $M_h = 0$; b) $M_h = M_d$; c) $M_h = 3M_d$. In the case of the bar mode, the unstable region lies above the corresponding curve; for the other (membrane) modes it lies below the curve.

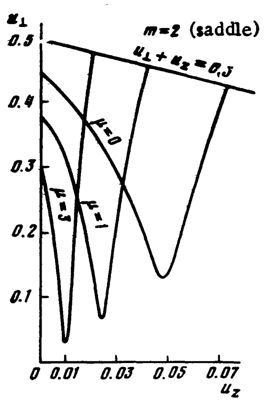

FIG. 2. Regions of instability in the $(u_\perp, u_z)$ plane for the saddle mode. Instability occurs in the domain between the line $u_\perp + u_z = 0.5$ and the curves representing selected halo masses $M_h = \mu M_d$.

These modes are of interest primarily because they are the ones of largest scale and, as a rule, should therefore by the most "hazardous" from the standpoint of loss of stability. Furthermore, it is natural to expect that the behavior of such large-scale modes as these, involving the system as a whole, will be governed by a small number of "global" system parameters. Hence the results we shall obtain from analysis of the simplest homogeneous models should not change too much if we turn to other, more complicated models. This contention is supported, for example, by the universality of the criterion for stability of flat galaxies against barlike perturbations.[7] On the other hand, an investigation of the stability against the modes mentioned above does not demand a specific distribution function for the system; it suffices to know the values of the moments (1). For the homogeneous ellipsoids we are considering, the expressions for the moments (1) will be determined uniquely if we merely assume that the velocity dispersion in the plane of rotation is isotropic. This assumption will in any event hold for systems with nearly circular orbits.[2,5]

We shall follow the standard technique of stability analysis.[2,10] From the equations for the Lagrangian displacements X, Y, Z, we first express these quantities in terms of the perturbed potential Φ_1. We then calculate the density perturbation ρ_1 and the normal displacement of the boundary. Upon solving Poisson's equation and comparing the resulting potential with the initial potentials (2)-(4), we arrive at a system of equations linear in the coefficients that occur in the potential. By setting the determinant of this system equal to zero we obtain the dispersion relation we require.

The dispersion equations[1)] for the modes (2)-(4) have been solved numerically. The results are plotted[2)] in Figs. 1 and 2.

In Figs. 1a-c the abscissa is the flattening c/a of the spheroid, and the ordinate is the quantity, γ^2, which describes the contribution of centrifugal forces to the equilibrium. The quantity $1-\gamma^2$ is proportional to the velocity dispersion of the stars in the plane of rotation. The various curves in Fig. 1 correspond to selected values of the parameter $\mu = M_h/M_d$, the ratio of the halo mass to the spheroid mass. It is evident from Figs. 1a-c that in the disk limit each of the modes will be unstable if the stellar velocity dispersion is large enough. As the thickness of the spheroid increases the region of instability first grows, that is, increasingly "cool" systems become unstable; but then the instability disappears rather quickly. The most significant effect is the dependence of the limits of the instability region on the mass of the halo: if the halo is sufficiently massive, systems with a small velocity dispersion in their plane may become unstable. Here we see the destabilizing influence of the halo.

The curves drop off sharply for a thickness c at which, in the homogeneous model we have used, resonances occur between the oscillation frequencies of the stars with respect to z and in the plane of rotation (x, y). Modes m = 0 and m = 2 have a resonance $\omega_0 = 2\Omega$, while mode m = 1 has a resonance $\omega_0 = 3\Omega$. It is, of course, the idealization of the model that makes the influence of these resonances so strong. For actual differentially rotating systems the curves will be smoother.

As Figs. 1a-c demonstrate, the saddle mode m = 2 is not affected so much by the resonance. But this is the very mode that has the largest region of instability with respect to γ^2. We also observe that as the halo mass grows the region of instability with respect to the parameter c/a becomes narrower; that is, only highly flattened systems will be unstable. For the mode m = 2, however, even if the halo is very massive ($M_h \approx 13M_d$) the ellipsoid will remain unstable for axial ratios up to $c/a \approx 1/15$, a typical thickness for the flat component of a spiral galaxy. Thus the saddle mode stands out among the other membrane-type modes. One will recall that a similar situation arises[7] for perturbations that do not bend the plane of symmetry: in such cases as well the barlike mode m = 2 is distinguished.

3. UNIVERSALITY OF STABILITY CRITERIA FOR LARGE-SCALE MODES

In order to apply the results we have obtained to real systems, it is convenient to represent them in a more universal form, one that does not depend on the special properties of the model. For this purpose we introduce the parameters

$$u_r = U_r/W,\quad u_\varphi = U_\varphi/W,\quad u_z = U_z/W,\quad t = T/W, \tag{5}$$

where $U_{r,\varphi,z}$ denote the kinetic energy of random stellar motion along the r, φ, z axes, respectively; T is the rotational energy of the galaxy; and $W = |W_d| + 2U_0$ represents its potential energy, with W_d the interaction energy of the member stars among one another and U_0 the energy for interaction of the stars with the halo. For a homogeneous spherical halo, $U_0 = -\frac{1}{2}\int \rho_d \Omega_h^2 (r^2+z^2)\,dV$, the integration extending over the volume of the galaxy; ρ_d denotes the star density. The virial theorem implies that

$$2(U_r+U_\varphi+U_z+T) = |W_d| + 2U_0, \tag{6}$$

so that

$$u_r+u_\varphi+u_z+t = 1/2. \tag{7}$$

For our model the parameters $u_{r,\varphi,z}$ and t are uniquely related to the quantities μ, c/a, γ^2, and we can outline the instability zone in the $(u_\perp, u_z)$ plane, where $u_\perp = u_r + u_\varphi = 2u_r$ (see Fig. 2).

As a basis for subsequent application of our criteria to real systems, we shall take the hypothesis that the limits of stability of a galaxy in the variables $u_\perp$, u_z depend only on the parameter $\mu = M_h/M_d$ and just weakly on the model. As mentioned above, Ostriker and Peebles[7] have made a similar claim for the barlike mode. They give arguments indicating that the barlike mode will be excited if $t > t_{cr} = 0.14 \pm 0.03$, and they regard the quantity t_{cr} as conserved despite variations in the rotation curve and the halo mass. We have tested this hypothesis for the model described in Sec. 2, and we find that the proposed criterion remains valid only in the disk limit. The dependence on the halo mass is fairly strong, so that the uncertainty extends beyond the limits indicated above.

We are, however, able to formulate a criterion for excitation of the barlike mode for a system of arbitrary thickness, although admittedly we no longer have universality with respect to halo mass:

$$\tfrac{1}{2}u_\perp + \beta(\mu)\,u_z < \alpha_{cr}(\mu), \tag{8}$$

where $\alpha_{cr}(\mu) = 0.25\,[1-(4-a)^3/27a^2]$ with $a = \tfrac{3}{2}/(1+4\mu/3\pi)$. Here β depends not on the flattening but on the mass of the halo. For selected values of μ we find $\beta(0) = 0.69 \pm 0.01$, $\beta(0.2) = 0.93 \pm 0.03$, $\beta(0.4) = 1.26 \pm 0.07$. The criterion (8) agrees with the Ostriker–Peebles criterion for $c = u_z = \mu = 0$; in particular, we may infer that the model becomes stable (even for a disk of zero thickness) if $\mu \approx 1.1$. For an axial ratio $c/a \approx 1/20$ the model will be stable if $\mu > 1.0$.

We turn now to the bending perturbations. Instability criteria analogous to the condition (8) may here be formulated only in the case of the dome mode m = 0:

$$\tfrac{1}{2}u_\perp + \beta(\mu)\,u_z > \alpha_{cr}(\mu). \tag{9}$$

We can obtain the quantity $\alpha_{cr}(\mu)$ from the disk limit of the dispersion relation, derived in the Appendix:

$$(\omega - m\gamma\Omega)^2 = \Omega_d^2(4\Gamma_n^m - 2) - \tfrac{1}{3}\Omega^2(1-\gamma^2)\times$$
$$\times[(2n+m-1)(2n+m) - m^2 - 2] + \Omega_h^2, \tag{10}$$

where $\Omega_h^2 = GM_h/a^3$; the Γ_n^m are expressed by Eq. (A16). The dome mode corresponds to m = 0, n = 2. For this mode Eq. (10) gives $\omega^2 = \tfrac{5}{3}\Omega_d^2 - \tfrac{10}{3}(1-\gamma^2)\Omega^2 + \Omega_h^2$ and for $\alpha_{cr} = \tfrac{1}{4}(1-\gamma^2)$ we obtain

$$\alpha_{cr} = \tfrac{3}{16}(1+8\mu/15\pi)/(1+4\mu/3\pi). \tag{11}$$

In the case of the modes m = 1 and m = 2, the criteria cannot be written as a simple linear dependence on $u_\perp$, u_z. A dependence of this type holds approximately only in the range of small u_z.

To confirm our hypothesis that the stability criteria are universal, we have investigated on a computer the stability of a disk with various rotation curves for the case of the domelike mode. In the first model considered, the disk has a surface density of the form[11]

$$\sigma = M/(2\pi R^2)\sum_{k=1}^{n} b_k^{(n)}\xi^{2k-1}, \tag{12}$$

where $\xi = \sqrt{1-r^2/R^2}$, M and R denote the mass and radius of the disk, and

$$b_1^{(n)} = (2n+1)/(2n-1),$$
$$b_k^{(n)} = [4(k-1)(n-k+1)/(2k-1)(2n-2k-1)]\,b_{k-1}^{(n)}.$$

The disk has an angular rotational velocity Ω such that

$$\Omega^2(r) = \gamma^2\,\frac{2n+1}{4n}\,\frac{\pi GM}{R^2}(1-\xi^{2n})/r^2, \tag{13}$$

where $\gamma^2 \le 1$ is defined as in Sec. 2. In the case n = 1 this model coincides with the model we have discussed above, in which the criterion for instability with respect to the dome mode takes the form $u_\perp = \tfrac{1}{2}(1-\gamma^2) > \tfrac{3}{8}$ or $t < t_{cr} = 0.125$. We find that for $n \le 7$,

$$0.101 \le t_{cr} \le 0.125. \tag{14}$$

For the second model,[12]

$$\sigma = \sigma_0 \exp(-\alpha r), \tag{15}$$

$$\Omega^2(r) = \gamma^2\pi G\sigma_0\alpha[I_0(\alpha r/2)K_0(\alpha r/2) - I_1(\alpha r/2)K_1(\alpha r/2)], \tag{16}$$

where the I_n, K_n are the corresponding cylindrical functions. In this case the domelike mode becomes unstable for

$$t < t_{cr} = 0.120. \tag{17}$$

We see, then, that t_{cr} (or u_{cr}) proves to be approximately the same for models that differ very greatly.

4. DISCUSSION

Let us apply the results above to estimate an upper limit on the mass of the halo. Setting $c/a \approx 0.05$–0.1, $(\overline{v_r^2})^{1/2} \approx 35$ km/sec, $V = \Omega r \approx 200$ km/sec, and taking as an estimate $(1-\gamma^2) \simeq (1-6)\,\overline{v_r^2}/V^2 \simeq (0.03$–$0.18)$, we find $\gamma^2 \approx 0.82$–0.97. Figure 1 shows that for such values of γ^2 and c instability will set in if $\mu \approx 3$–4. For larger values of the halo mass, the disk of the galaxy will become unstable.

Evidently, then, bending and barlike perturbations can simultaneously be stabilized only if the halo has a mass confined to a rather narrow interval. Adopting as a lower limit[7] a value $M_h \approx (1$–$2.5)\,M_d$, we obtain for the range of admissible halo masses

$$(1.0-2.5) \lesssim M_h/M_d \lesssim (3-4). \tag{18}$$

One should recognize, however, as pointed out in Sec. 2, that if the halo mass considerably exceeds the upper limit (18), the galaxy can still be stabilized by virtue of its finite thickness. For the same reason increasingly short-wavelength membrane modes will rapidly become stabilized when finite flattening is taken into account.

The estimate we have obtained for the halo mass is a consequence of the hypothesis that the Galaxy is stable against bending. We know that the galactic plane is curved. Following Hunter and Toomre,[11] we may conjecture that the curvature has nothing to do with instability but is caused by the influence of the Magellanic Clouds. However, there is an alternative point of view: the warp might represent a late phase in the development of unstable membrane perturbations. In this event we would be confronted with a nonlinear problem for determining the level to which this instability has been stabilized and comparing that level with the observations.

For elliptical galaxies in which the rotation is weak, we can use the criterion for stability against bending to determine the maximum flattening c/a. If the halo mass is zero, this value (see Fig. 1a) will be of order $c/a \approx 0.3$. We obtain roughly the same estimate from the condition of stability against the barlike mode.[13] As the mass M_h increases the lower limit on c/a will diminish.

APPENDIX: DERIVATION OF EQUATIONS FOR MEMBRANE OSCILLATIONS OF AN INFINITELY THIN COLLISIONLESS DISK

For a thin disk we may take the perturbed potential in the form $[\mathbf{r} = (x, y), \mathbf{v} = (v_x, v_y)]$

$$\Phi_1 = a(\mathbf{r}, t)z. \tag{A1}$$

Let $Z(\mathbf{v}, \mathbf{r}, t)$ denote the Lagrangian displacements of stars in the middle of the plane $z = 0$. For simplicity we shall regard the undisturbed potential in the plane $z = 0$ as quadratic in r; thus $\Phi_0 = \Omega_0^2 r^2/2$ [a generalization can readily be made to the case of arbitrary $\Phi_0(r)$]. We then have for Z the equation[2,10]

$$(\hat{D}^2+\omega_0^2)Z = -\partial\Phi_1/\partial z = -a, \tag{A2}$$

where ω_0 denotes the frequency of vertical oscillation of stars in the undisturbed state and

$$\hat{D} = \partial/\partial t + \mathbf{v}\partial/\partial\mathbf{r} - \Omega_0^2\mathbf{r}\partial/\partial\mathbf{v}. \tag{A3}$$

In view of the small thickness of the disk, $\omega_0^2 \gg \Omega_0^2$, and the solution of Eq. (A2) may be written in the form

$$Z = -\frac{1}{\omega_0^2}\left(1 - \frac{\hat{D}^2}{\omega_0^2}\right)a. \tag{A4}$$

For $\hat{D}^2 a$ we easily obtain

$$\hat{D}^2 a = [(\partial/\partial t + \mathbf{v}\partial/\partial\mathbf{r})^2 - \Omega_0^2\mathbf{r}\partial/\partial\mathbf{r}]a. \tag{A5}$$

Averaging the expression (A4) with the stellar distribution function $f_0(\mathbf{r}, \mathbf{v})$, which in turn is obtained by averaging the distribution function $f_0(\mathbf{r}, \mathbf{v}, z, v_z)$ of the ellipsoid with respect to z, v_z], we find

$$\bar{Z} = \frac{a}{\omega_0^2} - \frac{1}{\omega_0^4}\left\{\left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}\frac{\partial}{\partial \mathbf{r}}\right)^2 + \frac{P_{ik}}{\sigma}\frac{\partial^2 a}{\partial x_i\,\partial x_k} - (\Omega_0^2 - \Omega^2)x_i\frac{\partial a}{\partial x_i}\right\}, \tag{A6}$$

where the "pressure" tensor

$$P_{ik} = \int \bar{f}_0 v_i' v_k'\, d\mathbf{v}', \tag{A7}$$

$\mathbf{v} = \bar{\mathbf{v}}(\mathbf{r}) + \mathbf{v}'$, $\bar{\mathbf{v}}(\mathbf{r})$ denotes the velocity of the centroid, $\mathbf{v}(\mathbf{r}) = v_\varphi(\mathbf{r}) = \Omega r$, and $\mathbf{v}'$ represents the peculiar velocities of the stars. Sums are to be taken over repeated indices; $i, k = 1, 2$; and $x_1 = x$, $x_2 = y$.

The condition of equilibrium in the plane of rotation gives

$$\frac{\partial P_{rr}}{\partial r} + \frac{P_{rr} - P_{\varphi\varphi}}{r} = -\sigma\left(\frac{\partial\Phi_0}{\partial r} - \Omega^2 r\right) = -\sigma r(\Omega_0^2 - \Omega^2). \tag{A8}$$

Substituting $\Omega_0^2 - \Omega^2$ from Eq. (A8) into Eq. (A6), we obtain the final expression

$$\bar{Z} = \frac{1}{\omega_0^2}\left\{a - \frac{1}{\omega_0^2}\left[\left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}\frac{\partial}{\partial \mathbf{r}}\right)^2 a + \frac{1}{\sigma}\frac{\partial}{\partial x_i}P_{ik}\frac{\partial a}{\partial x_k}\right]\right\}. \tag{A9}$$

We solve Eq. (A9) for a to obtain

$$a = \omega_0^2\bar{Z} + \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}\frac{\partial}{\partial \mathbf{r}}\right)^2\bar{Z} + \frac{1}{\sigma}\frac{\partial}{\partial x_i}P_{ik}\frac{\partial \bar{Z}}{\partial x_k}. \tag{A10}$$

On the other hand, Poisson's equation gives

$$a(r) = \omega_0^2\bar{Z} + F_{gr}(z), \tag{A11}$$

where[11]

$$F_{gr}(\bar{z}) = G\iint d\mathbf{r}'\,\frac{\sigma(\mathbf{r}')[\bar{Z}(\mathbf{r}') - \bar{Z}(\mathbf{r})]}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{A12}$$

From Eqs. (A11) and (A12) we obtain the equation we require for the vertical displacements $h \equiv \bar{Z}$ of the disk:

$$\left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}\nabla\right)^2 h = F_{gr}(h) - \frac{1}{\sigma}\frac{\partial}{\partial x_i}\left(P_{ik}\frac{\partial h}{\partial x_k}\right). \tag{A13}$$

Equation (A13) will take a form of this type for a disk with an arbitrary rotation law.

The dispersion relation is readily derived for Maclaurin disks with a distribution function

$$f_{0\gamma} = \frac{1}{2\pi\sqrt{1-\gamma^2}}[(1-\gamma^2)(1-r^2/R^2) - v^2]^{-1/2}. \tag{A14}$$

It has the form[2,14]

$$(\omega - m\gamma\Omega_0)^2 = \Omega_0^2\{4\Gamma_n^m - 2 - {}^1/_3(1-\gamma^2)[(2n+m)(2n+m-1) - m^2 - 2]\}. \tag{A15}$$

The eigenfunctions are here given by

$$h(\mathbf{r}) = \frac{1}{\xi}P^m_{2n+m-1}(\xi)e^{im\varphi}; \qquad \xi = \sqrt{1-r^2/R^2},$$

$$\Gamma_n^m = \frac{(2n-1)!(2n+2m-1)!}{2^{2m+4n-3}[(n-1)!(n+m-1)!]^2}. \tag{A16}$$

Equation (A15) can be generalized to the case of superposed Maclaurin disks:[3)]

$$f_0 = \int_{-1}^{1} A(\gamma)f_{0\gamma}'\,d\gamma, \tag{A17}$$

where $f_{0\gamma}'$ denotes the function $f_{0\gamma}$ expressed in an inertial reference frame (the equations given previously[2,14] contain errors):

$$(\omega - m\bar{\gamma}\Omega_0)^2 = \{4\Gamma_n^m - 2 - {}^1/_3(1-\overline{\gamma_2})[(2n+m-1)(2n+m) + m^2 - 2] - m^2(\overline{\gamma^2} - \bar{\gamma}^2)\}\Omega_0^2, \tag{A18}$$

with

$$\overline{\gamma^n} = \int_{-1}^{1} A(\gamma)\gamma^n\,d\gamma.$$

There is a trivial generalization to the case of a disk with a halo:

$$(\omega - m\bar{\gamma}\Omega_0)^2 = \Omega_d^2(4\Gamma_n^m - 2) - {}^1/_3\Omega_0^2\{(1-\overline{\gamma^2}) \times [(2n+m-1)(2n+m) - m^2 - 2] - m^2(\overline{\gamma^2} - \bar{\gamma}^2)\} + \Omega_h^2, \tag{A19}$$

where $\Omega_0^2 = \Omega_d^2 + \Omega_h^2$.

3)These equations are rather cumbersome and will not be given explicitly