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WHITE DWARF SEISMOLOGY

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ABSTRACT

A cool, 1 $M_\odot$, ^{56}Fe white dwarf model is analyzed to determine the effects of a shear-supporting crystalline core upon the nonradial oscillation periods in the Cowling approximation. Both spheroidal and toroidal modes with $l = 1$, 2, and 3 have been calculated. Of the spheroidal oscillations, only the g-modes are seriously shifted in frequency. The toroidal oscillations, which have nonzero frequencies because of the nonzero shear modulus of the solid core, have periods intermediate between those of the p- and g-modes. None of the low-order oscillation modes have frequencies comparable to those observed in the ZZ Ceti white dwarfs.

Subject headings: stars: interiors — stars: pulsation — stars: white dwarfs

I. INTRODUCTION

Over the past several years high-speed photometric observations have shown that a number of white dwarfs exhibit periodic light variations, with periods ranging between a few tens of seconds (in the cataclysmic variable systems) and 1000 seconds (in the isolated ZZ Ceti white dwarfs; cf. McGraw 1977; Nather 1978; Van Horn 1978, and references therein). However, theoretical investigations have as yet neither identified the precise modes observed nor satisfactorily explained the cause of variability in any of these objects. These are important problems because stellar oscillations provide built-in probes of the deep interior of a star. Thus, it should be possible, in principle, to infer something about the internal structure of a white dwarf by analyzing oscillation spectra—an approach which has been notably powerful and informative in geophysical seismology. However, this approach has so far not been overly successful in application to white dwarfs because the theoretical periods derived are too short to match the observations. Attempts to address the problems posed by white dwarf oscillation have included studies of possible nonlinear coupling of presumably unstable high-overtone nonradial modes with low-order modes (Dziembowski 1977), nonlinear coupling of rotationally split g-mode oscillations (Wolff 1977), and the suggested excitation of toroidal "r-modes" (Papaloizou and Pringle 1978). None of the above attempts has been at all conclusive.

The purpose of this report is to explore another effect that may occur in one class of variable white dwarfs, namely, the effects of solid, crystalline cores upon nonradial oscillations in white dwarfs. It has been pointed out by Van Horn and Savedoff (1976, hereafter VHS; see also Van Horn 1978) that the onset of core crystallization takes place quite rapidly around an effective temperature that in 1 $M_\odot$, ^{12}C white dwarfs is $\sim$10,000 K (Lamb and Van Horn 1975). This is precisely the region of temperature occupied by the ZZ Ceti white dwarf variables (McGraw 1977) which display periods ranging from about 100 to 1000 s. As we show below (see also VHS), the effect of a solid core which can support shear stresses is to decrease nonradial oscillation periods; thus the longer periods of the ZZ Ceti variables are made even more inexplicable. However, the large effect of crystallization on the g-mode periods shows that the solid core effects cannot be ignored in future studies of the ZZ Ceti oscillations.

In § II we review very briefly the dynamic consequences of a solid core and introduce the necessary physics. A complete mathematical description is deferred to the Appendix which the reader should consult for details. Section III summarizes our results and conclusions.

II. EFFECTS OF AN ELASTIC CORE

For a fluid star, the types of modes associated with small-amplitude, nonradial perturbations are well understood (cf. the review by Cox 1976). However, if portions of the stellar material can support shear strains and stresses, new classes of motion are possible. As discussed in the Appendix, these are the *toroidal* modes, which are dominated by transverse motion (perpendicular to the radial coordinate). In conformance with geophysical nomenclature these modes are noted by ${}_nT_l$, where n is the number of nodes in the transverse component of displacement and l is the "angular momentum quantum number." The remaining modes are *spheroidal* and have fluid counterparts

[1] Operated jointly by the National Bureau of Standards and the University of Colorado.

in the p-, f-, and g-modes of conventional nonradial stellar oscillation theory. They are denoted by ${}_nS_l$, where n is the number of nodes in the radial (r) component of displacement. Thus, for example, the f-mode, which for simple fluid stars such as homogeneous white dwarfs has no nodes in its radial displacement eigenfunction, would be denoted by ${}_0S_l$. Similarly, p-modes would have the designations ${}_1S_l$, ${}_2S_l$, etc. Gravity modes, which are not often encountered in solid Earth geophysics, will be designated by ${}_{-1}S_l$, ${}_{-2}S_l$, where the minus subscript is intended to signal a g-mode (or one corresponding to it in the case with finite shear).[2]

If we assume a Hooke's law relationship between stress and strain, then it can easily be shown that there are two characteristic velocities of wave propagation (excluding, for now, gravity waves) if the material properties of the undisturbed configuration are homogeneous and isotropic. These velocities depend on the adiabatic bulk modulus $K = \Gamma_1 p$ and the shear modulus, or rigidity, μ. Here Γ_1 is the usual adiabatic exponent (see Landau and Lifshitz 1970) and p is the pressure. For convenience we replace the bulk modulus by λ, which is one of the Lamé elastic coefficients (the other being μ) defined by $\lambda = K - 2\mu/3$. The two velocities are then

$$v_P = [(\lambda + 2\mu)/\rho]^{1/2}$$

for pressure (P) waves, and

$$v_S = [\mu/\rho]^{1/2}$$

for shear (S) waves. Since we assume transverse isotropy, no distinction need be made between the horizontal and vertical variants of these waves, unlike the more complex situations in geophysical seismology (Takeuchi and Saito 1971, and further references in the Appendix).

For nonradial modes the local characteristic frequencies associated with the P- and S-wave modes are

$$\sigma_P^2 \approx \frac{l(l+1)(\lambda + 2\mu)}{r^2\rho}$$

and

$$\sigma_S^2 \approx \frac{l(l+1)}{r^2\rho}\mu\,.$$

In the one model which we have analyzed in detail (see later) a typical ratio of μ/λ in the crystalline core is around 0.04, implying that we may expect shear oscillation periods to be about 5 times those of pressure waves. In addition, the P-waves should closely resemble, and have periods within a few percent of conventional p-mode stellar oscillations.

The gravity (g) modes of stellar oscillation, however, are significantly affected by the inclusion of shear. In a naive picture, the large transverse motions characteristic of g-modes should be even more easily excited in or near the shear-supporting core and thus should, in some respects, resemble toroidal modes. The latter, from the above discussion, have periods some 5 times longer than pressure modes but, for white dwarfs, are still shorter than g-mode periods. Hence, we expect g-mode periods to shorten when shear is included. Conclusions similar to the above were reached previously by VHS on the basis of a short-wavelength analysis of the equations of motion.

To permit quantitative calculations of the effects mentioned above, we must know when core crystallization occurs, and we must have an approximation for the shear modulus of the solid white dwarf core. As a result of the extensive Monte Carlo calculations, primarily by J.-P. Hansen and his co-workers, it is now known that crystallization of a plasma takes place when the Coulomb parameter is $\Gamma \gtrsim 160$ (Pollock and Hansen 1973; Lamb and Van Horn 1975). Here Γ is defined by

$$\Gamma = \frac{(Ze)^2}{akT}\,,$$

where a is the radius of a sphere containing one single ion of charge $+Ze$. The value of Γ at the point of crystallization is currently known to an accuracy of about 15–20%.

To approximate the shear modulus, we note that the shear strength of the lattice must be determined by the Coulomb forces that bind it together, while the bulk modulus is determined, for white dwarf matter, primarily by the compressibility of the degenerate electrons. We thus expect

$$\frac{\mu}{K} \sim \frac{(Ze)^2/a}{Z\epsilon_{\mathrm{Fermi}}}\,,$$

where $\epsilon_{\mathrm{Fermi}}$ is the characteristic energy per electron. More sophisticated calculations yield

$$\mu \sim 0.37n\frac{(Ze)^2}{a}$$

where $n = 1/(4/3)\pi a^3$ (cf. Fuchs 1936; Mott and Jones 1958; Kugler 1969; Lamb 1976; Pollock and Hansen 1973; Pollock 1977) is the ion number density. Through a programming error, we have actually used a value of μ twice this great; the difference is unimportant for the purpose of this exploratory study.

III. MODEL RESULTS AND CONCLUSIONS

Since the present study is exploratory, we have chosen to analyze a representative iron white dwarf model from the evolutionary sequence of Savedoff, Van Horn, and Vila (1969). The pulsational properties of this sequence are well understood so that changes in these properties due to shear are easily apprehended (see Van Horn, Richardson, and Hansen 1972; Osaki and Hansen 1973; Hansen, Cox, and Van Horn 1977, Hansen, Cox, and Carroll 1978). The major drawback to the use of the iron models is that crystallization effects were not taken in account in computing the evolution, so that the artificial crystalline core is not

[2] These designations for g-modes have been suggested to us by Dr. Martin Smith of CIRES (Boulder).

self-consistent. This should not be important in the present context, however.

The particular model investigated here in detail is of 1 $M_\odot$, $\log T_e = 3.826$, and $\log L/L_\odot = -4.194$. (A similar model of 0.63 $M_\odot$ was also examined, but only in a cursory way; the results were qualitatively the same as for the 1 $M_\odot$ model). The extent and properties of the crystalline core were determined by computing Γ and μ. This model is quite cool and thus has an extensive core comprising all but about 0.1% of the total mass (5% of the radius). In this sense we have chosen nearly the most extreme configuration. Less evolved models (or those having less total mass) would show lesser effects due to crystallization.

The mathematical and computational details are summarized in the Appendix, and our results are given in Table 1 for l = 1, 2, and 3. Listed are periods (in seconds) for the modes p_2, p_1, f, g_1, and g_2 computed under the assumption that the white dwarf is fluid throughout; the toroidal modes ${}_0T_l$, ${}_1T_l$, and ${}_2T_l$; and the spheroidal modes ${}_{-2}S_l$, ${}_{-1}S_l$, ${}_0S_l$, ${}_1S_l$, and ${}_2S_l$. The last set are those which reduce to the fluid case of the first set as either μ or crystalline core mass approaches zero (this last was checked by direct computation).

As anticipated, the toroidal mode periods are ~4–5 times longer than those of the corresponding spheroidal modes ${}_2S_l, \ldots, {}_0S_l$ (cf. § II). Note that the mode ${}_0T_1$ does not exist; such a mode can be shown to violate angular momentum conservation. None of the periods obtained seem relevant to observed white dwarfs, nor, if the modes were excited, could they be detected at the stellar surface—at least not with the physics assumed here. If the envelope were extremely viscous, perhaps the shear waves could be transmitted to the surface, especially since the transverse displacements peak strongly near the core boundary. However, it can easily be shown that viscosity is important only if the product $\sigma \times$ viscous coefficient $\sim\mu$ or λ. For reasonable estimates of the viscous coefficient, this would require frequencies many orders of magnitude larger than any reported here. A complete treatment of viscous effects would entail a full nonadiabatic investigation—a noteworthy project in any event.

By inspection of Table 1 it appears that the only spheroidal modes which show any appreciable influence from shear are the g-like modes—especially g_1 where a factor of 2 decrease in period is computed. The expectation of § II is also confirmed in that transverse (and radial) displacements are enhanced in the core when shear is present. This is shown in Figures 1 and 2. (The eigenfunctions for pressure modes are essentially the same as for the fluid case.) Note, however, that if our estimate of μ were seriously in error (say too small by a factor of 10–100), then g_1 would decrease in period until it began to interfere with the f- and p-like modes, producing a very complicated frequency spectrum. In this regard we should point out that the presence of shear introduces numerous complexities in the interpretation of what modes are seen. In those calculations where μ was reduced to very low values the eigenfunctions of radial (and transverse) displacement displayed many nodes in the solid core whereas for the nominal value of μ the structure was very simple. On the other hand, as μ approached zero, the eigenfunction structure was very oscillatory but the amplitude of those oscillations decreased until, in the fluid extreme, their behavior was smooth. A good example of this behavior was also demonstrated by Denis (1975) for very simple

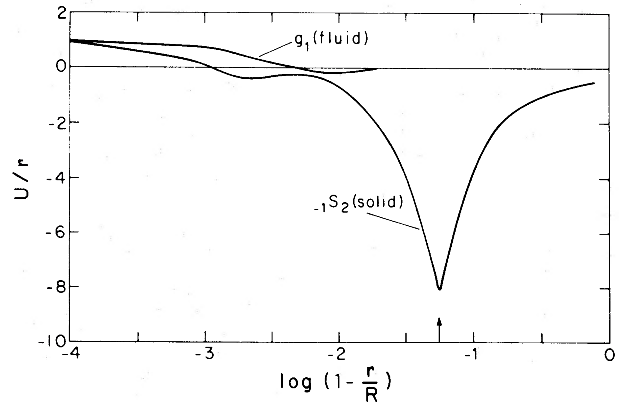

FIG. 1.—The radial displacements U/r (see the Appendix for further details) are plotted against $\log(1 - r/R)$ for l = 2, g_1 (fluid) and ${}_{-1}S_2$ modes. The edge of the crystalline core is indicated by an arrow.

TABLE 1

PERIODS (in seconds) FOR 1 $M_\odot$ MODEL

A. FLUID MODES

l	p_2	p_1	f	g_1	g_2
1......	1.044	1.547	...	193.8	270.3
2......	0.936	1.329	2.313	111.9	156.1
3......	0.863	1.198	2.047	79.1	110.4

B. CRYSTALLINE MODES

l	${}_2S_l$	${}_1S_l$	${}_0S_l$	${}_{-1}S_l$	${}_{-2}S_l$	${}_2T_l$	${}_1T_l$	${}_0T_l$
1......	1.024	1.513	...	99.8	193.9	4.73	7.70	...
2......	0.911	1.293	2.212	53.4	112.0	4.05	6.03	11.67
3......	0.837	1.164	1.932	36.7	79.2	3.61	5.14	8.77

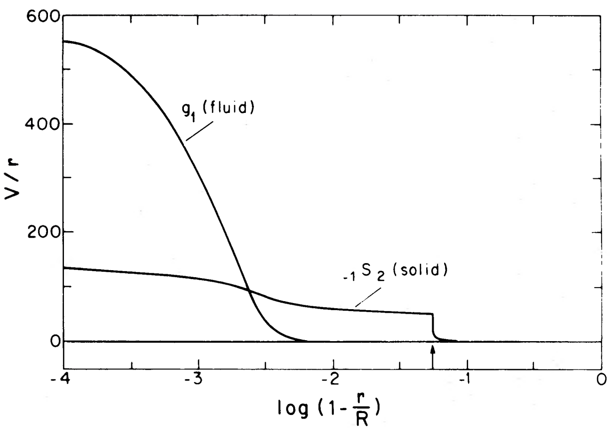

FIG. 2.—Plotted are the transverse displacements V/r for the same modes as in Fig. 1.

homogeneous stellar models. Thus, a cursory examination of the eigenfunctions may not easily reveal what kind of mode has been obtained. Furthermore, the ordering of periods may not be a fail-safe indicator of mode type (see again, Denis 1975). The caution is that the frequency space of solutions must be searched carefully in order that interesting modes not be missed. Our calculations used several safeguards in this respect, but then again our stellar model was rather simple.

In more realistic models for ZZ Ceti variables there would be important ionization zones (not considered here) in the envelope which would affect both g-mode frequencies and their material displacements. It may be possible that some of these, with long periods, might be trapped very near the surface (see Hansen 1979), in which case crystallization effects, occurring deep in the core, might not play a significant role in changing frequencies. Such models are now under study.

In summary, with our present models (see above) we find no way of obtaining such long periods as are observed in the ZZ Ceti white dwarfs. However, the present study does point out the necessity of including solid, crystalline core effects if cool white dwarfs are the subject of study. Any interior diagnostics based on periods, especially for the g-modes, must account for shifts in pulsation frequency when shear is present.

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APPENDIX

A procedure for obtaining the linearized pulsation equations for stars which contain a solid region capable of supporting shear has been given by VHS, which the reader is advised to consult for details. In addition, the present authors have found the review article by Takeuchi and Saito (1971) to be very useful in explaining the details of the problem from a geophysical point of view. In what follows we will give only the essentials of the analysis and review the computational techniques we have used to obtain the numerical results.

In the Cowling approximation, the linearized forms of the continuity and momentum equations yield the spherical coordinate wave equations;

$$\rho\sigma^2\xi_r = \rho\frac{\partial\chi}{\partial r} - \Gamma_1 pA\alpha - \mu\left[-\frac{1}{\mu}\frac{\partial}{\partial r}(\tfrac{2}{3}\mu\alpha) + \frac{2}{\mu}\frac{d\mu}{dr}\frac{\partial\xi_r}{\partial r} + \frac{\partial\alpha}{\partial r} + \nabla^2\xi_r - \frac{2}{r^2}\xi_r - \frac{2}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\xi_\theta) + \frac{\partial\xi_\phi}{\partial\phi}\right)\right], \tag{A1}$$

$$\rho\sigma^2\xi_\theta = \frac{\rho}{r}\frac{\partial\chi}{\partial\theta} - \mu\left[-\frac{1}{\mu r}\frac{\partial}{\partial\theta}(\tfrac{2}{3}\alpha\mu) + \frac{1}{\mu}\frac{d\mu}{dr}\left(\frac{1}{r}\frac{\partial\xi_r}{\partial\theta} + \frac{\partial\xi_\theta}{\partial r} - \frac{\xi_\theta}{r}\right)\right.$$
$$\left. + \frac{1}{r}\frac{\partial\alpha}{\partial\theta} + \nabla^2\xi_\theta + \frac{2}{r^2}\frac{\partial\xi_r}{\partial r} - \frac{1}{r^2\sin^2\theta}\left(\xi_\theta + 2\cos\theta\frac{\partial\xi_\phi}{\partial\phi}\right)\right], \tag{A2}$$

and

$$\rho\sigma^2\xi_\phi = \frac{\rho}{r\sin\theta}\frac{\partial\chi}{\partial\phi} - \mu\left[-\frac{1}{\mu r\sin\theta}\frac{\partial}{\partial\phi}(\tfrac{2}{3}\mu\alpha) + \frac{1}{\mu}\frac{d\mu}{dr}\left(\frac{1}{r\sin\theta}\frac{\partial\xi_r}{\partial r} + \frac{\partial\xi_\phi}{\partial r} - \frac{\xi_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\phi}\right.$$
$$\left. + \nabla^2\xi_\phi + \frac{1}{r^2\sin^2\theta}\left(2\sin\theta\frac{\partial\xi_r}{\partial\phi} + 2\cos\theta\frac{\partial\xi_\theta}{\partial\phi} - \xi_\phi\right)\right]. \tag{A3}$$

Here, ξ_r, ξ_θ, and ξ_ϕ are the components of the vector Lagrangian radius variation, σ is the frequency where a time dependence of $e^{i\sigma t}$ is assumed,

$$\chi = -\frac{\Gamma_1 p\alpha}{\rho} - \frac{1}{\rho}\frac{dp}{dr}\xi_r,$$

$\alpha = \nabla \cdot \xi$ is the dilatation, and A, which is one measure of convective stability, is given by

$$A = \frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{\Gamma_1 p}\frac{dp}{dr}\,.$$

In the above, the stress tensor is of the (Cartesian) form

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}\,,$$

with $e_{\alpha\beta}$ being the strain tensor (see, e.g., Landau and Lifshitz 1970) and in our case, the Lamé coefficients λ and μ are functions of r only.

As is well known (Takeuchi and Saito 1971; Bolt and Derr 1969), the above equations admit of two physically distinct classes of modes. The first, better known to those doing nonradial stellar pulsation studies, are the spheroidal modes which, for the case of a fluid, are the usual g-, f-, and p-modes. These are characterized by the vanishing of the radial component of $\nabla \times \xi$ (Aizenman and Smeyers 1977, for example). This condition allows the simple and well-known separation of variables in spherical harmonics

$$\xi_r = U(r)Y_l^m(\theta, \phi)\,, \qquad \xi_\theta = V(r)\frac{\partial}{\partial\theta}Y_l^m\,, \qquad \xi_\phi = \frac{V(r)}{\sin\theta}\frac{\partial}{\partial\phi}Y_l^m\,,$$

where $U(r)$ and $V(r)$ are the vertical and horizontal components of displacement.

Once the separation is performed, it is convenient to introduce the new variables[3]

$$z_1 = \frac{U}{r}\,, \qquad z_2 = \lambda\alpha + 2\mu\frac{dU}{dr}\,, \qquad z_3 = \frac{V}{r}\,,$$

and

$$z_4 = \frac{\mu}{r}\left(\frac{dV}{dr} - \frac{V}{r} + \frac{U}{r}\right).$$

The new variables z_2 and z_4 are related to the vertical and horizontal tractions. We then obtain the fourth order system (in space), corresponding to equations (A1)–(A3), given in a slightly different form by Alterman, Jarosch, and Pekeris (1959);

$$\begin{aligned}
rz_1' &= -(1 + 2\lambda\delta)z_1 + \delta z_2 + \lambda\hat{l}\delta z_3\,,\\
rz_2' &= (-\sigma^2\rho r^2 - 4\rho g r + 4\pi G\rho^2 r^2 + 4\mu\beta\delta)z_1 - 4\mu\delta z_2 + (\hat{l}\rho g r - 2\mu\beta\delta\hat{l})z_3 + \hat{l}z_4\,,\\
rz_3' &= -z_1 + z_4/\mu\,,\\
rz_4' &= (g\rho r - 2\mu\beta\delta)z_1 - \lambda\delta z_2 + \{-\rho\sigma^2 r^2 + 2\mu\delta[\lambda(2\hat{l} - 1) + 2\mu(\hat{l} - 1)]\}z_3 - 3z_4\,.
\end{aligned}$$

Here the prime denotes the first radial derivative, $\delta = (\lambda + 2\mu)^{-1}$, $\beta = 3\lambda + 2\mu$, $\hat{l} = l(l + 1)$, and g is the local gravity.

The fluid case may be obtained by setting μ equal to zero in the above. The resulting system is second order (as is usual in the fluid case using the Cowling approximation) with $z_4 = 0$ and an algebraic relation

$$z_3 = (r\rho\sigma^2)^{-1}(g\rho z_1 - z_2/r)\,.$$

Furthermore, it may be easily shown that the relations between the variables z_i in the fluid case and the "Dziembowski variables" y_i (see, for example, Osaki and Hansen 1973 for a complete description) are

$$z_1 = y_1\,, \qquad z_3 = \frac{g}{\sigma^2 r}y_2\,, \qquad \text{and} \qquad z_2 = rg\rho(y_1 - y_2)\,.$$

The boundary conditions for the fluid envelope are given in Osaki and Hansen (1973) and are two in number: one relating y_1 and y_2 at the surface and the other being a normalization ($y_1 = 1$) at the surface. For the central solid core the simplest tack is to expand all z_i in power series in r from the origin. This last procedure reveals that there are four classes of solutions, which vary as $r^{l-2,\,l,\,-(l+1),\,-(l+3)}$ and of which only the first two (regular solutions) are of interest. A general (regular) solution is then a linear superposition of these two. Thus, near the origin, we write

$$z_i = a_i r^{l-2} + a_i' r^l\,;$$

[3] Note that these transformations are computationally most convenient for $l = 2$. For other values of l, the z_i should be scaled by appropriate factors of r (see Crossley 1975).

explicit expressions for a_i and a_i' are given by Crossley (1975). Out of these eight coefficients, only two are independent (chosen to be a_1 and a_4' here); and these, plus the eigenvalue σ^2, make a total of three quantities which must be determined. They are fixed by requiring continuity of z_1, and the tractions z_2 and z_4 across the fluid-solid interface. The tangential component of displacement, z_3, need not be continuous. These three conditions determine, in effect, the three unknowns.

In the numerical calculations a high-order Runge-Kutta scheme coupled with spline interpolation of physical quantities is used to integrate the two independent solutions for the z_i (i.e., those going as r^{l-2} and r^l) out to the solid-fluid interface with arbitrary values of a_1 and a_4' and a guessed value for σ^2. The variables y_1 and y_2 (and thus z_1, z_2, and z_3) are integrated inward from the surface to the same point with the same choice of the eigenvalue. The coefficients a_1 and a_4' are then determined by requiring that z_4 be zero and that z_1 be continuous at the interface. However, in general, z_2 will not be continuous unless σ^2 is chosen properly. To do this, a strategy similar to that used in the "fitting method" of stellar structure calculations is applied. Convergence is found to be quite rapid.

The second class of oscillations are the *torordial* modes which, if there were no shear (or rotation), would all have zero frequency. These are characterized by the property that ξ_r, $\nabla\cdot\boldsymbol{\xi}$, and χ all vanish; that is, only transverse motions are possible. Their separation is

$$\xi_\theta = \frac{V}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}, \qquad \text{and} \qquad \xi_\phi = -V\frac{\partial Y_l^m}{\partial\theta}.$$

Defining the new variables

$$z_1 = \frac{V}{r^2}, \qquad \text{and} \qquad z_2 = \frac{\mu}{r}\left(\frac{dV}{dr} - \frac{V}{r}\right),$$

we find (see again Alterman, Jarosch, and Pekeris 1959)

$$rz_1' = -z_1 + \frac{1}{\mu}z_2, \qquad \text{and} \qquad rz_2' = [\mu(\hat{l} - 2) - \sigma^2\rho r^2]z_1 - 4z_2.$$

Regular solutions near the origin go as r^{l-2}, and they require the central boundary condition

$$(l-1)z_1 = \frac{1}{\mu}z_2.$$

Since the transverse traction (proportional to z_2) must vanish just past the solid core-fluid envelope boundary, we must have $z_2 = 0$ there and, for normalization, we take $z_1 = 1$ at that point as well. The fluid envelope, since it cannot support shear waves, does not partake of the motion. Any number of numerical techniques can be applied to this rather simple problem.

REFERENCES

Aizenman, M. L., and Smeyers, P. 1977, *Ap. Space Sci.*, **48**, 123.
Alterman, Z., Jarosch, H., and Pekeris, C. L. 1959, *Proc. Roy. Soc., A* **252**, 80.
Bolt, B. A., and Derr, J. S. 1969, *Vistas Astr.*, **11**, 69.
Cox, J. P. 1976, *Ann. Rev. Astr. Ap.*, **14**, 247.
Crossley, D. J. 1975, *Geophys. J. Roy. Astr. Soc.*, **41**, 153.
Denis, C. 1975, *Mém. Soc. Roy. Sci. Liège*, **8**, 253.
Dziembowski, W. 1977, *Acta Astr.*, **27**, 1.
Fuchs, K. 1936, *Proc. Roy. Soc., A* **153**, 622.
Hansen, C. J. 1979, in *Proc. Nonradial and Nonlinear Stellar Pulsation: A Workshop* (University of Arizona Special Report), in press.
Hansen, C. J., Cox, J. P., and Carroll, B. W. 1978, *Ap. J.*, **226**, 210.
Hansen, C. J., Cox, J. P., and Van Horn, H. M. 1977, *Ap. J.*, **217**, 151.
Kugler, A. E. 1969, *Ann. Phys.*, **53**, 133.
Lamb, D. Q. 1976, private communication.
Lamb, D. Q., and Van Horn, H. M. 1975, *Ap. J.*, **200**, 306.
Landau, L. D., and Lifshitz, E. M. 1970, *Theory of Elasticity* (2d ed.; Reading: Addison Wesley).
McGraw, J. T. 1977, Ph.D. thesis, University of Texas (Austin).
Mott, N. F., and Jones, H. 1958, *The Theory of the Properties of Metals and Alloys* (New York: Dover), pp. 147ff.
Nather, E. 1978, *Pub. A.S.P.*, **90**, 477.
Osaki, Y., and Hansen, C. J. 1973, *Ap. J.*, **185**, 277.
Papaloizou, J., and Pringle, J. E. 1978, *M.N.R.A.S.*, **182**, 423.
Pollock, R. 1977, personal communication.
Pollock, R., and Hansen, J.-P. 1973, *Phys. Rev. A*, **8**, 3110.
Savedoff, M. P., Van Horn, H. M., and Vila, S. C. 1969, *Ap. J.*, **155**, 221.
Takeuchi, H., and Saito, M. 1971, *Meth. Comp. Phys.*, **11**, 217.
Van Horn, H. M. 1978, in *Proc. Conference on Current Problems in Stellar Pulsation Instabilities*, ed. D. Fischel, J. R. Lesh, and W. M. Sparks (NASA Special Publication).
Van Horn, H. M., Richardson, M. B., and Hansen, C. J. 1972, *Ap. J.*, **172**, 181.
Van Horn, H. M., and Savedoff, M. P. 1976, in *Proc. Solar and Stellar Pulsation Conference*, ed. A. N. Cox and R. G. Deupree (LASL Report LA-6544-C), p. 109 (VHS).
Wolff, C. L. 1977, *Ap. J.*, **216**, 784.

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