THE ASTROPHYSICAL JOURNAL, 228:240–256, 1979 February 15

MASSES AND RADII OF WHITE-DWARF STARS. III. RESULTS FOR 110 HYDROGEN-RICH AND 28 HELIUM-RICH STARS

HARRY L. SHIPMAN
University of Delaware
Received 1978 April 11; accepted 1978 August 23

ABSTRACT

This paper provides masses and radii for a large sample of white-dwarf stars, with radii listed in tables and shown in a figure. Radii were computed from photometry, parallaxes, and model atmospheres, and masses from the mass-radius relation. The mean radius of the sample of H-rich stars is 0.0127 $R_\odot$ with an estimated error, arising from calibration uncertainties, of 5%. The mean radius of the sample of He-rich stars is 0.0111 $R_\odot$, not significantly different from that for the H-rich stars. The most serious limitation on these results is the possible existence of selection effects, which are evaluated in an approximate way. The true mean radius of H-rich white-dwarf stars lies between the figure given by the raw data, corresponding to a mass of 0.55 $M_\odot$, and a value of 0.0103 $R_\odot$ (or $M = 0.75\ M_\odot$) obtained after corrections, with the smaller radius more likely.

Subject headings: stars: atmospheres — stars: white dwarfs

I. INTRODUCTION

White-dwarf stars, an endpoint of stellar evolution, play a significant role in astrophysics. Determination of their physical properties can affect work in closely related fields like the study of stellar evolution and less closely related fields like the study of the mass budget of the Galaxy. The other final states of stellar evolution, neutron stars and black holes, are objects that cannot be analyzed by traditional spectroscopic techniques, and so the properties of white-dwarf stars can be determined more easily from the observations.

Since white-dwarf stars follow a mass-radius relation, the determination of the radii of a sample of white-dwarf stars permits an estimate of the mean mass and mean radius of these objects, provided that one makes some reasonable assumptions about the composition. Such an estimate can help solve two problems relating to other areas of astrophysics. Clarification of the value of the white-dwarf mean mass and some estimate of its range can shed some light on the amount of mass loss that takes place in the red giant stage (Auer and Woolf 1965; Eggen and Greenstein 1965; Weidemann 1977*a*; and intermediate papers cited in Weidemann 1977*a*). Further, knowledge of the mean mass of white-dwarf stars is important for understanding the evolution of the Milky Way Galaxy's stellar population.

Several years ago, I published a list of white-dwarf radii based on the photometry available at that time (Shipman 1972, hereafter Paper I). Paper I listed 51 hydrogen-rich stars with parallaxes and useful photometry, along with nine helium-rich stars of various spectral types. Eleven more hydrogen-rich stars were assigned radii a few years later (Shipman 1977*a*, hereafter Paper II), making a total of 62 H-rich and nine He-rich stars which have had radii published in the literature. With the availability of a large volume of multichannel photometry from Palomar (Greenstein 1976), many more U.S. Naval Observatory parallaxes, and a set of model atmospheres covering the region 8000 K $\leq T_{\rm eff} \leq$ 16,000 K (McGraw and Shipman 1978), it seemed worthwhile and timely to calculate radii for a new sample of white-dwarf stars, comprising all stars in the literature with known distances, photometry, and spectral types. This paper constitutes such a survey. A forthcoming paper (Shipman and Sass 1979) uses various two-color diagrams (e.g., Strömgren $u - y$ versus $b - y$) to compute radii and compares the radii determined here with those determined from a variety of other methods, including the results of Wehrse (1975) and Trimble and Greenstein (1972).

Preliminary results of this work were reported earlier (Shipman 1977*b*). This paper presents the details of the preliminary results, and also treats a larger data base. Thus there are some minor differences between the results here and the preliminary ones. Concurrently with the preliminary presentation of this work, Weidemann (1977*b*) presented the parallel work of Koester, Schulz, Weidemann, and Wehrse of the Kiel group who, using independently calculated model atmospheres, obtained results for the hydrogen-rich stars that agree with those here.

The basic method for calculating stellar radii, used earlier in Papers I and II, is briefly described in § II. This paper claims that systematic errors in the method are approximately 5% for the hydrogen-rich stars, and § II also contains a discussion of systematic errors in support of that statement. Procedures strictly relevant to the calculations for the H-rich stars are given in § III, and the results are in § IV. Procedures and results for the He-rich stars are discussed in § V, and selection effects are treated in § VI. Readers interested primarily in results will find them in §§ IV, V*b*, and VI; in tabular form in Tables 4–6, and illustrated in Figure 3.

II. THE BASIC METHOD

a) *Fundamentals*

The determination of stellar radii in this paper is based on the fundamental relation

$$f_\nu = 4\pi H_\nu R^2/D^2 ,$$

or

$$R = D(f_\nu/4\pi H_\nu)^{1/2} , \quad (1)$$

between the monochromatic Eddington flux H_ν at a stellar surface, the stellar radius R, its distance D, and the flux f_ν measured at the top of the Earth's atmosphere. In all cases fluxes in this paper are measured in ergs cm^{-2} s^{-1} Hz^{-1}.

For precise work, some care needs to be taken in setting the zero point of the magnitude system. The principal magnitude system used here is the one defined by the multichannel spectrometer and used at Palomar, designated MC. Greenstein (1976) states that $V_{\rm MC} = V_{\rm J} - 0.05$, where $V_{\rm J}$ is the V-magnitude in the Johnson system. I have verified this relation for the white-dwarf stars with parallaxes considered in this paper. Breger (1976) discusses the problems involved in calibrating photometry of white-dwarf stars and other faint stars with reference to a star like Vega, tens of thousands of times brighter. The calibration used here is strictly valid only for white-dwarf stars. Taking Vega as the fundamental standard, with a Johnson V-magnitude of $+0.04$ (Breger 1976), we have $V_{\rm MC}$(Vega) $= -0.01$. Various investigators have measured the absolute flux from Vega in the V-band. Three recent results are from Oke and Schild (1970), corrected by Hayes and Latham (1975), and by Tug, White, and Lockwood (1977); they give (respectively) the flux as (3.43, 3.57, and 3.56) $\times 10^{-20}$ ergs cm^{-2} s^{-1} Hz^{-1}. An average value for this flux is 3.52×10^{-20} ergs cm^{-2} s^{-1} Hz^{-1}. Using this value and $V_{\rm MC}$(Vega) $= -0.01$ to define the zero point of the white-dwarf MC magnitude scale, where magnitudes are defined as $m = -2.5 \log f +$ constant where the constant is tied into the Johnson system and the fundamental calibration of Vega as defined above, equation (1) can be cast into the numerically convenient form

$$2 \log R/R_\odot = -\log H_\nu + 2 \log D - 5.262 - m_\nu/2.5 . \quad (2)$$

This method for measuring stellar radii was first introduced by Gray (1967). While, in a sense, it is more or less equivalent to using temperatures and bolometric corrections, it is far more direct. Further, equation (1) is expressed in a form that makes it clear where the uncertainties come from. Because this paper claims to produce radii for hydrogen-rich stars with a small systematic uncertainty, this form of the radius equation is most useful. Of course, the measurement of the radius of an individual star and its associated uncertainty depends crucially on the parallax of that star and its accuracy, but the accuracy of the measurement of the mean radius of a group of stars depends less on parallaxes of individual stars and more on the systematic errors that enter the application of equation (1).

The application of equations (1) and (2) to real observational data introduces several systematic uncertainties. This section deals with those uncertainties which are common to the determination of radii of stars of all spectral types: systematic parallax errors (affecting D), uncertainties in the absolute flux calibration (affecting f_ν), and uncertainties in the color calibration (affecting the stellar color which determines H_ν). The remaining source of error comes from the model-atmosphere calibration of the color–H_ν relation, which is discussed in §§ III and V for the different chemical classes of white-dwarf stars.

b) *Flux Calibration*

The three separate measurements of the absolute flux from Vega have a standard deviation of the mean of about 1%. However, additional uncertainties arise because of problems in calibrating secondary standards, as discussed by Breger (1976). An uncertainty in the absolute flux scale of 2%, equal to the uncertainty quoted by Hayes and Latham (1975), is adopted here. Since the stellar radius $R \sim f^{1/2}$ from equation (1), the systematic uncertainty in the mean radius arising from an uncertain flux calibration is 1%.

The absolute calibration enters the stellar radius determination in another way. A measured stellar color has to be interpreted in absolute terms in order to use a model-atmosphere relation between color and stellar flux H_ν in order to determine H for a given star. For the monochromatic colors $g - r$ $(=2.5 \log f_{6944}/f_{4716})$ and $u - v$ $(2.5 \log f_{5405}/f_{3571})$, Hayes and Latham estimate errors of 0.03 magnitudes or 3%. (The subscripts to the f's indicate central wavelengths in angstroms.) Again, the calibration of Tug, White, and Lockwood justifies this error estimate. The dependence of H_ν on color indicates that 0.03 mag in one of these colors corresponds to an uncertainty in H_ν of 6%, if $g - r$ is used as a surface brightness parameter for the cooler stars and $u - v$ for the hotter ones. Since $R \sim H_\nu^{1/2}$, the uncertainty in R resulting from the uncertainty in the relative flux calibration is approximately 3%.

c) *Trigonometric Parallaxes*

The uncertainty of the radius of an individual star is, in most cases, dominated by the uncertainty in the measurement of its distance. Most of the white-dwarf stars covered in this paper have distances that come from trigonometric parallaxes, many of them from the U.S. Naval Observatory. Harrington and Kallarakal (1978) show that the external errors of the Naval Observatory parallaxes are the same as the internal errors, showing that systematic effects are very small, probably less than 0″001. A more serious problem comes from the correction from relative to absolute parallax. The Naval Observatory publishes relative parallaxes; van Altena's (1974) tables were used to correct relative parallaxes to absolute parallaxes. Since white-dwarf stars are faint, the reference stars, having the same magnitudes as the parallax stars (Riddle 1970), are also faint and the corrections are small, averaging approximately 0".002. A reasonable estimate is that the systematic uncertainty in the correction from relative to absolute for the entire sample of stars is half that figure. (The uncertainty in the correction for any individual star can be considerably larger.) The average parallax for the 75 stars in the sample with trigonometric parallaxes is 0".03, so that an uncertainty in the correction from relative to absolute parallax of 0".001 corresponds to an uncertainty of 3% in the average radius.

In a preliminary report of this work (Shipman 1977*b*), the parallaxes were taken directly from the catalog of McCook and Sion (1977). Following the Naval Observatory, McCook and Sion list relative parallaxes for all stars. The McCook-Sion catalog used the parallax list of the Kiel group (Sion, private communication), but it was unfortunately not explicitly stated that the parallaxes listed therein were relative parallaxes. As a result, the mean radius found here differs by about 4% from the mean value found in the preliminary report. All parallaxes used here have been checked with the original source in order to avoid propagation of errors, and all are absolute parallaxes. Where two stars are members of a common-proper-motion system, the mean parallax of both stars was adopted.

The one remaining source of systematic error yet to be discussed is the model-atmosphere calibration of the H_v versus color relation. Since the problems involved in calculating this relation differ for the two major compositional classes of white-dwarf stars, these uncertainties are discussed separately in §§ III and V.

III. PROCEDURES FOR THE HYDROGEN-RICH STARS

Most white-dwarf stars are of spectral type DA, showing hydrogen lines in their spectra. An additional group of white-dwarf stars which can be classified as possibly H-rich are the very cool stars, type DC, DF, or DK, stars sufficiently cool that hydrogen lines would not be visible in their spectra (Paper II). This section discussed the computation of the radii of these two groups of stars, the most accurately determined radii.

a) *Stars with Scans*

Two useful colors for determining fluxes from DA stars are the monochromatic colors $g - r$ and $u - v$, observed by Greenstein (1976) with the Palomar multichannel spectrometer. The *b* band (central wavelength 4255 Å) was not used because it lies between Hγ and Hδ, where the stellar spectrum is curved, affected by overlapping hydrogen lines. At the level of precision sought here, observational uncertainties in locating the center of the spectrometer band and theoretical uncertainties in calculating the spectrum and folding it with the multichannel bandpass preclude the use of this color. The difference $u - v$ was used as a flux indicator at high temperatures ($T_{\rm eff} \geq 16{,}000$ K), and $g - r$ was used at low temperatures ($T_{\rm eff} \leq 8000$ K). Between these two temperatures, $g - r$ is perhaps the best temperature indicator, but there is a slight gravity dependence of the H_{5405} ($=H_v$) versus $g - r$ relation and a strong gravity dependence of $u - v$ (which will be used in Shipman and Sass 1979 to determine surface gravities). For stars with multichannel colors, in this temperature range, $8000\ \text{K} < T_{\rm eff} < 16{,}000$ K, temperatures and hence fluxes were determined by interpolation in the ($u - v$, $g - r$)-plane.

b) *Stars with Intermediate- or Broad-Band Colors*

Yet there are many stars which have published parallaxes and which do not have published multichannel observations. Broader-band systems can be used to determine the Eddington flux for these stars, but they have to be calibrated. Two approaches to calibration have been used. In one, used by Strittmatter and Wickramasinghe (1971) and Wehrse (1975), fluxes from the model atmospheres are convolved with a filter transmission function, normalized to some zero point, and converted to magnitudes to define a Strömgren or Johnson color for a given model. This approach has some shortcomings. The Strömgren and particularly the Johnson filters include wavelength regions where Balmer lines affect the shape of the spectrum, and the accuracy of the model spectrum and that of the filter transmission function. A potentially more serious problem is that of zero points. Calibrations of the *UBV* and *uvby* systems, to be related to absolute photometry, must be carried through some system of fundamental standard stars. The accepted energy distribution for the most fundamental standard star, Vega, has changed through the years, although recent work shows concordance between different workers (§ II*b*). The calibrations of the *UBV* and *uvby* systems generally refer to rather old work, that of Matthews and Sandage (1963) for the *UBV* system and that of Matsushima (1969) and Olson (1974) for the *uvby* system. These calibrations predate the Hayes-Latham calibration.

With the extensive list from Greenstein (1976) of multichannel colors that are tied to some well-defined absolute calibration, it is now possible to empirically calibrate the Johnson and Strömgren systems. Here as elsewhere in this work the colors published in Greenstein (1976) were corrected to the Hayes-Latham calibration. Define a relationship

$$\mathrm{CI_{BB}} = a(\mathrm{CI_{MC}}) + b \qquad (3)$$

between a broad-band color $\mathrm{CI_{BB}}$ and a multichannel color index $\mathrm{CI_{MC}}$. A least-squares technique was then used to determine the various coefficients *a* and *b* by fitting broad-band and multichannel observations of various stars where both types of data were available.

TABLE 1

COLOR TRANSFORMATIONS

Broad-Band Color	Multichannel Color	a	b	r^2	Error of an Individual MC Color	Number of Stars
Johnson $U - V$.......	$u - v$	0.953	−0.758	0.944	0.06	49
Johnson $B - V$.......	$g - r$	0.591	+0.334	0.926	0.15	49
Strömgren $u - y$......	$u - v$	1.093	0.240	0.995	0.035	24
Strömgren $b - y$......	$g - r$	0.448	0.220	0.972	0.066	24

The fitting procedure spanned the entire range of available colors, since MC colors are available for the coolest (WD 2054−05 = EG 202) and hottest (WD 0501+52 = Gr 247) stars listed here. This procedure works only when the two-color systems, multichannel and intermediate- or broad-band, measure the same physical quantity (slope of the Paschen continuum or Balmer jump). Because the broad-band filters inevitably include some part of a spectrum line in an extremely broad-lined star like a white-dwarf star, it is important to use a homogeneous data set when using broad-band observations, a point emphasized strongly by Graham (1972). Graham's Strömgren colors and the Palomar *UBV* colors were used when possible. The various coefficients a and b are listed in Table 1, along with the correlation coefficients r^2. The next to last column lists the rms difference between the multichannel color predicted by equation (3) and the observed color, an estimate of the uncertainty introduced when a Strömgren or Johnson color is used to predict an MC color. Examination of these errors showed no systematic dependence of error on color, no need for a quadratic term in equation (3). These transformations are valid only for the DA stars; in addition, they are probably only strictly valid for Strömgren or Johnson colors measured in a manner consistent with the Palomar measurements (Johnson colors) or Graham's measurements (Strömgren colors).

The size of the rms errors in Table 1 deserves some comment. Greenstein (1976) and Graham (1972) both cite errors of 0.02 mag for an individual observation, and so one would naively expect an rms error of 0.028 mag for the transformations involving the Strömgren colors if the Strömgren and MC colors had equal baselines. The data of Table 1 show that the $u - y$ colors are almost as good as the observers claim. The $b - y$ colors seem to have larger uncertainties; but when one considers that the $b - y$ color measures a shorter segment of the Paschen continuum than the $g - r$ color, the rms error is again about as good as one could ask for. Errors in the Johnson colors are somewhat larger, probably reflecting the effect of lines on the wider-band Johnson filters. These results show that for white-dwarf stars, properly calibrated Strömgren colors are as useful as MC colors for determining the photometric properties of an individual star. The Johnson colors are less accurate for individual stars, but should be suitable for calculating the mean radius of a group of stars.

In this paper, the Strömgren colors (where available) or Johnson colors were used to predict monochromatic $u - v$ or $g - r$ colors for each star without multichannel observations, using the relations in Table 1. These colors then provide a temperature and a flux H_v, with $u - v$ used for $T_{\rm eff} > 16{,}000$ K and $g - r$ used for $T_{\rm eff} \leq 16{,}000$ K. The errors introduced by the use of intermediate- or broad-band photometry can be assessed by comparing mean radii of a given sample derived by both methods. The 49 H-rich stars with MC photometry had a mean radius that was 1.5% less when it was computed using Strömgren or Johnson photometry than it was when the multichannel photometry was used. Evidently uncertainties introduced by using the broader-band photometry are small.

c) *Relation between Color and Flux*

Central to this entire procedure is an accurately calibrated relationship between color and flux, which is calculated using model atmospheres. The relationship used here is summarized in Table 2 and graphically illustrated in Figure 1. Some of the models used to define this relation have been published previously (Shipman 1971; Paper I; Paper II), but some gaps and inconsistencies regarding the treatment of hydrogen line blanketing have been coped with in producing the relationship in Table 2. Additions to previously published results are in the range 8000 K $\leq T_{\rm eff} \leq$ 16,000 K, where a new set of pure hydrogen, line-blanketed, convective model atmospheres (McGraw and Shipman 1978) were used to supplant the grid of Paper I. Line-blanketed models at $T_{\rm eff}$ = 27,500, 35,000 and 50,000 K were calculated to replace the unblanketed models with $T_{\rm eff} \geq$ 30,000 K published in Paper I. At 27,500 and 35,000 K, the inclusion of line blanketing had the effect of making a line-blanketed model look like an unblanketed model with a $T_{\rm eff}$ 2000 K higher than that of the blanketed model. At $T_{\rm eff}$ = 50,000 K line blanketing produced no effect at the few percent level, verified by comparing the blanketed model with earlier unblanketed models.

How accurate is the relationship between flux and color presented in Table 2? The accuracy of these computations depends both on the adequacy of the numerical procedures involved and on the accuracy of the input physics. The numerical procedures can be tested by comparing the calculation of model atmospheres for similar parameters by different authors. Auer and Shipman (1977) briefly mentioned the results of one such comparison. Here the Auer-Mihalas and ATLAS (Kurucz 1970) computer programs were used

TABLE 2

THE FLUX VERSUS COLOR RELATION

$T_{eff}/10^3$ K	$u - v$	$g - r$	H (λ5405 Å)	
4.0....	1.850	1.030	1.47E−06	
5.0....	1.432	0.779	3.28E−06	
6.0....	0.958	0.458	7.26E−06	
8.0....	0.452	0.042	2.06E−05	
9.0....	0.411	−0.096	3.13E−05	
9.5....	0.419	−0.169	3.78E−05	
10.0....	0.428	−0.228	4.52E−05	
10.5....	0.445	−0.280	5.24E−05	
11.0....	0.448	−0.329	6.02E−05	
11.5....	0.447	−0.373	6.78E−05	
12.0....	0.439	−0.409	7.41E−05	
12.5....	0.414	−0.438	7.95E−05	
13.0....	0.396	−0.465	8.44E−05	
13.5....	0.347	−0.492	9.13E−05	
14.0....	0.294	−0.512	9.76E−05	
15.0....	0.228	−0.545	1.08E−04	
16.0....	0.145	−0.567	1.19E−04	
20.0....	−0.121	−0.641	1.70E−04	
25.0....	−0.339	−0.694	2.38E−04	
27.5....	−0.415	−0.711	2.82E−04	
35.0....	−0.614	−0.755	4.49E−04	
50.0....	−0.711	−0.770	6.49E−04	
70.0....	−0.766	−0.779	8.51E−04	
100.0....	−0.807	−0.795	1.14E−03	
Effects of Convection				
10.0....	0.43	−0.23	4.52E−05	$l/H = 1$
10.0....	0.45	−0.23	4.36E−05	$l/H = 0.6$
10.0....	0.47	−0.23	4.19E−05	$l/H = 0.3$
10.0....	0.43	−0.22	4.57E−05	$l/H = 1.5$

completely independently, and conversations indicate that these programs have been rewritten from different points of view from the foundations (opacities, numerical procedures) on up. The resulting $u - v$ colors differed by approximately 2%, probably attributable to the inclusion of line blanketing in ATLAS and its lack of inclusion in the Auer-Mihalas program. A comparison of an ATLAS model at $T_{eff} = 15{,}000$ K, $\log g = 8$, $n(\mathrm{He})/n(\mathrm{H}) = 0.144$ calculated at Delaware with a similar model by Wickramasinghe (1972) shows differences of 0.005 in $g - r$, 0.01 (mag) in $u - v$, and 1.6% in absolute flux level.

Recalling that the derived radius $R \sim H^{1/2}$, these comparisons show that numerical inaccuracies in the model programs produce errors in the derived radii of 1%, far less than the errors introduced by calibration uncertainties. Note that the visual flux at 5405 Å changes by less than 1% when $\log g$ is changed from 8.0 to 7.5, so that we can ignore the gravity dependence of H_v on $\log g$ when computing radii.

Yet given the high confidence level ($\approx 1\%$) in the numerical accuracy of existing computer programs, what of the physics that underlies any calculated model stellar atmosphere? For hydrogen-rich white-dwarf stars, nature is quite benevolent and provides us with an element, hydrogen, in which we have considerable confidence. We know the continuous absorption coefficient of hydrogen to high accuracy. Metallic lines play relatively small roles in the game of high-temperature white-dwarf hydrogen model atmospheres, and consequently they can be neglected. Densities in these atmospheres are high enough that, at the temperatures considered for all but a few of the stars dealt with in this paper, LTE is a valid approximation for the continuum. I have verified this approximation up to $T_{eff} = 50{,}000$ K by explicit calculations, and Peterson and Greenstein (1973) have included line transitions and verified this conclusion for lower temperatures. Thus, with the exception of the low-temperature models ($T_{eff} \lesssim 5000$ K) where there are some questions regarding the appropriate physics to use in the equation of state (see below) and the pressure-induced-dipole (Paper II) opacity, we can have a reasonable ($< 2\%$) degree of confidence in the models.

There are a few potential uncertainties that should be considered. One arises from the adequacy of the treatment of hydrogen line blanketing, one from the treatment of the equation of state, and one from the treatment of convection. Let us consider these *seriatim*.

Concerning hydrogen line blanketing, Wickrama-

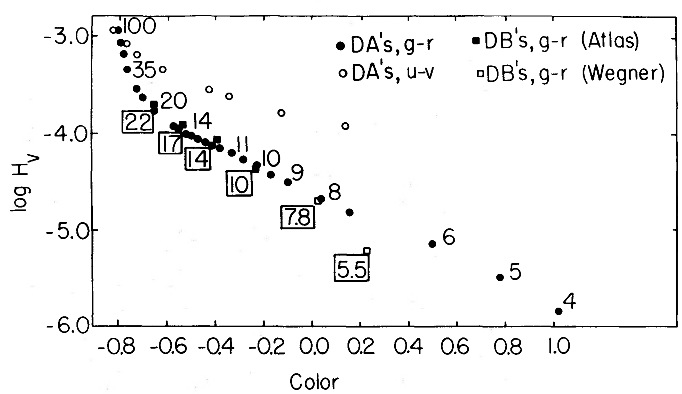

FIG. 1.—The relation between visual flux H_V and various colors. Numbers by points indicate model effective temperatures in units of 10^3 K; numbers in boxes refer to the helium-rich models for the DB stars. The coincidence of the relations between H_V and $g - r$ is noteworthy; its consequences for analysis of white-dwarf stars are explored further in § V*a*, and its relevance to the analysis of main-sequence and giant stars are explored in Shipman (1978).

singhe (1972) used a simpler treatment of hydrogen line blanketing, which, on the basis of the published frequencies, used a sparser frequency set in the region of the Lyman lines than the Delaware (McGraw and Shipman 1978) models used for the flux relation in Table 2. The concordance of two models, computed for identical chemical composition, indicates that the two different treatments of hydrogen line blanketing produce essentially the same results and that, therefore, the treatment used here is adequate. In fact, an examination of the frequency points tabulated by Wickramasinghe (1972) indicates that the difference between the two models is in the right direction and of the right order of magnitude if one seeks to explain it by appealing to the sparseness of Wickramasinghe's frequency set.

Another concern is the equation of state. At high temperatures, the gas in the stellar atmosphere can be treated as an ideal gas but one has to worry about the lowering of the ionization potential caused by the close packing of the atoms and ions. The ATLAS program currently in use at Delaware uses the same, simple Debye-Hückel approach that I have used in all model-atmosphere calculations so far. It is important to include this effect (Shipman 1971), but preliminary estimates indicate that at high temperatures, at least, it is less important that this effect be given an elaborate treatment. For example, the inclusion of the Stewart-Pyatt (1966) corrections to the Debye-Hückel approach shows that the inclusion of this approach lowers the indicated surface gravity of a $T_{\mathrm{eff}} = 16{,}000$ K model by 0.05 in the logarithm. Since the emergent flux H_v is insensitive to gravity at this temperature, the approximate treatment of pressure ionization should not produce significant errors in the calculation of white-dwarf radii presented in this paper.

Yet at lower temperatures, problems do arise. Here, one cannot be sure of the dominant constituent of an atmosphere, since neither H nor He produces spectral lines in white-dwarf atmospheres below temperatures of approximately 6000 K. Further, Böhm *et al.* (1977) have recently shown that a very cool He white-dwarf star would be degenerate at the stellar surface. The H-rich models do not, apparently, reach such extreme conditions, but the equation of state cannot be regarded as that of an ideal gas at very cool temperatures. Examination of the region of the temperature-pressure plane traversed by the model atmospheres above Rosseland optical depth unity shows that for $T_{\mathrm{eff}} \geq 6000$ K, the ATLAS ideal-gas equation of state agrees with the more detailed calculations of Fontaine, Graboske, and Van Horn (1977) within 5% at $\log g = 8$.

Perhaps the greatest source of uncertainty in these models, and in the color-flux relation of Table 2, is the treatment of convection. At and below $T_{\mathrm{eff}} = 12{,}000$ K, convection carries a substantial fraction of the energy flux in the stellar atmosphere, altering the temperature gradient at optical depths near unity. Extensive numerical experiments were performed with a $T_{\mathrm{eff}} = 10{,}000$ K, $\log g = 8$ model, and the resulting colors and fluxes for various values of the convective efficiency parameter l/H are shown in the last lines of Table 2. Evidently the flux H_v is reduced by 7%, for constant $g - r$ color, if the l/H value is reduced from the conventional 1.0 to 0.3, a minimally realistic value (Mullan, private communication). Increasing l/H above 1.0 does not produce a significant effect. A similar investigation at $T_{\mathrm{eff}} = 6000$ K produces similar results. If the true energy transport of convection is in fact represented by the mixing-length theory with $l/H = 0.3$, the radii of stars with $T_{\mathrm{eff}} < 12{,}000$ K would be underestimated by $3\frac{1}{2}$%. Since these stars constitute approximately half of the sample of H-rich stars, the mean radius of the entire sample has a 2% uncertainty attributable to uncertainties in the treatment of convection.

Still another question to address is the appropriateness of using pure hydrogen model atmospheres to define the flux-color relation. Very few white-dwarf stars show both H and He in their spectra, and analyses of the high-temperature H-rich stars (see, for example, Auer and Shipman 1977 and Paper I) show that the helium abundance of these stars is extremely low. One would guess that the cooler H-rich stars were also He-deficient; but even if they were not, the effect of adding He to a white-dwarf atmosphere is one of changing the surface gravity. Since the flux-color relation is gravity independent, the helium abundance possible problem is of no consequence for the present investigation.

In addition, one worries about metal abundance effects. Wehrse (1975) showed that the Strömgren $u - b$ color at $T_{\mathrm{eff}} = 9000$ K was changed considerably in going from pure H to 1/100 of the solar metal abundance, a value in excess of the metal abundances observed for cooler DA stars (Paper II). But even this change would not affect the calibration of the flux-color relation used here, since at $T_{\mathrm{eff}} = 9000$ K it is the slope of the Paschen continuum, through a $g - r$, Strömgren $b - y$, or Johnson $B - V$ color that is used to give H_v.

To recapitulate: The flux-color relation in Table 2, used for the H-rich stars, comes from pure H models calculated using the ATLAS code (Kurucz 1970). Hydrogen line blanketing, both Lyman and Balmer lines, was included where appropriate ($T_{\mathrm{eff}} \leq 50{,}000$ K). Changes to the ATLAS code were described in Paper II; the treatment of convection was changed, and additional opacities (H_2^- and H_2 pressure-induced-dipole) were added. The presently known uncertainties have been discussed, and the problems that produce significant effects (5% in H_v for a given color) are confined to uncertainties in the equation of state for $T_{\mathrm{eff}} < 6000$ K and uncertainties in the treatment of convection. All other published model-atmosphere calculations are also subject to these uncertainties.

IV. HYDROGEN-RICH STARS: RESULTS

a) *Radii*

The only source of observational data yet to be discussed in the parallax list. Most parallaxes were taken from the Naval Observatory catalogs (Riddle 1970; Routly 1972; Harrington *et al.* 1975; Dahn *et al.* 1976; Dahn 1978); trigonometric parallaxes from other sources are listed in Table 3. Stars with distances determined from cluster or common-proper-motion (cpm) system membership are identified in the tables of radii, Tables 4–6, and literature references are provided in the notes to Table 4.

Table 4 includes all stars which have hydrogen lines in their spectra. One star in Table 4, WD 1917−07 (=EG 131), has very weak H lines for its color and probably is dominantly He. It is also included in Table 6, but since the H_v versus $g - r$ relation is rather similar for the two types of stars its listed radii is not too different in the two tables.

Table 5 includes stars which are too cool to show H lines in their spectra but which are probably H-rich. I regard them as probably H-rich for two reasons. One is that most higher-temperature white-dwarf stars are also H-rich. Another is that the application of a He-rich flux-color relation would produce improbably large radii for these stars ($R \geq 0.02\ R_\odot$). One can test the presumption that these stars are H-rich, or at least have a high enough abundance of electron donors so that they follow an H-rich flux-color relation, by calculating the mean radius of these probably H-rich stars in Table 5. It turns out to be 0.0121 ± 0.0008 $R_\odot$, not significantly different from the mean radius of 0.0127 $R_\odot$ derived for the entire sample of Tables 4 and 5, The histogram of white-dwarf radii, shown in Figure 2, graphically illustrates the similarity of radii in Tables 4 and 5. While the probably H-rich stars of Table 5 are separated from the confirmed H-rich (DA) stars in Figures 2 and 3, in the subsequent discussion they are treated as a group.

Various columns in Tables 4 and 5 provide the following information: the star identification on the Villanova system (McCook and Sion 1977), the EG/Gr number (where the numbers N are to be considered as EG numbers for N less than 202 and Gr numbers otherwise), the spectral class, the radius and mass in solar units, the value of log g (in cgs units), a number indicating which color was used to determine H_v, the temperature in 10^3 K, the parallax, the probable error of the parallax, and the fractional uncertainty in the radius. Note that mean parallax errors are listed, following astrometric tradition. The radius errors should be treated as estimated errors since they are largely based on two independent evaluations of a calibration or calculations of model atmospheres. The precepts that the radius errors are based on are discussed in § IV*b* below.

TABLE 3

PARALLAXES OF WHITE-DWARF STARS

Star	π_{abs}	Source	π_{abs}(adopted)
0046+05......	0.230 ± 0.003	Gatewood and Russell 1974*	0.230 ± 0.003
0208+39......	0.0422 ± 0.007	van Altena and Vilkki 1973	...
	0.065 ± 0.004	Routly 1972	0.059 ± 0.005
0413−07......	0.207 ± 0.002	Heintz 1974*	0.207 ± 0.002
0426+58......	0.183 ± 0.003	Strand 1977*	0.183 ± 0.003
0612+17......	0.033 ± 0.003	Heintz 1973	...
	0.023 ± 0.003	Dahn 1978	0.028 ± 0.003
0642−16......	0.375 ± 0.004	Jenkins 1963*	...
	0.3756 ± 0.003	Gatewood and Gatewood 1978	0.3756 ± 0.003
0727+48......	0.092 ± 0.004	van Altena and Stone 1973	...
	0.086 ± 0.002	Strand *et al.* 1976	...
	0.092 ± 0.004	Heintz 1976	0.089 ± 0.003
0738−17......	0.164 ± 0.008	Jenkins 1963 (no. 1813)	...
	0.102 ± 0.014	Jenkins 1963 (no. 1810.1)	0.148 ± 0.007
0839−32......	0.108 ± 0.007	Jenkins 1963*	0.108 ± 0.007
1142−64......	0.203 ± 0.008	Jenkins 1963*	0.203 ± 0.008
1257+03......	0.078 ± 0.011	Jenkins 1963*	0.078 ± 0.011
1314+29......	0.016 ± 0.003	Margon *et al.* 1976*b*	0.016 ± 0.003
1544−37......	0.069 ± 0.008	Jenkins 1963*	...
	0.075 (cpm)	Wegner 1973	...
	0.079 (cpm)	Eggen and Greenstein 1965	0.0731 ± 0.007
1716+02......	0.028 ± 0.0028	Dahn *et al.* 1976	...
	0.023 ± 0.007	Heintz 1976	0.0272 ± 0.003
1756+82......	0.059 ± 0.005	Riddle 1970	...
	0.063 ± 0.005	van Altena 1971	0.061 ± 0.004
2032+24......	0.079 ± 0.002	Lippincott 1974	...
	0.069 ± 0.003	Routly 1972	...
	0.072 ± 0.004	Wagman 1967	0.071 ± 0.002
2153−51......	0.064 ± 0.012	Jenkins 1963*	0.064 ± 0.012
2248+29......	0.0428 ± 0.009	van Altena and Vilkki 1973	...
	0.050 ± 0.005	Routly 1972	0.0486 ± 0.005
2341+32......	0.058 ± 0.004	Harrington *et al.* 1975	...
	0.050 ± 0.002	Heintz 1976	0.0516 ± 0.002
2359−43......	0.120 ± 0.007	Jenkins 1963	0.120 ± 0.007

* These sources cite parallaxes which are the averages of previous determinations.

1979ApJ...228..240S

TABLE 4

RADII OF HYDROGEN-RICH STARS

STAR NO.	EG/GR	SP.CL	R	M	G	COLOR	T	PI	P.E.(PI)	DEL(R)/R
0011+00	2	DA	0.0104	0.75	8.3	4	9.4	0.0330	0.0040	0.24
0030+44	306	DA	0.0172	0.25	7.4	1	9.6	0.0111	CPM 4	
0033+01	4	DA	0.0081	0.96	8.6	1	11.2	0.0280	0.0050	0.19
0101+04	7	DAS	0.0151	0.36	7.6	1	8.6	0.0480	0.0040	0.11
0126+10		DA	0.0189	0.20	7.2	4	9.2	0.0290	0.0040	0.24
0133-11	10	DA	0.0318	0.05	6.1	4	10.0	0.0170	0.0030	0.27
0135-05	11	DAS	0.0197	0.18	7.1	1	7.3	0.0840	0.0040	0.08
0205+25	15	DA	0.0133	0.50	7.9	2	20.3	0.0310	0.0030	0.11
0208+39	168	DAS	0.0128	0.54	8.0	1	7.4	0.0590	0.0050	0.11
0220+22	18	DAN	0.0068	1.09	8.8	2	16.3	0.0216	CPM 1	
0232+03	20	DA	0.0239	0.11	6.7	1	60.0	0.0110	0.0040	0.39 CM5
0257+08		DAWK	0.0111	0.68	8.2	4	7.3	0.0370	0.0030	0.22
0349+24	25	DAWK	0.0081	0.96	8.6	1	29.8	0.0077	PLEIADES	
0352+09	26	DA	0.0120	0.60	8.1	1	15.1	0.0230	0.0015	0.09 H
0401+25	28	DA	0.0109	0.70	8.2	1	13.5	0.0370	0.0040	0.12
0406+16	29	DA	0.0085	0.92	8.5	1	15.3	0.0230	0.0015	0.09 H
0413-07	33	DA	0.0124	0.56	8.0	1	16.9	0.2070	0.0020	0.04
0421+16	36	DA	0.0113	0.66	8.2	3	19.4	0.0230	0.0015	0.08 H
0425+16	37	DA	0.0110	0.69	8.2	3	23.9	0.0230	0.0015	0.08 H
0431+12	39	DA	0.0108	0.71	8.2	1	20.9	0.0230	0.0015	0.08 H
0438+10	42	DAN	0.0098	0.80	8.4	1	28.8	0.0230	0.0015	0.08 H
0501+52	247	DAWK	0.0164	0.29	7.5	1	61.9	0.0230	0.0030	0.19
0518+33	43	DAE	0.0134	0.49	7.9	4	8.6	0.0200	CPM 1	
0612+17	46	DA	0.0109	0.70	8.2	2	26.7	0.0280	0.0030	0.12
0642-16	49	DA	0.0074	1.03	8.7	1	29.0	0.3756	0.0030	0.08
0644+37	50	DA	0.0094	0.83	8.4	1	22.2	0.0700	0.0030	0.06
0752+36	345	DAWK	0.0103	0.76	8.3	1	7.9	0.0310	0.0040	0.15
0816+38	58	DA,F	0.0100	0.78	8.3	4	8.0	0.0256	0.0040	0.26
0827+32	249	DAS	0.0084	0.94	8.6	4	8.1	0.0450	0.0040	0.22
0836+20	59	DA	0.0141	0.43	7.8	4	10.6	0.0062	PRAESEPE	
0836+19	60	DA	0.0093	0.85	8.4	4	13.3	0.0062	PRAESEPE	
0837+19	61	DA	0.0118	0.62	8.1	4	13.6	0.0062	PRAESEPE	
0839-32	62	DA,FS	0.0156	0.33	7.6	4	9.3	0.1080	0.0070	0.21
0913+44	64	DAS	0.0182	0.22	7.3	5	9.4	0.0190	CPM 1	
0921+35	65	DA	0.0142	0.42	7.8	1	12.5	0.0157	CPM 1	
0930+29	324	DA	0.0101	0.77	8.3	1	8.3	0.0320	0.0040	0.14
0943+44	67	DA	0.0170	0.26	7.4	1	14.5	0.0310	0.0040	0.14
0955+24	69	DA	0.0105	0.74	8.3	5	8.7	0.0420	0.0040	0.14
1055-07	74	DA	0.0084	0.93	8.6	5	8.1	0.0850	0.0050	0.12
1104+60	75	DAN	0.0137	0.46	7.8	2	19.2	0.0240	0.0040	0.18
1105-04	76	DA	0.0126	0.55	8.0	1	15.3	0.0460	CPM 1	
1121+21	79	DA,F	0.0123	0.57	8.0	1	7.4	0.0730	0.0040	0.09
1143+32	185	DA	0.0124	0.57	8.0	3	16.5	0.0330	0.0040	0.13
1147+25	84	DA	0.0116	0.63	8.1	5	10.5	0.0220	0.0040	0.21
1244+14	92	DAS	0.0099	0.79	8.4	1	10.4	0.0227	CPM 4	
1314+29	98	DAWK	0.0135	0.48	7.9	1	60.0	0.0160	0.0030	0.23
1327-08	99	DA	0.0144	0.41	7.7	1	12.8	0.0650	0.0040	0.08
1334-16	101	DA	0.0083	0.94	8.6	2	20.3	0.0186	CPM 1	
1344+10	360	DAF	0.0111	0.68	8.2	1	7.6	0.0500	0.0040	0.10
1354+34	173	DA	0.0100	0.77	8.3	5	13.6	0.0145	CPM 2	
1408+32	189	DA	0.0123	0.58	8.0	2	18.2	0.0260	0.0040	0.16
1455+29	298	DA,F	0.0118	0.62	8.1	4	8.6	0.0300	0.0040	0.24
1510+56	112	DA	0.0183	0.22	7.3	1	9.3	0.0121	CPM 1	
1544+00	113	DAWK	0.0094	0.83	8.4	3	39.0	0.0100	CPM 1	
1544-37	114	DAS	0.0198	0.17	7.1	4	8.1	0.0731	0.0070	0.22
1555-08	174	DA	0.0154	0.34	7.6	5	12.6	0.0190	CPM 2	
1559+36	115	DA	0.0111	0.68	8.2	1	11.2	0.0360	0.0050	0.15
1609+13	117	DAS	0.0067	1.10	8.8	4	9.6	0.0550	0.0050	0.22
1637+33	120	DAS	0.0131	0.52	7.9	4	9.7	0.0340	0.0040	0.23
1647+59	368	DA	0.0124	0.57	8.0	1	11.8	0.0820	0.0040	0.08
1655+21	197	DASS	0.0136	0.48	7.9	1	9.4	0.0440	0.0040	0.11
1659-53		DA	0.0157	0.32	7.6	4	11.8	0.0364	0.0050	0.24 CM:
1706+33	198	DA	0.0102	0.76	8.3	1	13.1	0.0170	CPM 2	
1716+02	123	DA	0.0148	0.38	7.7	5	11.6	0.0272	0.0030	0.15
1736+05	371	DA	0.0132	0.51	7.9	1	8.6	0.0240	0.0040	0.18
1743-13	175	DA	0.0186	0.21	7.2	4	10.6	0.0250	CPM 2	
1756+82	199	DASS	0.0135	0.48	7.9	1	7.5	0.0610	0.0040	0.09
1826-04	126	DAS	0.0139	0.45	7.8	4	9.4	0.0350	0.0040	0.23
1855+33	127	DAN	0.0126	0.55	8.0	4	10.5	0.0310	0.0040	0.24
1911+13	130	DAN	0.0148	0.38	7.7	4	11.4	0.0316	CPM 1	
1917-07	131	DAWK	0.0107	0.71	8.2	3	11.4	0.0980	0.0040	0.08
1932-13	132	DA	0.0125	0.56	8.0	2	17.2	0.0107	CPM 1	
1935+27	277	DA	0.0149	0.37	7.7	4	10.5	0.0560	0.0040	0.21
1943+16	134	DA	0.0113	0.66	8.2	2	20.9	0.0240	0.0040	0.18
1953-01	135	DA	0.0111	0.68	8.2	5	8.2	0.0830	0.0040	0.11
2032+24	139	DA	0.0122	0.59	8.0	1	21.5	0.0710	0.0020	0.05
2059+19	377	DAWK	0.0140	0.44	7.8	1	6.8	0.0260	0.0040	0.17
2111+26		DA	0.0145	0.40	7.7	4	9.3	0.0320	0.0040	0.24
2124+55	143	DA	0.0093	0.84	8.4	1	14.1	0.0300	0.0040	0.14
2126+73	144	DA	0.0153	0.35	7.6	1	15.4	0.0420	0.0040	0.11
2136+22		DA	0.0110	0.69	8.2	4	11.0	0.0250	0.0040	0.26
2136+82	147	DA	0.0158	0.32	7.6	3	17.6	0.0320	0.0060	0.19
2149+02	150	DA	0.0131	0.52	7.9	2	18.3	0.0420	0.0030	0.09
2246+22	155	DA	0.0085	0.93	8.6	1	10.5	0.0520	0.0040	0.10
2248+29	283	DAS,F?	0.0160	0.31	7.5	1	5.6	0.0486	0.0050	0.12
2253-08	178	DA,F	0.0139	0.45	7.8	4	7.4	0.0220	CPM 2,3	
2326+04	159	DA	0.0094	0.84	8.4	1	11.9	0.0740	0.0040	0.09
2329+26	161	DAN	0.0124	0.57	8.0	4	9.7	0.0270	0.0040	0.25
2341+32	162	DA	0.0134	0.49	7.9	1	12.5	0.0516	0.0020	0.07
2359-43	165	DAS	0.0040	1.31	9.4	2	18.7	0.1200	0.0070	0.08

TABLE 5

RADII OF COOL, PROBABLY HYDROGEN-RICH STARS

STAR NO.	EG/GR	SP.CL	R	M	G	COLOR	T	PI	P.E.(PI)	DEL(R)/R
0213+42	16	DC	0.0109	0.70	8.2	1	5.5	0.0520	0.0040	0.10
0433+27	40	DC	0.0106	0.72	8.3	4	5.7	0.0600	0.0030	0.21
0552-04	45	DK	0.0124	0.57	8.0	1	4.4	0.1560	0.0050	0.07
0553+05	290	DCP	0.0115	0.64	8.1	1	5.7	0.1250	0.0040	0.07
0727+48	52	DC	0.0149	0.37	7.7	1	4.3	0.0890	0.0030	0.07
0912+53	250	DCP	0.0101	0.77	8.3	1	7.5	0.0990	0.0030	0.07
1039+14	72	DC	0.0111	0.68	8.2	1	7.2	0.0240	0.0030	0.14
1257+03	95	DC	0.0096	0.82	8.4	1	5.3	0.0780	0.0110	0.16
1334+03	100	DK	0.0183	0.22	7.3	1	4.1	0.1190	0.0040	0.07
1625+09	327	DC	0.0084	0.94	8.6	1	6.9	0.0470	0.0060	0.14
1633+57	258	DC-F	0.0117	0.62	8.1	1	6.3	0.0680	0.0030	0.08
1705+03		DF-DG	0.0104	0.74	8.3	4	7.1	0.0570	0.0050	0.22
1748+70	372	DXP	0.0100	0.78	8.3	1	5.3	0.1660	0.0040	0.07
1818+12		DC	0.0209	0.15	7.0	4	6.4	0.0250	0.0050	0.28
1917+38	375	DC	0.0095	0.83	8.4	1	6.8	0.0850	0.0040	0.08
2011+06	138	DC?	0.0099	0.78	8.3	4	7.6	0.0430	0.0030	0.21
2054-05	202	DC	0.0180	0.22	7.3	1	3.4	0.0640	0.0050	0.10
2207+14	302	DC	0.0090	0.87	8.5	4	8.4	0.0400	0.0040	0.23
2312-02		DF	0.0112	0.67	8.2	4	6.6	0.0380	0.0060	0.26

NOTES TO TABLES 4 AND 5

Parallax sources.—U.S. Naval Observatory catalogs (Riddle 1970; Routly 1972; Harrington *et al.* 1975; Dahn *et al.* 1976), along with the Fifth Catalog, supplied kindly in advance of publication by Dahn (1978). Stars from the Fifth Catalog are 0126+10, 0257+08, 0433+27, 1818+12, 1826−04, 1943+16, 2011+00, and 2111+26. Stars with parallaxes from other sources are listed in Table 3.

Distances from common-proper-motion companions.—Coded by a "CPM" in the parallax error column, these are taken from the following: (1), Eggen and Greenstein 1965; (2), Eggen and Greenstein 1967; (3), Wegner 1973; (4), Greenstein 1976; and (5), Liebert and Margon 1977. For 2253−08, an average of the values of $\pi_{abs} = 0.0251$ (Wegner 1973) and $\pi_{abs} = 0.0219$ (Eggen and Greenstein 1967) was used; the small, old trigonometric parallax of $\pi_{abs} = 0.044 \pm 0.006$ from Jenkins (1963) is probably wrong.

Cluster membership.—The parallax of stars in the Hyades was taken to be $\pi_{abs} = 0.023 \pm 0.0015$, suggested by Klemola *et al.* (1975) as the "standard" value, and corresponding to a distance modulus of $m - M = 3.19$. Distances of the Pleiades and Praesepe are from Allen (1973). Curiously, the Allen (1973) value of 160 pc for the distance of Praesepe, agreeing with Eggen's (1965) value, is different from the value used in Eggen and Greenstein (1965).

Names.—0126+10 = G2-40, W72, LTT 10525 (Wickramasinghe *et al.* 1975); 0257+08 = G76-48 (Hintzen and Strittmatter 1974); 1659−53 = BPM 24601, L268-92 (Wegner 1973), 1705+03 = G139-13 (Hintzen and Strittmatter 1974); 1818+12 = G141-2, L1208-132, LTT 15423 (Hintzen and Strittmatter 1974); 2111+26 = G187-32, LTT 16224 (Hintzen and Strittmatter 1974); 2136+22 = G26-18 (Wickramasinghe *et al.* 1975); 2312−02 = G157-35, LHS 3917 (Hintzen and Strittmatter 1974).

ADDITIONAL NOTES

WD 0232+03 (=EG 20, Feige 24).—This star is an EUV source. The adopted temperature of 60,000 K is in the range specified by Margon *et al.* (1976*a*) determined from the EUV fluxes. The *V*-magnitude is from Holm (1976) and is corrected for the contribution from the red companion.

WD 0642−16 (Sirius B).—The temperature of 29,000 K was adopted as being the best value consistent with the following: $T_{eff} = 32{,}000 \pm 1000$ K, essentially from the hydrogen line profiles on an unblanketed temperature scale (Greenstein, Oke, and Shipman 1971); $T_{eff} = 27{,}000 \pm 6000$ K, unblanketed temperature scale, from *Copernicus* satellite observations of the ultraviolet flux (Savedoff *et al.* 1976); and $T_{eff} \leq 29{,}000$ K (blanketed temperature scale, corresponding to ~30,000 K on an unblanketed temperature scale) from the failure to find EUV radiation in a rocket flight (Cash *et al.* 1978).

WD 0727+48 (=EG 52, G107-70).—This is a double white dwarf system (Strand *et al.* 1976). The observed magnitudes have been reduced by 0.75 in order to account for the duplicity. Strand *et al.* noted no magnitude difference between the two components.

WD 1314+29 (=EG 98, HZ 43).—The temperature is in the range allowed by the EUV observations (Auer and Shipman 1977).

The colors in column (7) of Tables 4–6 are coded as follows: (1), multichannel colors; (2), Johnson $U - V$ (Palomar colors used if available); (3), Strömgren $u - y$; (4), Johnson $B - V$; and (5), Strömgren $b - y$. Stars with no listed parallax or radius error have had distances determined from membership in a cluster or in a cpm system. The masses, column (5), and surface gravities, column (6), were obtained by using the radii and the Hamada-Salpeter (1961) mass-radius relation for carbon configurations. Recent calculations (for example, Lamb and Van Horn 1975) show that this relation provides masses that vary by less than 1%

TABLE 6

RADII OF THE HELIUM-RICH STARS

STAR NO.	EG/GR	SP.CL	R	M	G	COLOR	T	PI	PI ERROR
0007+30	1	DC	0.0111	0.68	8.2	1	8.1	0.0204	CPM 1
0038+55	245	DC	0.0104	0.74	8.3	4	12.7	0.0436	0.0040
0046+05	5	DG	0.0138	0.46	7.8	1	5.5	0.2300	0.0030
0115+15	9	DC	0.0093	0.84	8.4	5	11.1	0.0630	0.0040
0426+58	180	DC	0.0115	0.64	8.1	1	7.1	0.1830	0.0030
0437+13	316	DB	0.0104	0.74	8.3	1	16.4	0.0230	0.0015
0551+12	44	DC	0.0099	0.79	8.3	4	12.7	0.0200	0.0050
0615-59		DB	0.0084	0.94	8.6	4	16.9	0.0400	CPM 3
0625+10	47	DF	0.0094	0.84	8.4	4	9.2	0.0221	CPM 1
0706+37	51	C2	0.0091	0.86	8.5	5	7.9	0.0440	0.0040
0738-17	54	DF	0.0096	0.82	8.4	1	7.8	0.1460	0.0070
0751+57	322	DC	0.0131	0.51	7.9	1	9.7	0.0290	0.0040
0802+38	346	DF P	0.0102	0.76	8.3	1	11.5	0.0240	0.0040
0856+33	182	C2 P	0.0072	1.06	8.8	1	10.0	0.0490	0.0040
0912+53	250	+DCPOL	0.0128	0.54	8.0	1	6.4	0.0990	0.0030
1039+14	72	+DC	0.0153	0.35	7.6	1	5.9	0.0240	0.0030
1115-02	78	DC ?	0.0107	0.72	8.2	4	11.4	0.0270	0.0050
1142-64	82	C2	0.0105	0.73	8.3	4	9.5	0.2030	0.0080
1425+54	295	+DBP	0.0124	0.57	8.0	4	16.3	0.0180	0.0040
1626+36	119	DF PEC	0.0133	0.50	7.9	1	8.5	0.0620	0.0050
1705+03		+DF-DG	0.0123	0.57	8.0	4	6.3	0.0570	0.0050
1900+70	129	DXPOL	0.0080	0.98	8.6	1	13.9	0.0780	0.0040
1917-07	131	+DAVWK	0.0127	0.54	8.0	1	10.6	0.0980	0.0040
2059+31	262	DC	0.0134	0.49	7.9	1	9.6	0.0290	0.0040
2107+42	334	DC	0.0118	0.62	8.1	1	13.5	0.0149	CPM 4
2129+00	145	DB	0.0120	0.60	8.1	1	15.2	0.0230	0.0040
2140+20	148	C2 WK	0.0140	0.44	7.8	1	8.6	0.0760	0.0040
2153-51		C2	0.0071	1.06	8.8	4	9.8	0.0640	0.0120

NOTES TO TABLE 6

Parallax sources.—Same as Table 4.

Common-proper-motion companions.—Notation similar to that of Table 4.

Colors.—(1), flux derived from multichannel color; (4), from Johnson $B - V$ color; (5), from Stromgren $b - y$ color.

INDIVIDUAL STARS

WD 0046+05 (=EG 5, vMa 2).—Temperature from Wegner 1972.

WD 0426+58 (=EG 180, Stein 2051 B).—Temperature from Liebert 1976.

WD 0437+13 (=Gr 316).—Noted by Greenstein 1974 as a possible Hyades member. The reasonable radius obtained here on the presumption that it is a Hyades member supports but does not confirm its membership.

WD 0738−17 (=EG 54, L745-46 A).—Temperature from Wegner 1972.

WD 0853+33 (=EG 182).—Temperature from Grenfell 1972.

WD 1626+36 (=EG 119, Ross 640).—Temperature from Liebert 1977.

from more modern ones. Stars without an EG number are given other names in the Notes to Table 4, along with references to the discovery papers. Stars with temperatures determined in part from space observations are also listed in the notes.

b) *Uncertainties*

This section provides an explanation of how the radius uncertainties in the last column of Tables 4 and 5 are calculated. The basic formula is determined by using standard error propagation techniques and equation (1):

$$\frac{\Delta R}{R} = \left[\left(\frac{\frac{1}{2}\Delta f_v}{f_v}\right)^2 + \left(\frac{\frac{1}{2}\Delta H_v}{H_v}\right)^2 + \left(\frac{\Delta \pi_{\text{abs}}}{\pi_{\text{abs}}}\right)^2\right]^{1/2} . \quad (4)$$

The first and third terms in equation (4) need little comment; for most stars, the third is the dominant source of error in the radius, although in some cases (Sirius B, for example) where the V-magnitude and hence the stellar flux is difficult to determine, the first term can be an important contributor. For any individual star, the second term can be written

$$(\Delta H_v/H_v)^2 = (0.015)^2 + a_i^2[(\Delta c_i)^2 + (0.03)^2] + C(0.07)^2 , \quad (5)$$

where the a_i's are the slope of the color-flux relations

$$\ln H_v = a_i c_i + b_i \quad (6)$$

for individual colors c_i, and $C = 1$ if convection

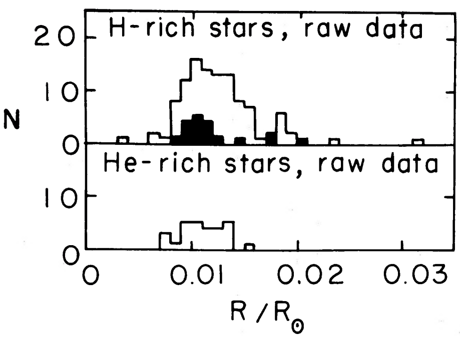

FIG. 2.—Histograms of the radius distributions for the two main classes of H-rich stars. The black histogram in the upper panel refers to the possibly H-rich stars of Table 5; the whole histogram refers to the combined samples of Tables 4 and 5. There is no evidence of a mass or radius difference between the H-rich stars, the possibly H-rich stars, and the He-rich stars.

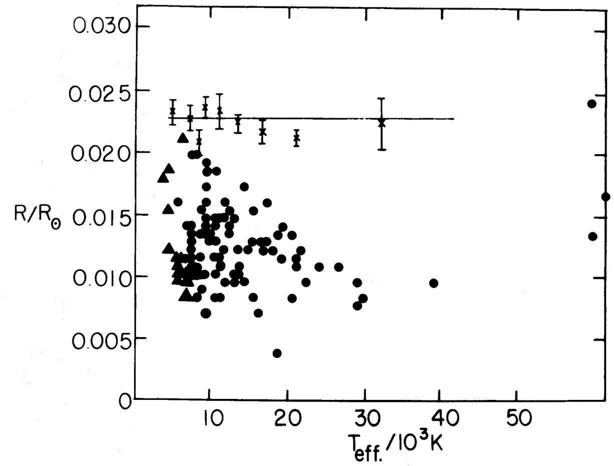

FIG. 3.—The radius of each of the H-rich stars in Table 4 (*dots*) and the possibly H-rich stars in Table 5 (*triangles*) is plotted against the stars effective temperature. Crosses indicate normal points, means of groups of 8–17 stars in various temperature ranges shifted 0.01 $R_\odot$ upward. The line through the crosses refers to the mean radius of 0.0127 $R_\odot$.

seriously affects the determination of H_v (i.e., if $T_{\rm eff} \lesssim 12{,}000$ K) and $C = 0$ otherwise. For $T_{\rm eff} < 16{,}000$ K, $a_i = 2.6$ for multichannel $g - r$ colors, and for $T_{\rm eff} \geq 16{,}000$ K, $a_i = 1.7$ for multichannel $u - v$ colors. The numerical values assigned in equation (5) have been discussed above; the first comes from inherent numerical uncertainties in model atmosphere programs, the second from the choice of model to fit a given star and including both systematic and random errors in the photometry, and the third from the possible uncertainties regarding the proper treatment of convection. If other information such as space observations is available to better determine the choice of model, the second term can be smaller for a given star.

The radius uncertainties listed in Table 4 were computed using equations (4)–(6). Random errors in the differential photometry were taken as 0.03 mag for colors from the multichannel spectrometer (Greenstein 1976) and from Table 1 for stars where broader-band colors were used to determine $u - v$ or $g - r$ colors. For WD 0642−16 (Sirius B) and for stars with $T_{\rm eff} \gtrsim 50{,}000$ K, the errors were estimated from reasonable estimates of temperature uncertainties (1500 K for Sirius B and 10,000 K for the hot stars). The radius errors should be regarded as estimated errors, not standard (or even probable) errors, since calibration uncertainties are estimated by comparing results of two different investigations in most cases. Readers critically interested in the values for particular stars are invited to make their own uncertainty estimates following the precepts outlined by equations (4)–(6).

The mean radius of the 110 hydrogen-rich stars listed in Tables 4 and 5 is 0.0127 $R_\odot$. The uncertainty of this number is determined by the systematic uncertainties entering the determination of individual stellar radii, since with such a large sample random errors become negligible when an average radius is taken. To summarize the systematic errors, they are expressed as $\Delta R/R$:

1% arising from the uncertain absolute calibration of f_v (§ II*b*).

3% arising from uncertain calibration of colors (§ II*b*).

3% arising from uncertainties in the correction from relative to absolute parallax (§ II*c*).

1% or less arising from numerical inaccuracies in the model atmospheres (§ III*c*).

2% arising from uncertainties in the treatment of convection, averaged over the entire sample (§ III*c*).

Assuming that these uncertainties are statistically independent, the uncertainty in the mean radius of this particular sample of hydrogen-rich stars is 5%. The extent to which this sample represents the true distribution of white-dwarf radii will be discussed in § VI in connection with selection effects.

b) *Discussion*

While further data can undoubtedly refine the values of some of the individual radii in Table 4, let us ask what astrophysical insights can be gained from this information. What does it all mean, beyond a compilation of stellar properties?

One star in Table 5 worthy of remark is the double white dwarf G107-70 (WD 0727+48, EG 52), recently discovered to be a binary and discussed by Strand, Dahn, and Liebert (1976). The radius in Table 5 is based on the assumption that each component is equally bright and has the same color, justified by their comment that there is no observable magnitude difference between the two components. Were each star equally massive, the mass of each would be 0.9 $M_\odot$, corresponding to a radius of approximately 0.009 $R_\odot$, far smaller than the radius listed in the table which applies to each star. Given the current data, it is not possible to track down the cause of this discrepancy, but it highlights the need for further investigations of this important system.

An astrophysically interesting question that is of considerable importance to stellar evolution studies is that of the range of observed white-dwarf masses. Studies of binary systems show masses ranging from $0.43 \pm 0.02\ M_\odot$ for 40 Eri B (Heintz 1974; the error is a probable error) to $1.054 \pm 0.026\ M_\odot$ for Sirius B (Gatewood and Gatewood 1978; a 1 σ error). Applying 2 σ variations in appropriate directions, one can conclude that data from binary stars indicate that white-dwarf masses could vary from 0.37 to 1.11 $M_\odot$, corresponding to radii of 0.015 $R_\odot$ and 0.0065 $R_\odot$, respectively. Let us ask whether the data of Table 4 can support a larger range of white-dwarf radii.

Starting with the upper end of the radius distribution, shown graphically in Figure 3, we note that Table 4 contains a reasonable number of stars with radii that are, apparently quite large, corresponding to low masses. However, closer examination shows that these stars have radii that are sufficiently uncertain that a radius of 0.015 $R_\odot$ is within the 2 σ bounds of probability. Five of these apparently large stars are stars with distances determined from membership in a cpm system. For four of these cpm systems (WD 0030+44, WD 0913+44, WD 1510+56, and WD 1743−13) a 20% error in the presumed distance is all that is needed to bring the radius down to 0.015 $R_\odot$; such an error is quite reasonable. The distance of the fifth large star with a cpm distance, WD 0232+03 (Feige 24), is quite uncertain, and it is difficult to estimate just how much of the visual flux is due to the white dwarf and how much is due to the M dwarf companion.

For another group of stars with large radii (WD 0126+10, WD 0133−11, WD 1544−37, and WD 1818+12) the temperatures are determined from $B - V$ color indices, which are poor indicators of the slope of the Paschen continuum, the quantity of fundamental importance (Table 1). Application of the precepts developed in the previous section show that, again, with the application of 2 σ errors in the right direction, the radii of these stars can be brought below 0.015 $R_\odot$.

The one remaining large star is WD 0135−05 (EG 11, L870-2). This star has emission in the line cores (Greenstein *et al.* 1977). The large radius of this star was noted earlier in Paper I. The new Naval Observatory parallax, kindly supplied in advance of publication by Dr. Dahn, alleviates the situation to some extent, reducing the deduced radius from the previously published value of 0.039 to 0.0197 $R_\odot$. Virtually all of the change is attributable to the new parallax; the Hayes-Latham calibration provides a slightly higher temperature, reducing the radius a little. If this error of $\Delta R/R = 0.081$ is treated as a mean error, there is a statistical probability of 1 in 100 that the true radius of L870-2 is 0.015 $R_\odot$. With a sample of 110 stars it is not surprising that one star is so far from the mean. In short, there is no reason to believe from the data in Table 4 that white-dwarf stars exist with radii larger than 0.015 $R_\odot$, corresponding to masses of $\geq 0.37\ M_\odot$.

Again, in the same spirit, let us ask whether the radius list of Table 4 contains any evidence that white dwarfs more massive or smaller than Sirius B exist. The only star with a radius significantly less than 0.0065 $R_\odot$ is the last entry in the table, WD 2359−43, EG 165. This star has a radius that is 6 standard deviations below the radius corresponding to the largest permissible mass for Sirius B and 8 standard deviations below the most probable radius of Sirius B. Unfortunately only a Johnson color is available for this object. The parallax should be reliable; although no recent parallaxes are available, the two old parallaxes of 0.109 ± 0.016 (Lembang) and 0.135 ± 0.007 (Cape) would both have to be a factor of 2 too large if the photometry is correct and if the radius were to be 0.008 $R_\odot$. In view of the limited data that are available it would probably be premature to assert that this star is in fact smaller and therefore more massive than Sirius B, but this star is an interesting object and is worthy of further photometric and astrometric study.

The data of Table 4 reaffirm the conclusion that mass loss is important in the red giant stage, a conclusion first pointed out by Eggen and Greenstein (1965) and Auer and Woolf (1965). Weidemann (1977*a*) provides a fairly extensive discussion. There does seem to be some evidence that the larger parent stars produce larger white-dwarf stars. The star in the Pleiades, descending from a star of mass $\sim 6\ M_\odot$, has a mass of 0.96 $M_\odot$; Sirius B, with a parent mass of 3 $M_\odot$, has a mass of 1.05 $M_\odot$; and EG 29, the most massive Hyades white-dwarf star at 0.92 $M_\odot$, has a parent mass of 4.0 $M_\odot$. The parent masses are given in Weidemann (1977*a*) and Sweeney (1976). However, plots of white-dwarf masses versus parent masses failed to provide evidence of any well-defined relationship between white-dwarf mass and parent mass.

One question that has appeared fairly often in the white-dwarf literature is that of the accuracy of the temperature scale. Trimble and Greenstein (1972) argued that because the hot stars have systematically small radii, the temperature scale of Paper I, substantially the same one as the one used here, needed adjustment. Figure 3 is a plot of the radius of individual stars versus $T_{\rm eff}$. Crosses show normal points, average radii where the averages were taken over groups of stars in various temperature ranges, shifted by 0.01 in $R/R_\odot$. This figure requires some discussion.

The most obvious trend in the figure is a trend in the upper envelope of the radius distribution, increasing at both high and low $T_{\rm eff}$. The normal points, however, show no monotonic or even parabolic trend of mean radius with $T_{\rm eff}$. At low temperatures the upper envelope goes to higher values of R just because there are more stars in the sample. At high temperatures the statistics are small and the radii uncertain; the highest-temperature normal point covers the range 25,000 K $\leq T_{\rm eff} \leq$ 62,000 K.

Within the high-temperature range, though, there is a rather noticeable concentration of small stars with temperatures between 22,000 K and 40,000 K. Is this a cause for concern with the accuracy of the temperature scale? I believe that it is not. Three of these seven stars are in the Hyades and Pleiades clusters, where the small radii of the white-dwarf stars could be the result of the large progenitor masses in these clusters. A fourth small star in this temperature range is Sirius B, where the radius derived here agrees with both the astrometrically determined mass and the gravitational redshift. The average radius of the three remaining stars in this temperature range is 0.0104 $R_{\odot}$, not significantly different from the sample mean for a sample of three stars.

It is worth comparing the mean white-dwarf radius of 0.0127 $R_{\odot}$ with other determinations. Moffett, Barnes, and Evans (1978) have used the empirical Barnes-Evans relation, a relation between stellar flux and $V - R$ derived from lunar occultation observations of main-sequence and high-luminosity stars which Wesselink (1969) originally noticed, to determine a mean radius of 0.013 $R_{\odot}$ for a smaller sample of stars. Weidemann (1977*b*) presented the work of the Kiel group that provided a mean radius of 0.0126 $R_{\odot}$. Agreement between these two investigations and the present one is gratifying, for the three investigations used different methods of calibrating the flux-color relation.

V. THE HELIUM-RICH STARS

a) *Procedure*

It is possible to use the basic method of equation (1) to determine radii for the helium-rich white-dwarf stars, but it is not quite as easy. The grid of model atmospheres for the hotter ($T_{\mathrm{eff}} \geq 10{,}000$ K) DBs is sparse but sufficient, but for the cooler stars of various spectral types (DC, DF, DG), models are few. Models are difficult to calculate for these stars because convection is stronger, the opacity less certain, and modifications to the equation of state potentially more troublesome, particularly at low temperatures. Nevertheless, it is worth trying the method to see how successful it is.

The flux versus color relation for the hotter DB stars, with $T_{\mathrm{eff}} \geq 10{,}000$ K, was derived from the ATLAS models of Paper I. Two model atmospheres by Wegner (1972) were used to extend the relation to the temperature characteristic of van Maanen 2. The model atmosphere at $T_{\mathrm{eff}} = 6000$ K presented in Paper I was computed for a solar metal abundance composition that is unlikely to be representative of most DC stars. The H_v versus color relation for these various models is shown in Figure 1.

It is worth pausing to note one aspect of Figure 1. The H_v versus $g - r$ relation is rather similar for the H-rich and He-rich stars. Following the suggestion provided by this figure, I have investigated Barnes, Evans, and Parsons's (1976) claim that a similar relation between surface brightness and $V - R$ color is a rather good indicator of surface brightness and $V - R$ color is a rather good indicator of surface brightness, showing that the H_v versus $V - R$ relation is the same for giants ($\log g = 2$), main-sequence stars, and white-dwarf stars with both H and He compositions at the level of 20% in H_v for $T_{\mathrm{eff}} > 4000$ K. This point is discussed more extensively in Shipman (1978).

Further, Greenstein (1976) had noted that the multichannel color $g - r$ was a good indicator of absolute visual magnitude M_V for both the H-rich and the He-rich white-dwarf stars. This paper shows that both compositional classes of white-dwarf stars have the same average radius. This being the case, a H-rich and a He-rich white-dwarf star with similar $g - r$ (and hence similar H_v) will have similar absolute visual magnitudes M_v, since

$$M_v = -2.5 \log [(4\pi)^2 R^2 H_v + C] , \qquad (7)$$

where C is a normalizing constant. Figure 1 thus bears out Greenstein's empirical correlation, showing that for white-dwarf stars, $g - r$ is a good predictor of H_v and therefore of M_v, provided that the mean radii of the two groups of stars are similar. The results here show that they are.

How accurately determined is the flux versus color relation for the He-rich stars? At high temperatures, a comparison of the DB models of Wickramasinghe (1972) with the ATLAS models again shows agreement within a percent or so. The ATLAS models differ from those of Bues (1970) in that Bues neglected convection and so the two model sets are not comparable; since convection *is* important in He-rich white-dwarf atmospheres, the ATLAS models should be more accurate.

However, the sensitivity of the H_v-color relation to variations in the presumed physics has not yet been fully explored. A preliminary estimate, based on the changes in deduced temperatures provided by convective and radiative He-rich models, is that uncertainties in the H_v-color relation are roughly 20% at most, corresponding to a 10% uncertainty in the deduced radius. This source of uncertainty, if in fact this large, dominates the others that were discussed in § II. Because of the preliminary nature of this estimate, no errors are provided in tabular form for the He-rich stars. A reasonable guess at the uncertainty in a particular radius is

$$\frac{\Delta R}{R}\,(\text{He-rich star}) = \left[(0.1)^2 + \left(\frac{\Delta\pi_{\mathrm{abs}}}{\pi_{\mathrm{abs}}}\right)^2\right]^{1/2} . \qquad (8)$$

As with the H-rich stars, it is desirable to try to extend the list of He-rich stars beyond those which have published multichannel colors, but it is not possible to use the transformation relations derived earlier for the H-rich stars. The same procedure was followed in calibrating the Johnson and Strömgren systems, in that least-squares fits between multichannel colors (corrected to the Hayes-Latham calibration) and broader band systems defined the calibration. There are 24 stars which are or could be He-rich with measured multichannel and Johnson colors, excluding stars with strong carbon bands, WD 1748+70 (G240-72, Gr 372), a magnetic star with a broad absorption dip in the V band, and van Maanen 2. With these stars, the strong spectral lines that are not found in other He-rich stars would seriously affect the results. These stars define the relationship

$$(B - V)_{\mathrm{Johnson}} = 0.334 + 0.836(g - r) \qquad (9)$$

with a correlation coefficient $r^2 = 0.96$. The rms deviation of an individual $g - r$ color from that predicted by equation (9) is 0.093 mag. The fit is different from that derived for the H-rich stars because of the strong effect of Hγ on the B-band of DA stars. The seven He-rich stars with measured Strömgren and multichannel colors define the relation

$$(b - y)_{\text{Strömgren}} = 0.286 + 0.553(g - r)\,, \quad (10)$$

with $r^2 = 0.964$ and an rms deviation of 0.073 mag.

b) *Results*

Table 6 provides a list of radii for the He-rich stars. The description of the column headings and contents for Table 4 also applies here. DC stars where the colors provided temperatures cooler than 5500 K were excluded, since the $[H_v, (g - r)]$-relation of Figure 1 has not been calibrated for such cool stars. The mean radius for these stars is 0.0111 $R_\odot$, and the distribution of radii (Fig. 2) is not significantly different from the radius distribution of the H-rich stars. In view of the small sample size, the small parallaxes of these stars, and uncertainties in the calibration of the H_v-color relation, this difference cannot be regarded as significant. Thus the suggestion of Bues (1970), reiterated by Weidemann (1977*a*), that the DB stars have lower masses, or larger radii, than the DA stars is not confirmed by these results. In fact, if there is any systematic difference, it is in the other direction. The problem lies with Bues's models which do not include convection. Again, there is no credible evidence for a star with a smaller radius (or larger mass) than Sirius B or a larger radius (smaller mass) than 40 Eri B. The mass and radius ranges of the H-rich and He-rich stars overlap.

A number of cool stars which show neither hydrogen nor helium spectral features are excluded from Table 6 because their derived temperatures on a He-rich temperature scale make them too cool. Many of these stars (WD 0213+42, WD 0552−04, WD 0553+05, WD 0727+48, WD 1257+03, WD 1334+03, WD 1633+57, WD 1748+70, WD 1917+38, and WD 2054−05) are included in Table 5. The radii of these stars turn out to be much more reasonable if one assumes that they are hydrogen-rich; with an extrapolation of the He-rich $[H_v, (g - r)]$-relation, their apparent radii exceed 0.02 $R_\odot$.

VI. SELECTION EFFECTS

In previous sections the systematic uncertainties that limited the accuracy of the mean radius determination were explored, and the conclusion was that the mean radius of the DA stars in Table 4 was within 5% of the true mean radius of that particular sample. Now one asks: To what extent can we believe that that sample is representative of all DA stars? A similar question can be asked in connection with Table 5. Two selection effects are evaluated in this section.

The first is the well-known selection effect that arises when any sample is cut off by some limiting parallax. Here, stars were included only if their parallax was at least 4 times the mean error—a cutoff also implemented in the Naval Observatory's parallax program, where stars with small parallaxes derived after a few plates were taken are dropped from the list. Using the classic prescription of converting from an observed parallax distribution to a true one (see, for example, Mihalas and Routly 1968, eq. [4-19]), one can write

$$\frac{\langle R_{\text{observed}}\rangle}{\langle R_{\text{true}}\rangle} = \frac{\langle 1/\pi_{\text{observed}}\rangle}{\langle 1/\pi_{\text{true}}\rangle}\,, \quad (11)$$

since R is proportional to D or to $1/\pi$ (eq. [1]). An application of this precept to the parallax distribution of Table 4 indicates that this selection effect causes the true radius to be underestimated by 1%, so one can conclude that this selection effect is unimportant for the DA stars. The sample of He-rich stars is too small to allow for a meaningful evaluation of this effect; but since the parallax distributions are similar (as far as small-number-statistics allows one to tell), it seems unlikely that this effect will be important for them.

A more troublesome selection effect is produced by the tendency for the brighter and hence larger stars to be preferentially included in spectroscopic and astrometric observing programs. Suppose that the white-dwarf sample is cut off at some limiting magnitude m, for simplicity. Then stars of radius R will be visible to a distance

$$D = R(4\pi H_v/f_v)^{1/2} \quad (12)$$

by inverting equation (1). Since the number of stars in a sphere of radius D is proportional to D^3, far more large stars will be included in the sample than small stars. To obtain the true distribution of the number of white-dwarf stars of a given radius, one would have to divide by the volume of space being sampled; thus

$$\begin{aligned} N_{\text{true}}(R)dR &= N_{\text{obs}}(R)dR/D^3 \\ &= N_{\text{obs}}(R)dR/R^3(f_v/4\pi H_v)^{3/2}\,. \quad (13) \end{aligned}$$

The true distribution of radii includes far more small stars than the observed distribution, and so the sample mean radius is too large.

Thus, if the white-dwarf sample is magnitude-limited, selection effects will bias the radius estimate. Is the sample magnitude-limited? Figure 4 shows the loci of white-dwarf stars of minimal radii (taken to be 0.0065 $R_\odot$) at various magnitudes. Above and to the right of such a line, all white-dwarf stars, large and small, will be brighter than the magnitude corresponding to such a locus. If one can show that the list of white-dwarf stars embodied in Tables 4 and 5 is complete to some limiting magnitude m_{lim}, then one can conclude that a region in Figure 4 above and to the right of a locus of minimal radius corresponding to that magnitude will contain an unbiased sample of stars. To the left of such a line, the sample will be biased since the large stars are bright enough to be seen and the small stars are too faint to be included.

The shaded line provides a rough indication of the limits of the sample of Tables 4 and 5. One must ask

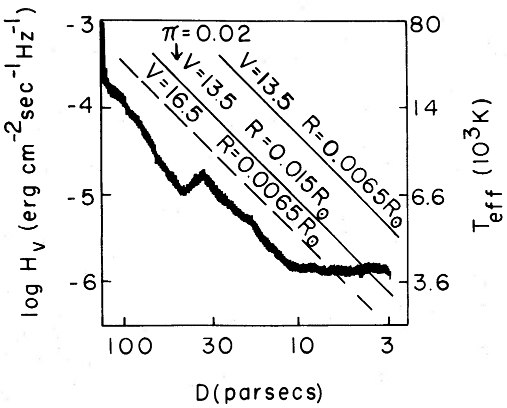

FIG. 4.—A graph which is useful in the evaluation of selection effects. Diagonal lines labeled with apparent m_V and radii show how distant a star of that radius and magnitude would be as a function of stellar surface flux (or, equivalently, of T_{eff}). A sample of white-dwarf stars complete to some limiting magnitude would be confined to the region of this diagram above and to the right of a line corresponding to that limiting magnitude and a minimal white-dwarf radius. The heavy line shows the actual limits of the sample in Tables 4–6. The sample is not complete, and therefore the possibility of selection effects should be considered; see § VI.

what the limiting magnitudes of white-dwarf spectroscopic and astrometric surveys are. White-dwarf stars are generally selected from proper-motion surveys, then investigated spectroscopically, and the spectroscopically interesting ones are placed on parallax programs. The two proper-motion surveys that, between them, cover the entire sky are the Bruce Proper Motion survey in the southern hemisphere (Luyten 1963) and the Lowell survey in the northern hemisphere (see, e.g., Giclas, Burnham, and Thomas 1971). These surveys have limiting magnitudes of 14.5 and 16.5, respectively, and list all stars of proper motion greater than 0″.3 yr^{-1} (Bruce) and greater than 0″.27 yr^{-1} (Lowell). However, Sion and Liebert (1977) showed that the sample of spectroscopically confirmed white-dwarf stars, a larger sample than the sample considered here (of stars with spectra and parallaxes) is incomplete for magnitudes fainter than $V = 13.5$. It is not yet established why this is so. It is possible that a significant number of white-dwarf stars fainter than $V = 13.5$ are too distant to have proper motions that would bring them within the range of the surveys. But even if the sample of spectroscopically identified and astrometrically investigated white-dwarf stars is complete to a limiting magnitude of $V = 16.5$, Figure 4 shows that a substantial proportion of the sample will lie below and to the left of the limiting line that separates a complete sample from an incomplete one, and that that portion of the sample will be affected by selection effects. This section is written not with the presumption that selection effects do exist; the evidence indicates that they may exist and that we should be prepared to correct for them. Corrections will be estimated only for the sample of H-rich stars, since the He-rich stars are insufficiently numerous to make such corrections meaningful.

It is possible to compensate for this selection effect in two ways. The first method of compensation is based on the observation that the selection effect operates at virtually all temperatures where white-dwarf stars are included in the sample. It is only for temperatures hotter than 40,000 K that white-dwarf stars are sufficiently bright that the sample is distance-limited rather than magnitude-limited. We can then presume that the entire sample is magnitude-limited and obtain an average radius by using the true distribution of white-dwarf radii given by equation (13) to calculate the mean white-dwarf radius. Since the observed radius distribution in Figure 2 is broader than the true radius distribution, this averaging process was confined to stars with $0.007 < R/R_\odot < 0.015$. Such a correction provides a true mean radius for the DA stars of 0.0103 $R_\odot$, significantly less than the sample mean radius of 0.0127 $R_\odot$.

In a second attempt to compensate for selection effects, I tried truncating the sample of H-rich stars by including only those stars in a region of Figure 4 where all stars, large or small, would be found. Assuming completeness to $V = 13.5$, only five stars remain in the sample. The mean radius of these five stars is 0.0103 ± 0.0009 $R_\odot$, suggestively but not significantly ($2\frac{1}{2}\,\sigma$) different from the sample mean radius. Extending the sample to $V = 14.5$, where Sion and Liebert (1977) do not claim completeness, produces a mean radius of 0.0118 ± 0.0008 $R_\odot$. An alternative approach is to truncate the sample at $T_{eff} = 10{,}000$ K, because a variety of effects may inhibit the discovery of cool white-dwarf stars, and at a distance of 20 pc where an average white-dwarf star with $v_T = 48$ km s^{-1} (Sion and Liebert 1977) would have a proper motion of 0″.5 yr^{-1} and thus be included in lists of blue stars with large proper motion and perhaps receive spectroscopic and astrometric attention. Such a sample contains 13 stars and has a mean radius of 0.0110 ± 0.006 $R_\odot$, again suggestively but not conclusively different from the mean radius of the sample. Thus all attempts to correct for selection effects by limiting the sample to potentially unbiased groups of stars produce mean radii that are less than the mean radius derived from the raw data. The magnitude of these corrections is similar to that derived from the first method of making them.

I therefore conclude that selection effects might exist and that the mean radius of white-dwarf stars in the Galaxy lies between the figure derived from the entire sample of observed stars, 0.0127 $R_\odot$, and the value of 0.0103 $R_\odot$ derived from the largest reasonable corrections for selection effects. These radius values correspond to masses of 0.55 $M_\odot$ (raw data) and 0.75 $M_\odot$ (corrected value). Perhaps most important is the finding that these corrections are far larger than the uncertainties in white-dwarf radii produced by uncertainties in the calibration of the photometry, astrometry, or the flux-color relation.

An improvement in one's ability to correct for these effects awaits a more complete sample. Greenstein (1976) lists seven stars brighter than $V = 13.5$ with predicted parallaxes larger than 0.025 which do not have parallaxes measured. These stars have parallaxes that are predicted assuming a radius of 0.012 $R_\odot$ and Grenstein's empirical M_V-color relation. Were these stars added to the restricted, complete sample of stars with V brighter than 13.5, the sample would become one of 12 stars and allow for a more definitive test of selection effects. A measurement of the parallaxes of these stars (EG 76, 103, 184; Gr 308, 370, 378, and 399) would be most helpful. Gr 309 and Gr 336—a subdwarf B star and a possible dwarf nova—are excluded from this suggested list even though they are bright and have predicted large parallaxes, because they are not representative white-dwarf stars.

The statement that the selection effect does in fact bias the sample of white-dwarf stars of Tables 4 and 5 presumes that there is a true variation in the radii of white-dwarf stars. This statement is supportable on several grounds. Binary stars show a variation of a factor of 2.5 in mass, from 40 Eri B to Sirius B. The present results indicate a distribution in radii from 0.007 to 0.015 $R_\odot$ that cannot be accounted for by errors in parallaxes or photometry. This conclusion agrees with the results of the Kiel group (Weidemann 1977*b*). Thus, in spite of the statements by Weidemann that the mass and radius range of white-dwarf stars is "narrow" (see, e.g., Weidemann 1977*a*), it is broad enough, a factor of 2 in radius, to produce bias in a magnitude-limited sample of stars.

VII. SUMMARY AND CONCLUSIONS

The principal conclusions of this paper are the following:

1. The mean radius of a sample of 110 hydrogen-rich stars with parallaxes and good photometry is 0.0127 $R_\odot$ (Table 4, Fig. 3, § IV*b*).

2. The mean radius of a sample of 28 He-rich stars is 0.0111 $R_\odot$, which, considering the uncertainties, is not significantly different from that of the DA stars (Table 5, Fig. 3, § V).

3. Mass loss occurs before stars become white-dwarf stars.

4. The lack of a significant difference in radius between the H-rich and He-rich stars discourages explanations of the origin of these two broad classes of white-dwarf stars that are based on mass differences.

5. The most serious limitation on the accuracy of the radius determinations is the possibility that the sample is magnitude-limited and that selection effects cause the preferential inclusion of large stars in the sample. Selection effects are $\sim 20\%$, while uncertainties in the mean radii caused by calibration errors are 5% for the H-rich stars (§§ II–IV) and estimated at 10% for the He-rich stars (§ V).

6. Estimates of the correction for selection effects for the H-rich stars lead to the conclusion that the true mean radius of this group is probably the value of 0.0103 $R_\odot$ (obtained after corrections) and 0.0126 $R_\odot$ given by the raw data. These radii correspond to masses of 0.55 $M_\odot$ (raw data) and 0.75 $M_\odot$ (corrected sample).

Thanks to Conard Dahn and George Gatewood for supplying data prior to publication; Jesse Greenstein, John McGraw, and Gary Wegner for helpful and stimulating discussions; C. Sass for assistance; the referee for useful comments; and the National Science Foundation (AST 76-20723) for financial support.

REFERENCES

Allen, C. W. 1973, *Astrophysical Quantities* (3d ed.; London: Athlone Press).
Auer, L. H., and Shipman, H. L. 1977, *Ap. J.* (*Letters*), **211**, L103.
Auer, L. H., and Woolf, N. 1965, *Ap. J.*, **142**, 182.
Barnes, T. G., Evans, D. S., and Parsons, S. B. 1976, *M.N.R.A.S.*, **174**, 503.
Böhm, K.-H., Carson, T. R., Fontaine, G., and Van Horn, H. M. 1977, *Ap. J.*, **217**, 521.
Breger, M. 1976, *Ap. J. Suppl.*, **32**, 1.
Bues, I. 1970, *Astr. Ap.*, **7**, 91.
Cash, W., Bowyer, S., and Lampton, M. 1978, *Ap. J.* (*Letters*), in press.
Dahn, C. C. 1978, *Pub. U.S. Naval Obs.*, in press.
Dahn, C. C., *et al.* 1976, *Pub. U.S. Naval Obs.*, **24**, No. 3.
Eggen, O. J. 1965, in *Galactic Structure*, ed. A. Blaauw and M. Schmidt (Chicago: University of Chicago Press).
Eggen, O. J., and Greenstein, J. L. 1965, *Ap. J.*, **141**, 83.
———. 1967, *Ap. J.*, **150**, 927.
Fontaine, G., Graboske, H. C., Jr., and van Horn, H. M. 1977, *Ap. J. Suppl.*, **35**, 293.
Gatewood, G., and Gatewood, C. 1978, preprint.
Gatewood, G., and Russell, J. 1974, *A.J.*, **79**, 815.
Giclas, H. L., Burnham, R., Jr., and Thomas, N. G. 1971, *The G-Numbered Stars* (Flagstaff: Lowell Observatory).
Graham, J. 1972, *A.J.*, **77**, 144.
Gray, D. F. 1967, *Ap. J.*, **149**, 317.
Greenstein, J. L. 1974, *Ap. J.* (*Letters*), **189**, L131.
———. 1976, *A.J.*, **81**, 323.
Greenstein, J. L., Oke, J. B., and Shipman, H. L. 1971, *Ap. J.*, **169**, 563.
Greenstein, J. L., Boksenberg, A., Carswell, R. F., and Shortridge, K. 1977, *Ap. J.*, **212**, 186.
Grenfell, T. C. 1972, Ph.D. thesis, University of Washington.
Hamada, T., and Salpeter, E. E. 1961, *Ap. J.*, **134**, 683.
Harrington, R. S., *et al.* 1975, *Pub. U.S. Naval Obs.*, **24**, No. 1.
Harrington, R. S., and Kallarakal, V. V. 1978, *Bull. AAS*, **9**, 599.
Hayes, D. S., and Latham, D. W. 1975, *Ap. J.*, **197**, 593.
Heintz, W. D. 1973, *A.J.*, **78**, 780.
———. 1974, *A.J.*, **79**, 819.
———. 1976, *A.J.*, **81**, 555.
Hintzen, P., and Strittmatter, P. A. 1974, *Ap. J.* (*Letters*), **193**, L112.
Holm, A. V. 1976, *Ap. J.* (*Letters*), **210**, L87.
Jenkins, L. F. 1963, *General Catalogue of Trigonometric Stellar Parallaxes* (New Haven: Yale University Observatory).
Klemola, A. R., Harlan, E. A., McNamara, B., and Wirtanen, C. A. 1975, *A.J.*, **80**, 642.
Kurucz, R. L. 1970, *Smithsonian Ap. Obs. Spec. Rept.*, No. 309.
Lamb, D. Q., and Van Horn, H. M. 1975, *Ap. J.*, **200**, 306.
Liebert, J. W. 1976, *Ap. J.*, **210**, 715.
———. 1977, *Astr. Ap.*, **60**, 101.
Liebert, J., and Margon, B. 1977, *Ap. J.*, **216**, 18.
Lippincott, S. L. 1974, *A.J.*, **78**, 426.
Luyten, W. J. 1963, *Pub. Minnesota Obs.*, **2**, No. 15.
Margon, B., Lampton, M., Bowyer, S., Stern, R., and Paresce, F. 1976*a*, *Ap. J.* (*Letters*), **210**, L79.
Margon, B., Liebert, J., Gatewood, G., Lampton, M., Spinrad, H., and Bowyer, S. 1976*b*, *Ap. J.*, **209**, 525.
Matsushima, S. 1969, *Ap. J.*, **158**, 1137.
Matthews, T. A., and Sandage, A. R. 1963, *Ap. J.*, **138**, 30.
McCook, G. P., and Sion, E. M. 1977, *Villanova Univ. Obs. Contr.*, No. 2.
McGraw, J. T., and Shipman, H. L. 1978, in preparation.

Mihalas, D., and Routly, P. M. 1968, *Galactic Astronomy* (San Francisco: Freeman).
Moffett, T. J., Barnes, T. G., and Evans, D. S. 1978, preprint.
Oke, J. B., and Schild, R. E. 1970, *Ap. J.*, **161**, 1015.
Olson, E. C. 1974, *Pub. A.S.P.*, **86**, 80.
Peterson, D. M., and Greenstein, J. L. 1973, *Astr. Ap.*, **25**, 29.
Riddle, R. K. 1970, *Publ. U.S. Naval Obs.*, **20**, No. 3.
Routly, P. M. 1972, *Pub. U.S. Naval Obs.*, **20**, No. 6.
Savedoff, M. P., Van Horn, H. M., Wesemael, F., Auer, L. H., Snow, T. P., and York, D. G. 1976, *Ap. J.* (*Letters*), **207**, L45.
Shipman, H. L. 1971, *Ap. J.*, **166**, 587.
———. 1972, *Ap. J.*, **177**, 723 (Paper I).
———. 1977*a*, *Ap. J.*, **213**, 138 (Paper II).
———. 1977*b*, in *IAU Symposium No. 80, The H-R Diagram*, ed. E. R. Dyer and A. G. D. Philip (Dordrecht: Reidel).
———. 1978, in *Proceedings of the Fourth Los Alamos-Goddard Conference on Stellar Pulsation Instabilities*, ed. W. Sparks and D. Fischel (NASA), in press.
Shipman, H. L., and Sass, C. 1979, in preparation.
Sion, E. M., and Liebert, J. 1977, *Ap. J.*, **213**, 468.
Stewart, J. C., and Pyatt, K. D., Jr. 1966, *Ap. J.*, **144**, 1203.
Strand, K. Aa. 1977, *A.J.*, **82**, 745.
Strand, K. Aa., Dahn, C. C., and Liebert, J. W. 1976, *Bull. AAS*, **8**, 506.
Strittmatter, P. A., and Wickramasinghe, D. T. 1971, *M.N.R.A.S.*, **152**, 47.
Sweeney, M. A. 1976, *Astr. Ap.*, **49**, 375.
Trimble, V., and Greenstein, J. L. 1972, *Ap. J.*, **177**, 441.
Tug, S., White, N. M., and Lockwood, G. W. 1977, *Astr. Ap.*, **61**, 679.
van Altena, W. F. 1971, *A.J.*, **76**, 932.
———. 1974, *A.J.*, **79**, 826.
van Altena, W. F., and Stone, R. C. 1973, *A.J.*, **78**, 781.
van Altena, W. F., and Vilkki, E. U. 1973, *A.J.*, **78**, 201.
Wagman, N. E. 1967, *A.J.*, **72**, 957.
Wegner, G. V. 1972, *Ap. J.*, **172**, 451.
———. 1973, *M.N.R.A.S.*, **165**, 271.
Wehrse, R. 1975, *Astr. Ap.*, **59**, 169.
Weidemann, V. 1977*a*, *Astr. Ap.*, **59**, 411.
———. 1977*b*, in *IAU Symposium No. 80, The H-R Diagram*, ed. E. R. Dyer and A. G. D. Philip (Dordrecht: Reidel).
Wesselink, A. J. 1969, *M.N.R.A.S.*, **144**, 297.
Wickramasinghe, D. T. 1972, *Mem. R.A.S.*, **76**, 129.
Wickramasinghe, D. T., Hintzen, P., Strittmatter, P. A., and Burbidge, E. M. 1975, *Ap. J.*, **202**, 191.

HARRY L. SHIPMAN: Physics Department, University of Delaware, Newark, DE 19711