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RETURNING RADIATION IN ACCRETION DISKS AROUND BLACK HOLES*

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ABSTRACT

Returning radiation is radiation emitted by an accretion disk which returns to its surface due to gravitational focusing or the shape of the disk. While it is unimportant for thin disks around non-rotating black holes, returning radiation will have significant effects on the structure and appearance of thin disks around rotating holes with $a/M \approx 1$. We calculate the flux of the returning radiation as a function of radius for $a/M = 0.9$, 0.99, 0.9981, and 0.9999, treating the propagation of the radiation by general-relativistic geometric optics and treating the disk as thin. We also consider the effects of returning radiation on the spectrum of an accretion disk seen by a distant observer. The spectrum will be modified by the returning radiation at photon energies greater than about 10 keV and less than about 10 eV. The notch at 10 keV in the "pre-transition" spectrum of Cyg X-1 might be a "signature" of returning radiation.

Subject headings: black holes — X-rays: sources

I. INTRODUCTION

Not all the radiation emitted by an accretion disk around a black hole escapes to large distances: some of it goes down the hole and has an important effect on the way the hole evolves (Thorne 1974); some of it returns to the surface of the disk, where it is reabsorbed or scattered. In this paper, we calculate the flux of the returning radiation in a steady-state, very thin accretion disk.[1] In such a disk the flux of locally generated energy, measured at the surface of the disk in the rest-frame of the accreting gas, is determined by conservation laws and does not depend upon the poorly understood properties of the accreting gas (Page and Thorne 1974).

If the disk is thin, the flux of the returning radiation may be determined solely from the distribution of the locally-generated energy, by using geometric optics to follow photons in the gravitational field of the hole. Thus, the flux of the returning radiation will also be independent of the properties of the accreting gas. In a steady-state disk, the flux of radiation emitted from a point on the disk's surface must equal the sum of the fluxes of the locally generated energy and the returning radiation. Thus, returning radiation will have important effects on the appearance of the disk if the flux of returning radiation is comparable to the locally generated energy.

In § II we calculate the returning flux in the inner disk, where radiation returns due to gravitational focusing. The returning flux exceeds the locally generated flux near the inner edge of the disk, where energy generation in the disk ceases and the gas begins to plunge into the hole. There the returning radiation affects the structure of the disk: the inner radius shifts to larger radii when effects of returning radiation are considered.

In § III we calculate the effects of the returning radiation on the outer disk, where radiation returns due to the flared shape of the disk. Shakura and Sunyaev (1973) observe that, since the disk is flared, the returning flux must exceed that generated locally at sufficiently large radii. The Doppler effect and gravitational focusing concentrate the radiation from the inner disk toward the equatorial plane, which increases the returning flux (Cunningham 1975). The heating of the outer disk by the returning radiation has important effects on its structure at large radii ($r \gtrsim 10^4 M$).

In § IV we consider the effects of returning radiation on the observed spectrum of the disk. Returning radiation which is scattered by the disk will add a high-energy component to the spectrum, and this might explain the notch at 10 keV in the "pre-transition" spectrum of Cygnus X-1. The heating of the outer disk greatly changes the spectrum at energies below about 10 eV.

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[1] The standard, Newtonian model of such disks has been developed by Lynden-Bell and Rees (1972), Pringle and Rees (1972), and Shakura and Sunyaev (1973). Relativistic effects on the disk structure are analyzed by Novikov and Thorne (1973) and Page and Thorne (1974). This model has been successful in explaining certain features of the observed spectrum of Cygnus X-1 (Thorne and Price 1975; Eardley, Lightman, and Shapiro 1975).

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II. RETURNING RADIATION IN THE INNER DISK

Assume that the inner disk lies in the equatorial plane of the black hole, and is thin and (when time-averaged) axisymmetric and in a steady state.[2]

Gas in the disk flows nearly along circular geodesics, spiraling slowly inward and losing energy and angular momentum until it reaches the inner edge of the disk; then it plunges into the hole, losing little more energy or angular momentum. Thus, at the inner edge, energy generation becomes negligible. However, for reasonable values of the accretion rate, the gas inside the inner edge remains optically thick to electron scattering and shines due to returning radiation and its own internally stored energy. In the standard model, the inner edge lies at the radius of marginal stability for circular geodesics; we shall find that it lies at a slightly different radius when returning radiation is considered.

We find the returning flux at a point of the surface of the disk by following the returning photons back to their sources. This process is described in detail in Cunningham (1975) (hereafter, Paper I), where we discuss the appearance of the disk to a distant observer, a closely related problem. As in equation (12) of Paper I, we may express the flux F_o (ergs s^{-1} cm^{-2}) of the returning radiation across the disk's surface, as measured by an observer at rest with the accreting gas at radius r_o, as an integral of the observed intensity I_o (ergs s^{-1} cm^{-1} sr^{-1}) over solid angle Ω. Geometric optics relates I_o to the intensity I_e of the radiation, measured in the rest frame of the gas which emits it. The integration over solid angle may be expressed as an integration over the redshift of the returning radiation and its radius of emission r_e. Thus,

$$F_o(r_o) = \int \cos n_o I_o d\Omega$$
$$= \int \pi^{-1} \cos n_o g^4 l F_e(r_e)[\partial\Omega/\partial(g, r_e)]dg dr_e \,, \tag{1}$$

where g is the ratio of observed to emitted energies of returning photons [$g = (1 + z)^{-1}$ and $I_o = g^4 I_e$, where z is the redshift, defined in the usual manner]; $[\partial\Omega/\partial(g, r_e)]dg dr_e$ is the solid angle observed for a bundle of photons from r_e in dr_e with g in dg; and n_o is the angle between the disk's surface normal $e_{(z)}$ and the direction of the incoming photons, as measured by the observer. The function l relates the emitted intensity I_e to the emitted flux F_e:

$$l(r_e, n_e) = \pi I_e(r_e, n_e)/F_e(r_e) \,, \tag{2}$$

where n_e (a function of g and r_e) is the angle from the surface normal at which the radiation is emitted. Thus, l equals unity for isotropic emission; $l \approx \frac{1}{2} + \frac{3}{4}\cos n_e$ for the limb darkening characteristic of an electron-scattering atmosphere (Chandrasekhar 1950). The emitted flux F_e will be the sum of the locally generated flux F_g and the flux returning from other parts of the disk F_o:

$$F_e = F_o + F_g \,. \tag{3}$$

Since the returning radiation carries angular momentum, it exerts a torque on the gas that absorbs it. This torque must be counterbalanced by viscous stresses in the disk. Hence, returning radiation will influence the radial structure of the disk, including the rate of energy generation. Equation (4), which is derived in the Appendix, relates F_g to F_{g0}, the flux of the locally generated energy in the absence of returning radiation, and $T^{(\varphi)(z)}$, a component of the stress tensor of the returning radiation at the disk's surface:

$$F_g(r_e) = F_{g0}(r_e) - F_{g0}(r_{\rm in}) - (3/2)M^{1/2}r_e^{-7/2}\mathscr{C}^{-1}\int_{r_{\rm in}}^{r_e} \mathscr{D}^{1/2}T^{(\varphi)(z)}r^2 dr \,, \tag{4}$$

where $r_{\rm in}$ is the radius of the inner edge of the disk and $\mathscr{C}$ and $\mathscr{D}$ are functions of r, which approach unity for $r \gg r_{\rm ms}$ (eqs. [A7] of Appendix).

The stress $T^{(\varphi)(z)}$ may be expressed as an integral of the returning radiation over solid angle (see Misner, Thorne, and Wheeler 1973, chap. 22):

$$T^{(\varphi)(z)} = \int \cos\psi_o \cos n_o I_o d\Omega$$
$$= \int \pi^{-1}\cos\psi_o \cos n_o g^4 l[F_g(r_e) + F_o(r_e)][\partial\Omega/\partial(g, r_e)]dg dr_e \,, \tag{5}$$

[2] These are the assumptions of the standard model (Page and Thorne 1974). However, Eardley, Lightman, and Shapiro (1975) find that the innermost regions of the disk may be geometrically thick, in which case this study would need revision. Bardeen and Petterson (1975) consider disks not in the equatorial plane of the hole—but find that the inner regions of such disks ($r < 10^3 M$) are pulled into the equatorial plane by "dragging of inertial frames" and viscosity. The effects of the returning radiation in the outer regions of such disks would be substantially different from those described here, however.

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where ψ_o is the angle between the direction of the incoming photons and the direction of motion $e_{(\varphi)}$, as measured by the observer at rest in the accreting gas. The vectors $e_{(\varphi)}$ and $e_{(z)}$ are two of the basis vectors of the observer's rest frame; the full orthonormal tetrad is given by Novikov and Thorne (1973).

Henceforth, it will be convenient to use dimensionless quantities representing the radius, the fluxes, and the stress:

$$r^* = r/M\,, \tag{6a}$$

$$y = (r/r_+)^{-2}\,, \tag{6b}$$

$$f = F/F^*\,, \tag{6c}$$

$$s = T^{(\varphi)(z)}/F^*\,, \tag{6d}$$

$$F^* = (3/8\pi)\dot{M}_0 M r^{-3} = (0.6 \times 10^{26}\ \mathrm{ergs\ s^{-1}\ cm^{-2}})\dot{M}_0{}^* M^{*-2} r^{*-3}\,, \tag{6e}$$

$$\dot{M}_0{}^* = \dot{M}_0/(10^{17}\ \mathrm{g\ s^{-1}}) \tag{6f}$$

$$M^* = M/3M_\odot\,, \tag{6g}$$

where r_+ is the radius of the event horizon and $\dot{M}_0$ is the accretion rate. We shall find that f_o and f_g are slowly varying functions of radius and are of order unity in the inner disk. At large radii f_g equals unity: $F_g = F^*$.

We define T_f and T_s, the flux and stress transfer functions:

$$T_f(r_o, r_e) = (2\pi)^{-1} r_+{}^{-2} r_o{}^3 \int \cos n_o g^4 l[\partial\Omega/\partial(g, r_e)]dg\,, \tag{7a}$$

$$T_s(r_o, r_e) = (2\pi)^{-1} r_+{}^{-2} r_o{}^3 \int \cos \psi_o \cos n_o g^4 l[\partial\Omega/\partial(g, r_e)]dg\,. \tag{7b}$$

The integrations over g include all radiation emitted at r_e and returning at r_o; and ψ_o and n_o are functions of r_o r_e, and g determined by the geometric optics of the rays connecting r_e with r_o. The method used to evaluate these transfer functions is described in the appendix to Paper I.

In terms of the dimensionless parameters and the transfer functions, equations (1), (4), and (5) become

$$f_o(y_o) = \int_0^1 T_f(y_o, y_e)[f_o(y_e) + f_g(y_e)]dy_e\,, \tag{8a}$$

$$f_g(y_o) = f_{g0}(y_o) - f_{g0}(y_{\mathrm{in}}) - \frac{3}{4}(r_+/M)^{-1/2} y_o{}^{1/4}\mathscr{C}^{-1}\int_{y_o}^{y_{\mathrm{in}}} \mathscr{D}^{1/2} s y^{-1} dy\,, \tag{8b}$$

$$s(y_o) = \int_0^1 T_s(y_o, y_e)[f_o(y_e) + f_g(y_e)]dy_e\,. \tag{8c}$$

The generated flux in the absence of returning radiation f_{g0} is easily calculated. In fact, f_{g0} may be expressed analytically (Page and Thorne 1974). Thus, once the transfer functions are known, the equations (8) determine f_o and f_g implicitly. We convert them to a system of simultaneous linear equations:

$$\boldsymbol{f}_o = \boldsymbol{T}_f\cdot(\boldsymbol{f}_g + \boldsymbol{f}_o)\,, \tag{9a}$$

$$\boldsymbol{f}_g = \boldsymbol{f}_{g0} - \boldsymbol{A}\cdot\boldsymbol{s}\,, \tag{9b}$$

$$\boldsymbol{s} = \boldsymbol{T}_s\cdot(\boldsymbol{f}_g + \boldsymbol{f}_o)\,. \tag{9c}$$

Here, $\boldsymbol{f}_o$, $\boldsymbol{f}_g$, $\boldsymbol{f}_{g0}$, and $\boldsymbol{s}$ are n-tuples of the values of f_o, f_g, f_{g0}, and s for n values of y; $\boldsymbol{T}_f$ and $\boldsymbol{T}_s$ are $n \times n$ matrices of values of T_f and T_s; and $\boldsymbol{A}$ is an $n \times n$ matrix representing the integral over $\boldsymbol{s}$ in equation (8b). We can write the formal solution to equations (9) immediately:

$$\boldsymbol{f}_g = \{\mathbf{I} + \boldsymbol{A}\cdot\boldsymbol{T}_s\cdot[\mathbf{I} + (\mathbf{I} - \boldsymbol{T}_f)^{-1}\cdot\boldsymbol{T}_f]\}^{-1}\cdot\boldsymbol{f}_{g0}\,, \tag{10a}$$

$$\boldsymbol{f}_o = (\mathbf{I} - \boldsymbol{T}_f)^{-1}\cdot\boldsymbol{T}_f\cdot\boldsymbol{f}_g\,, \tag{10b}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

Numerical values for the transfer functions are given in Tables 1–4 for $a/M = 0.9$, 0.99, 0.9981, 0.9999, assuming isotropic emission. (We do not include $a/M = 0$ because returning radiation is unimportant in the case of a nonrotating hole.) These transfer functions were computed [i.e., $\partial\Omega/\partial(g, r_e)$, n_o, and ψ_o were evaluated as

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TABLE 1

THE TRANSFER FUNCTIONS $T_f(y_o, y_e)$ AND $T_s(y_o, y_e)$ FOR $a/M = 0.9$

y_e		0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
r_e/M		6.42	3.71	2.87	2.43	2.14	1.94	1.78	1.66	1.56	1.47
y_o	r_o/M					T_f					
0.05	6.42	0.333	0.322	0.321	0.337	0.378	0.405	0.295	0.095	0.007	0.000
0.15	3.71	0.222	0.226	0.228	0.245	0.289	0.351	0.325	0.134	0.012	0.000
0.25	2.87	0.194	0.197	0.193	0.201	0.239	0.304	0.320	0.154	0.015	0.000
0.35	2.43	0.188	0.186	0.178	0.180	0.206	0.267	0.304	0.169	0.019	0.000
0.45	2.14	0.191	0.187	0.172	0.168	0.189	0.241	0.289	0.182	0.022	0.000
0.55	1.94	0.198	0.193	0.173	0.165	0.179	0.228	0.280	0.197	0.026	0.000
0.65	1.78	0.206	0.202	0.178	0.166	0.175	0.195	0.275	0.211	0.031	0.000
0.75	1.66	0.214	0.211	0.184	0.169	0.173	0.210	0.270	0.224	0.036	0.000
0.85	1.56	0.223	0.221	0.192	0.173	0.174	0.207	0.267	0.237	0.041	0.000
0.95	1.47	0.231	0.232	0.199	0.178	0.176	0.206	0.265	0.248	0.047	0.000
y_o	r_o/M					T_s					
0.05	6.42	0.099	0.048	0.030	0.022	0.020	0.019	0.015	0.006	0.001	0.000
0.15	3.71	0.097	0.050	0.024	0.009	−0.002	−0.010	−0.115	−0.004	0.000	0.000
0.25	2.87	0.102	0.059	0.029	0.010	−0.007	−0.024	−0.032	−0.015	−0.001	0.000
0.35	2.43	0.100	0.071	0.038	0.015	−0.004	−0.028	−0.045	−0.026	−0.002	0.000
0.45	2.14	0.122	0.086	0.050	0.024	0.001	−0.027	−0.054	−0.038	−0.004	0.000
0.55	1.94	0.137	0.102	0.063	0.034	0.008	−0.024	−0.061	−0.052	−0.006	0.000
0.65	1.78	0.154	0.122	0.079	0.046	0.017	−0.039	−0.068	−0.069	−0.010	0.000
0.75	1.66	0.173	0.142	0.096	0.060	0.027	−0.014	−0.074	−0.087	−0.014	0.000
0.85	1.56	0.192	0.164	0.114	0.074	0.038	−0.008	−0.079	−0.107	−0.020	0.000
0.95	1.47	0.205	0.181	0.129	0.087	0.048	0.000	−0.084	−0.124	−0.026	0.000

functions of r_e, g] assuming the following world lines for the gas: (i) at $r > r_{\rm ms}$, circular geodesics; (ii) at $r < r_{\rm ms}$, inward spiraling geodesics which were circular at $r_{\rm ms}$.

The values of the transfer functions assuming the limb darkening of an electron-scattering atmosphere do not differ from the isotropic values by more than a few percent: the effects of limb darkening in this problem are negligible. So, we shall not present separate results for limb darkening.

TABLE 2

THE TRANSFER FUNCTIONS $T_f(y_o, y_e)$ AND $T_s(y_o, y_e)$ FOR $a/M = 0.99$

y_e		0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
r_e/M		5.10	2.95	2.28	1.93	1.70	1.54	1.42	1.32	1.24	1.17
y_o	r_o/M					T_f					
0.05	5.10	0.441	0.411	0.413	0.432	0.458	0.479	0.470	0.314	0.064	0.001
0.15	2.95	0.315	0.295	0.297	0.322	0.362	0.416	0.478	0.450	0.143	0.001
0.25	2.28	0.292	0.262	0.254	0.268	0.303	0.358	0.442	0.509	0.234	0.003
0.35	1.93	0.297	0.257	0.236	0.241	0.267	0.313	0.396	0.518	0.327	0.005
0.45	1.70	0.321	0.270	0.236	0.230	0.243	0.275	0.352	0.496	0.415	0.008
0.55	1.54	0.361	0.298	0.249	0.230	0.234	0.256	0.314	0.461	0.487	0.013
0.65	1.42	0.423	0.345	0.279	0.245	0.234	0.246	0.294	0.418	0.530	0.020
0.75	1.32	0.494	0.404	0.319	0.272	0.251	0.250	0.279	0.394	0.569	0.031
0.85	1.24	0.578	0.476	0.372	0.311	0.277	0.265	0.282	0.234	0.600	0.050
0.95	1.17	0.683	0.567	0.440	0.361	0.316	0.291	0.294	0.375	0.613	0.078
y_o	r_o/M					T_s					
0.05	5.10	0.130	0.056	0.037	0.034	0.036	0.041	0.044	0.032	0.007	0.000
0.15	2.95	0.138	0.059	0.027	0.014	0.009	0.010	0.014	0.018	0.008	0.000
0.25	2.28	0.149	0.071	0.032	0.012	0.001	−0.005	−0.007	−0.005	0.002	0.000
0.35	1.93	0.163	0.087	0.042	0.017	0.001	−0.011	−0.020	−0.029	−0.013	0.000
0.45	1.70	0.183	0.107	0.057	0.027	0.006	−0.009	−0.025	−0.046	−0.036	0.000
0.55	1 54	0.212	0.134	0.078	0.042	0.017	−0.003	−0.025	−0.059	−0.066	−0.001
0.65	1.42	0.256	0.173	0.108	0.066	0.035	0.011	−0.018	−0.061	−0.100	−0.003
0.75	1.32	0.323	0.232	0.152	0.101	0.062	0.032	0.000	−0.058	−0.138	−0.007
0.85	1.24	0.429	0.323	0.223	0.157	0.107	0.066	0.024	−0.042	−0.188	−0.017
0.95	1.17	0.601	0.471	0.338	0.247	0.181	0.125	0.069	−0.024	−0.244	−0.040

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TABLE 3

THE TRANSFER FUNCTIONS $T_f(y_o, y_e)$ AND $T_s(y_o, y_e)$ FOR $a/M = 0.9981$

y_e..........		0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
r_e/M........		4.75	2.74	2.12	1.80	1.58	1.43	1.32	1.23	1.15	1.09
y_o	r_o/M					T_f					
0.05	4.75	0.482	0.450	0.459	0.486	0.513	0.520	0.472	0.331	0.088	0.001
0.15	2.74	0.351	0.324	0.331	0.366	0.418	0.481	0.534	0.509	0.226	0.003
0.25	2.12	0.334	0.291	0.283	0.303	0.351	0.419	0.512	0.589	0.398	0.008
0.35	1.80	0.351	0.290	0.270	0.273	0.303	0.365	0.466	0.603	0.572	0.021
0.45	1.58	0.388	0.309	0.268	0.263	0.280	0.323	0.412	0.575	0.702	0.044
0.55	1.43	0.450	0.351	0.288	0.265	0.269	0.303	0.374	0.521	0.787	0.088
0.65	1.32	0.548	0.424	0.335	0.296	0.280	0.296	0.344	0.469	0.795	0.170
0.75	1.23	0.715	0.551	0.423	0.352	0.317	0.310	0.336	0.440	0.750	0.303
0.85	1.15	0.957	0.743	0.564	0.457	0.394	0.363	0.363	0.414	0.683	0.582
0.95	1.09	1.228	0.977	0.758	0.618	0.524	0.466	0.440	0.456	0.658	1.056
y_o	r_o/M					T_s					
0.05	4.75	0.140	0.056	0.039	0.038	0.042	0.048	0.049	0.039	0.011	0.000
0.15	2.74	0.152	0.059	0.026	0.013	0.011	0.015	0.023	0.031	0.018	0.000
0.25	2.12	0.168	0.074	0.031	0.010	0.000	−0.003	−0.001	0.010	0.016	0.001
0.35	1.80	1.189	0.094	0.044	0.015	−0.001	−0.011	−0.016	−0.014	0.000	0.001
0.45	1.58	0.215	0.118	0.060	0.026	0.004	−0.011	−0.024	−0.033	−0.027	0.000
0.55	1.43	0.252	0.151	0.085	0.044	0.016	−0.005	−0.024	−0.043	−0.061	−0.003
0.65	1.32	0.308	0.198	0.120	0.071	0.037	0.009	−0.016	−0.045	−0.094	−0.014
0.75	1.23	0.405	0.277	0.184	0.118	0.074	0.038	0.006	−0.057	−0.018	−0.042
0.85	1.15	0.593	0.429	0.295	0.209	0.147	0.098	0.053	0.003	−0.114	−0.134
0.95	1.09	0.957	0.734	0.543	0.413	0.318	0.243	0.173	0.096	−0.072	−0.441

Both of the transfer functions are surprisingly well behaved. Their slight variations with radius are easily explained. Consider first the flux transfer function. For a given observation radius r_o, this function is large for $r_e \gg r_o$, since then the returning radiation will be gravitationally blueshifted; it is large also for $r_e \ll r_o$, since then the returning radiation will be gravitationally focused, and a component of it will be blueshifted due to the Doppler effect of the relativistic orbital velocity of the emitting gas. This Doppler blueshift vanishes for emitting gas near the horizon, since there the gas will be flowing away from the observer and into the hole. Thus,

TABLE 4

THE TRANSFER FUNCTIONS $T_f(y_o, y_e)$ AND $T_s(y_o, y_e)$ FOR $a/M = 0.9999$

y_e..........		0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
r_e/M........		4.54	2.62	2.03	1.71	1.51	1.37	1.26	1.17	1.10	1.04
y_o	r_o/M					T_f					
0.05	4.54	0.510	0.479	0.496	0.531	0.559	0.550	0.450	0.247	0.061	0.001
0.15	2.62	0.378	0.344	0.358	0.403	0.468	0.543	0.574	0.460	0.187	0.004
0.25	2.03	0.366	0.309	0.304	0.336	0.393	0.484	0.589	0.618	0.395	0.015
0.35	1.71	0.392	0.313	0.288	0.299	0.340	0.424	0.555	0.690	0.625	0.038
0.45	1.51	0.445	0.342	0.292	0.285	0.313	0.377	0.500	0.696	0.785	0.112
0.55	1.37	0.531	0.399	0.322	0.295	0.302	0.347	0.443	0.651	0.957	0.266
0.65	1.26	0.673	0.502	0.387	0.337	0.322	0.348	0.405	0.564	0.974	0.900
0.75	1.17	0.927	0.690	0.520	0.428	0.381	0.371	0.405	0.524	0.901	1.473
0.85	1.10	1.372	1.047	0.800	0.648	0.554	0.499	0.486	0.536	0.786	2.286
0.95	1.04	2.791	2.211	1.716	1.386	1.162	1.009	0.899	0.846	0.914	2.295
y_o	r_o/M					T_s					
0.05	4.54	0.146	0.057	0.040	0.041	0.047	0.053	0.050	0.031	0.009	0.000
0.15	2.62	0.162	0.059	0.024	0.012	0.011	0.018	0.029	0.034	0.018	0.001
0.25	2.03	0.183	0.076	0.030	0.008	−0.003	−0.004	0.003	0.019	0.023	0.001
0.35	1.71	0.209	0.099	0.043	0.013	−0.004	−0.015	−0.017	−0.006	0.016	0.003
0.45	1.51	0.243	0.129	0.064	0.026	0.002	−0.016	−0.029	−0.030	−0.005	0.005
0.55	1.37	0.291	0.169	0.094	0.048	0.016	−0.009	−0.029	−0.049	−0.040	0.004
0.65	1.26	0.366	0.229	0.139	0.082	0.041	0.007	−0.022	−0.050	−0.075	−0.018
0.75	1.17	0.497	0.332	0.215	0.141	0.088	0.045	0.004	−0.040	−0.095	−0.083
0.85	1.10	0.722	0.519	0.364	0.264	0.190	0.130	0.074	0.013	−0.076	−0.268
0.95	1.04	1.650	1.279	0.966	0.752	0.600	0.479	0.373	0.265	0.125	−0.449

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the flux transfer function is zero for emitting gas at the horizon. For r_o and $r_e \gg r_+$, $T_f \propto r_e^{1/2} r_o^{5/2}(r_e + r_o)^{-5/2}$. Thus, T_f diverges if r_o and $r_e \to \infty$, although the divergence is integrable in equation (8a). We are not interested in using this formalism for very large r_o, however.

Increasing a/M decreases the radius of marginal stability. As a result, gas near the horizon tends to move along circular, rather than radial, trajectories if a/M is large. The radiation this gas emits to larger radii and the radiation it receives from larger radii will both be less Doppler redshifted than if the gas were flowing radially. Thus, increasing a/M increases the flux transfer function for small r_o and r_e.

The stress transfer function is the product of the flux transfer function and an average value of cos ψ_o for the returning radiation (ψ_o is the angle from the $\boldsymbol{e}_{(\varphi)}$ direction, the direction of motion in the rest frame). Thus, the stress transfer function is positive if most of the returning radiation is observed to come from the direction of motion. This is generally so, due to the large orbital velocity of the accreting gas. However, if both r_o and r_e are small, the transfer function may be slightly negative: the net positive angular momentum carried by the returning radiation balances the aberration caused by the observer's motion.

Numerical values for f_o, f_g, and f_{g0} are compared in Table 5 for the four values of a/M. These results, like Tables 1–4 for T_f and T_s, were computed assuming the inner edge of the disk lies at the radius of marginal stability $r_{\rm ms}$, a slightly erroneous assumption, which leads to fictitious, negative values of f_g very near the radius of marginal stability. Note that f_o is roughly constant with radius, while f_g drops to zero at the inner edge of the disk. By assumption we set $f_g = 0$ inside $r_{\rm ms}$; this means that we ignore not only viscous heating in the inward spiraling gas, but also emission produced by the gas's internal store of thermal energy.

The relative importance of returning radiation in the inner disk is very sensitive to the value of a/M: Returning radiation is negligible for $a/M \leq 0.9$; it exceeds or is comparable to that generated locally for $a/M \geq 0.999$. Thus, the appearance of disks around rapidly rotating holes will be rather different from that of disks around slowly rotating holes, due to returning radiation.

The difference between the actual locally generated flux f_g and that in the absence of returning radiation f_{g0} is only about 10 percent over most of the inner disk for $a/M = 0.9999$, much less for smaller values of a/M. Of course, very near the inner edge of the disk, there is a considerable fractional difference between f_g and f_{g0}. Here, returning radiation modifies the radial and vertical structure of the disk.

The angular momentum transported by the returning radiation slightly increases the inner radius $r_{\rm in}$ of the disk. The requirement that the viscous stress always *oppose* the shearing motion of the gas leads to an approximate

TABLE 5

THE RETURNING FLUX f_o, THE LOCALLY GENERATED FLUX f_g, AND THE LOCALLY GENERATED FLUX WITH NO RETURNING RADIATION f_{g0}

PARAMETER	y: 0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
					$a/M = 0.9$, $r_{\rm in} = 2.33\ M$, $r_{\rm ms} = 2.32\ M$					
r/M	6.42	3.71	2.87	2.43	2.14	1.94	1.78	1.66	1.56	1.47
f_o	0.019	0.014	0.012	0.011	0.012	0.012	0.012	0.013	0.013	0.013
f_g	0.297	0.139	0.050	0.003	0.000	0.000	0.000	0.000	0.000	0.000
f_{g0}	0.301	0.141	0.052	0.004	0.000	0.000	0.000	0.000	0.000	0.000
					$a/M = 0.99$, $r_{\rm in} = 1.46\ M$, $r_{\rm ms} = 1.45\ M$					
r/M	5.10	2.95	2.28	1.93	1.70	1.54	1.42	1.32	1.24	1.17
f_o	0.060	0.046	0.042	0.042	0.043	0.045	0.050	0.057	0.064	0.076
f_g	0.369	0.257	0.186	0.122	0.062	0.012	0.000	0.000	0.000	0.000
f_{g0}	0.379	0.266	0.194	0.130	0.069	0.016	0.000	0.000	0.000	0.000
					$a/M = 0.9981$, $r_{\rm in} = 1.24\ M$, $r_{\rm ms} = 1.23\ M$					
r/M	4.75	2.74	2.12	1.80	1.58	1.43	1.32	1.23	1.15	1.09
f_o	0.100	0.083	0.079	0.079	0.082	0.087	0.098	0.118	0.152	0.199
f_g	0.383	0.294	0.241	0.194	0.144	0.088	0.028	0.000	0.000	0.000
f_{g0}	0.398	0.308	0.256	0.210	0.162	0.105	0.041	0.000	0.000	0.000
					$a/M = 0.9999$, $r_{\rm in} = 1.09\ M$, $r_{\rm ms} = 1.08\ M$					
r/M	4.54	2.62	2.03	1.71	1.51	1.37	1.26	1.17	1.10	1.04
f_o	0.171	0.162	0.167	0.176	0.189	0.212	0.274	0.350	0.486	0.771
f_g	0.383	0.309	0.272	0.241	0.213	0.180	0.138	0.075	−0.012*	0.000
f_{g0}	0.406	0.330	0.294	0.268	0.245	0.220	0.185	0.129	0.026	0.000

* This negative value for f_g is fictitious; it arises from the erroneous assumption that $r_{\rm in} = r_{\rm ms}$ made in the calculation.

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relation between the stress s of the returning radiation and the value of the inner radius (see Appendix):

$$r_{\rm in}^* \approx r_{\rm ms}^* + (3/2)r_{\rm ms}^{*1/2}\mathscr{D}^{1/2}s\,, \tag{11}$$

where $r_{\rm ms}$ is the radius of marginal stability (the inner radius in the absence of returning radiation) and $\mathscr{D}$ is a function of $r_{\rm ms}$, defined in equation (A7). Having obtained approximate solutions for f_o and f_g, we evaluate s at $r_{\rm ms}$ and solve equation (11) for $r_{\rm in}$. Numerical values of the inner radius and the radius of marginal stability are compared in Table 5. Note that the outward shift of the inner edge of the disk is always extremely slight. Even for $a/M \approx 1$, the stress s is not large enough to overwhelm the $\mathscr{D}^{1/2}$ term in equation (11) [$\mathscr{D}(r_{\rm ms}) \rightarrow 0$ as $a/M \rightarrow 1$].

Thus, even though the flux of the returning radiation is significant compared to the locally generated energy for $a/M \approx 1$, its effects on the disk structure and the rate of energy generation are minor everywhere except very near the inner edge of the disk.

III. RETURNING RADIATION IN THE OUTER DISK

At very large radii, $r_o \gtrsim 1000\,M$, the flux of returning radiation will be larger than predicted by Table 5, since the disk has a flared shape, and it is the shape of the disk, rather than gravitational focusing, that causes radiation to return to the disk's surface. One might expect that, since the Rosseland mean free-free opacity is much greater than the electron-scattering opacity in this region, all the returning radiation would be absorbed and would heat the disk. However, because of the energy-dependence of the opacity, a large fraction of the returning radiation is scattered, rather than absorbed. Even so, the energy absorbed from the returning radiation has important effects on the disk structure.

We may express the flux of the returning radiation in the outer region as

$$F_o = (\cos n_o)L_0/4\pi r_o^2\,, \tag{12}$$

where n_o is the angle of the returning radiation with the surface normal and L_o is the "effective luminosity" of the inner disk, i.e., the luminosity a local observer would assign to it, assuming that it radiates isotropically. (Eq. [12] may be regarded as a definition of the effective luminosity.) Paper I describes how to evaluate L_o given the flux of the emitted radiation in the inner disk. Of course, the calculations of Paper I neglect returning radiation: the effective luminosities given there are underestimates for disks around rapidly rotating holes.

In Table 6 we give the effective luminosity in the plane of the disk for five values of a/M, assuming either isotropic emission or the limb darkening of a scattering atmosphere, and either including or ignoring returning radiation in the inner disk. The values given are for the ratio

$$L^* = L_o/\dot{M}_o\,, \tag{13}$$

which measures the efficiency of the disk in converting rest mass energy to returning radiation. Our claims that returning radiation in the inner disk is negligible for $a/M \lesssim 0.9$ and that limb darkening has little effect on the returning radiation are supported by Table 6.

If the disk remains thin and the returning radiation comes from the inward radial direction, then the angle n_o of the returning radiation is determined by the half-thickness h of the disk:

$$\cos n_o = dh/dr_o - h/r_o\,. \tag{14}$$

Thus, the amount of returning flux will depend upon the vertical structure of the disk, which in turn will be influenced by the returning radiation.

TABLE 6

THE RATIO L^* OF THE EFFECTIVE LUMINOSITY IN THE PLANE OF THE DISK TO THE ACCRETION RATE

Returning Radiation in the Inner Disk..........	Included	Included	Ignored	Ignored
Limb Darkening............................	Isotropic	Electron Scattering	Isotropic	Electron Scattering
a/M:				
0.0000..................................	0.0049	0.0038	0.0046	0.0036
0.9000..................................	0.0365	0.0321	0.0307	0.0273
0.9900..................................	0.1361	0.1246	0.0998	0.0922
0.9981..................................	0.2421	0.2222	0.1642	0.1518
0.9999..................................	0.4653	0.4315	0.2578	0.2427

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The returning radiation heats the disk. A returning photon of energy $E_\gamma = h\nu$ can interact with the gas in the disk (temperature T, density ρ) in several ways: by *Compton scattering* off an electron, for which the opacity and mean energy loss per scattering are

$$\kappa_{\rm es} = (0.40\ {\rm g^{-1}\ cm^2})\,, \qquad \Delta E_\gamma = -E_\gamma(E_\gamma/m_e c^2)\,; \tag{15a}$$

by *photoionization* of intermediate-mass elements, with an opacity

$$\kappa_{\rm pi} = (4 \times 10^2\ {\rm g^{-1}\ cm^2})(E_\gamma/{\rm keV})^{-7/2} \tag{15b}$$

(assuming Brown-Gould 1970 abundances, temperatures $T \leqslant 10^7$ K, and photon energies 10 keV $\leqslant E_\gamma \leqslant$ 200 keV); and by *ion-electron free-free* absorption, with an opacity

$$\kappa_{\rm ff} = (3\ {\rm g^{-1}\ cm^2})(E_\gamma/{\rm keV})^{-3}(\rho/{\rm g\ cm^{-3}})(kT/{\rm keV})^{-1/2} \tag{15c}$$

(assuming $E_\gamma \gg kT \geqslant 10$ eV). For conditions appropriate to binary X-ray sources (e.g., Cyg X-1, where photons from the inner disk have 20 keV $\leqslant E_\gamma \leqslant$ 200 keV, while the outer part of the disk has $\rho \ll 1\ {\rm g\ cm^{-3}}$, $kT \leqslant 1$ keV) Compton scattering and photoionization dominate over free-free absorption.

To find the energy deposited in the disk by the returning radiation, we find an approximate solution to the radiative transfer equation for the returning radiation:

$$(\cos\theta)dI/d\tau = -I + S\,, \tag{16}$$

where I is the intensity (ergs cm^{-2} s^{-1} sr^{-1}) of a beam of radiation at an angle θ with the surface normal, τ is the optical depth in the disk atmosphere, and S is the source function which describes the rate at which energy is injected into the beam. In this problem, the only process which creates photons with energies like those of the returning photons is Compton scattering, for which the source function is

$$S = (1 - \beta)J\,, \tag{17a}$$

$$\beta = [\kappa_{\rm pi} + (E_\gamma/m_e c^2)\kappa_{\rm es}]/(\kappa_{\rm pi} + \kappa_{\rm es})\,, \tag{17b}$$

assuming isotropic, slightly inelastic scattering, $E_\gamma/m_e c^2 \ll 1$; J is the average of I over solid angle. We solve the transfer equation by the two-stream method (Mihalas 1970), and find that

$$J \approx (F_o/4\pi)[2(1-\beta)(1+\beta^{1/2})^{-1}\exp(-2\beta^{1/2}\tau) + (\cos n_o)^{-1}\exp(-\tau/\cos n_o)]\,, \tag{18}$$

assuming that the disk is optically thick to the returning radiation and that $\cos n_o \ll 1$. The energy dE deposited in a volume element dV in time dt is

$$dE/dVdt = 4\pi\rho(\kappa_{\rm pi} + \kappa_{\rm es})\beta J. \tag{19}$$

Thus, the ratio of the energy flux F_d which is deposited in the disk to the total returning flux is

$$F_d/F_o = \int (dE/dVdt)dz/F_o = \beta^{1/2}\,. \tag{20}$$

This ratio is a function only of the energy of the returning radiation and is always of order unity; the minimum is $\beta^{1/2} \approx 0.25$, at $E_\gamma \approx 25$ keV. In the range 10 keV $\leq E_\gamma \leq$ 100 keV, $\beta^{1/2} \leqslant 0.5$. Accordingly, $F_d \approx F_o/3$ ($\beta \approx 1/9$) is a reasonable approximation, and will be used henceforth.

If the disk is optically thin, much less energy is deposited:

$$F_d/F_o \approx \beta\tau_T/\cos n_o \approx \tfrac{1}{9}\tau_T/\cos n_o\,, \tag{21}$$

where τ_T is the optical thickness of the disk to the returning radiation, measured vertically.

One can show, by examining the rates for various interactions between the energized electrons and the plasma, that the deposited energy gets thermalized and reradiated thermally. Thus, the surface temperature T_s must be

$$\begin{aligned} \sigma T_s^4 = F_d + F_g &\approx F^* + F_0/3 &&(\tau_T \gg \cos n_o) \\ &\approx F^* + \tfrac{1}{9}F_o\tau_T/\cos n_o &&(\tau_T \ll \cos n_o)\,. \end{aligned} \tag{22}$$

We shall assume the disk to be isothermal at a temperature $T \approx T_s$ and find its vertical structure. Actually, there must be a temperature gradient in the disk interior to transport the locally generated flux F_g. Clearly, this gradient will be unimportant when the deposited flux F_d exceeds F_g. When $F_d \ll F_g$, our isothermal model does not differ significantly from the model for the outer disk of Novikov and Thorne (1973), which does include a temperature gradient.

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The equations governing the disk's vertical structure are the *equation of state*,

$$p = 2\rho kT/m_{\rm H} , \tag{23a}$$

where p is the gas pressure (in the outer region of the disk, radiation pressure is negligible); the condition of *vertical pressure balance*,

$$dp/dz = -\rho R^{(z)}{}_{(t)(z)(t)}z = -\rho Mr^{-3}z , \tag{23b}$$

where $R^{(z)}{}_{(t)(z)(t)}$ is the gradient of the "vertical acceleration of gravity" in the rest frame of the gas; and the *viscosity relation*,

$$\alpha \int p dz = \int t_{(\varphi)(r)} dz = (2\pi)^{-1}\dot{M}_0 M^{1/2} r^{-3/2} , \tag{23c}$$

where $t_{(\varphi)(r)}$ is the stress in the gas, measured in the rest frame. See § 5.7 of Novikov and Thorne (1973) for discussion of these equations. Solution of equations (23) gives

$$\rho = \rho_c \exp(-z^2/h^2) , \tag{24a}$$

$$\rho_c = (1.4 \times 10^{13}\ \text{g cm}^{-3})\alpha^{-1}\dot{M}_0{}^* M^{*-2} r^{*-3} T_{\rm K}{}^{-3/2} , \tag{24b}$$

$$h = (0.27\ \text{cm}) M^* r^{*3/2} T_{\rm K}{}^{1/2} , \tag{24c}$$

$$\tau_{\rm es} = (2.6 \times 10^{12})\alpha^{-1}\dot{M}_0{}^* M^{*-1} r^{*-3/2} T_{\rm K}{}^{-1} . \tag{24d}$$

The optical thickness of the disk to electron scattering $\tau_{\rm es}$ is approximately equal to the total optical depth τ_T for the returning radiation.

There may be three regions of the outer disk: an *unheated region* in which the surface temperature is determined primarily by the rate of local energy generation, surrounded by an *optically thick reheated region* in which the temperature depends upon the flux of returning radiation and is sensitive to the shape of the disk surface, surrounded in turn by an *optically thin reheated region* in which the temperature depends upon the intensity of the returning radiation and is independent of the shape of the disk.

In the unheated region $\sigma T^4 = F^*$ and

$$T = (3.1 \times 10^7\ \text{K})\dot{M}_0{}^{*1/4} M^{*-1/2} r^{*-3/4} , \tag{25a}$$

$$\rho_c = (79\ \text{g cm}^{-3})\alpha^{-1}\dot{M}_0{}^{*5/8} M^{*-5/4} r^{*-15/8} , \tag{25b}$$

$$h = (1.5 \times 10^3\ \text{cm})\dot{M}_0{}^{*1/8} M^{*3/4} r^{*9/8} , \tag{25c}$$

$$\tau_{\rm es} = (8.4 \times 10^4)\alpha^{-1}\dot{M}_0{}^{*3/4} M^{*-1/2} r^{*-3/4} . \tag{25d}$$

These results agree very closely with the structure for the outer disk given in equation (5.9.6) of Novikov and Thorne (1973).

In the optically thick reheated region $\sigma T^4 \approx F_o/3$. In this region, properties of the disk, in particular the thickness h, depend upon f_o, which, in turn, depends upon the angle n_o of the returning radiation. Equation (14) relates n_o to dh/dr_o and, thus, gives a differential equation for n_o:

$$\cos n_o = 2.95 \times 10^{-3}\dot{M}_0{}^{*1/8} M^{*-1/4} L^{*1/8} r^* d[r^{*1/4}(\cos n_o)^{1/8}]/dr^* , \tag{26a}$$

which has the solution

$$\cos n_o = (1.25 \times 10^{-3}\dot{M}_0{}^{*-1/8} M^{*1/4} L^{*-1/8} r^{*-1/4} + Cr^{*7/4})^{-8/7} , \tag{26b}$$

where C is a constant of integration. We have implicitly assumed that the disk has a smooth surface, so that equation (14) is valid everywhere.

For a well-behaved solution, we must set $C = 0$. If $C < 0$, $\cos n_o$ will be greater everywhere than for the $C = 0$ solution, but at some large radius $\cos n_o$ will diverge; there equation (26a) will be violated. If $C > 0$, $\cos n_o$ will be less everywhere than for the $C = 0$ solution, but at some large radius $\cos n_o$ will be so small that reheating is negligible again. At this radius, however, a solution for the disk properties which joins smoothly onto the unheated solution (25) will violate equation (26a), since $\cos n_o$ is not negligible at large radii in the unheated solution.

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Thus, in the optically thick reheated region of a disk with a smooth surface,

$$T = (2.8 \times 10^6\ \mathrm{K})\dot{M}_0{*}^{2/7}M{*}^{-4/7}L{*}^{2/7}r{*}^{-3/7}\,, \tag{27a}$$

$$\rho_c = (2.9 \times 10^3\ \mathrm{g\ cm^{-3}})\alpha^{-1}\dot{M}_0{*}^{4/7}M{*}^{-8/7}L{*}^{-3/7}r{*}^{-33/14}\,, \tag{27b}$$

$$h = (4.5 \times 10^2\ \mathrm{cm})\dot{M}_0{*}^{1/7}M{*}^{5/7}L{*}^{1/7}r{*}^{9/7}\,, \tag{27c}$$

$$\tau_{\mathrm{es}} = (9.4 \times 10^5)\alpha^{-1}\dot{M}_0{*}^{5/7}M{*}^{-3/7}L{*}^{-2/7}r{*}^{-15/14}\,. \tag{27d}$$

The properties of the disk are quite different in this region from those in the unheated region: reheating has important effects on the disk structure.

The transition between the unheated region and the optically thick reheated region occurs where T(unheated) = T(reheated), at

$$r{*}_1 = (1.85 \times 10^3)\dot{M}_0{*}^{-1/9}M{*}^{2/9}L{*}^{-8/9}\,. \tag{28}$$

Note that this radius is rather small for a disk around a rapidly rotating hole.

In the optically thin reheated region $\sigma T^4 \approx 1/9F_0\tau_{\mathrm{K}}/\cos n_o$. Thus, the temperature is independent of $\cos n_o$. The corresponding solution for the vertical structure is

$$T = (1.8 \times 10^8\ \mathrm{K})\alpha^{-1/5}\dot{M}_0{*}^{2/5}M{*}^{-3/5}L{*}^{1/5}r{*}^{-7/10}\,, \tag{29a}$$

$$\rho_c = (5.7\ \mathrm{g\ cm^{-3}})\alpha^{-7/10}\dot{M}_0{*}^{2/5}M{*}^{-11/10}L{*}^{-3/10}r{*}^{-39/20}\,, \tag{29b}$$

$$h = (3.6 \times 10^3\ \mathrm{cm})\alpha^{-1/10}\dot{M}_0{*}^{1/5}M{*}^{7/10}L{*}^{1/10}r{*}^{23/20}\,, \tag{29c}$$

$$\tau_{\mathrm{es}} = (1.46 \times 10^4)\alpha^{-4/5}\dot{M}_0{*}^{3/5}M{*}^{-2/5}L{*}^{-1/5}r{*}^{-4/5}\,. \tag{29d}$$

The transition between the optically thick and the optically thin reheated regions occurs where T(thick) = T(thin), at

$$r{*}_2 = (4.55 \times 10^6)\alpha^{-14/19}\dot{M}_0{*}^{8/19}M{*}^{-2/19}L{*}^{-6/19}\,. \tag{30}$$

Since this transition occurs at such a large radius, the disk will probably not have an optically thin reheated region at all. A simple calculation of the flow of the accreting gas from the inner Lagrange point of the binary system (Cunningham 1973) indicates that the radius of the outer gas of the disk is, roughly,

$$r{*}_{\mathrm{out}} \lesssim 3 \times 10^6 M{*}^{-1} \ll r{*}_2\,. \tag{31}$$

To derive the structure of the optically thick reheated region, we assumed that the surface of the disk was very smooth. However, suppose small bumps sometimes form on the disk surface. At a bump, the angle n_o of the returning radiation differs from its normal value. The inner edge of the bump will receive more returning flux and will be heated and swell; the outer edge of the bump will cool and contract. Thus, the bump will move inward. Local inhomogeneities should produce ingoing ripples on the disk surface. This rippling will not occur in the optically thin region, if it exists, since there the rate of heating is independent of the angle n_o.

We must determine if the ripples grow with time. If so, the disk structure described by equation (27) is meaningless, and the optically thick reheated region will be unstable and in a state of chaotic motion. To do this we consider the time-development of a slight perturbation of the disk structure.

Suppose that in some region of the disk the surface temperature T_s and the internal temperature T_i are slightly different from the equilibrium temperature T of equation (27a):

$$T_s = (1 + \epsilon_s)T\,, \qquad \epsilon_s \ll 1\,, \tag{32a}$$

$$T_i = (1 + \epsilon_i)T\,, \qquad \epsilon_i \ll 1\,. \tag{32b}$$

The flux of energy F_i (ergs $\mathrm{s^{-1}\ cm^{-2}}$) flowing from the surface to the interior of the disk is approximately

$$F_i \approx (4/3)(\Delta\sigma T^4/\Delta z)\kappa^{-1}\rho^{-1} \approx (16/3)\sigma T^4 h^{-1}\kappa^{-1}\rho_c^{-1}(\epsilon_s - \epsilon_i)\,. \tag{33}$$

The heat capacity C (ergs $\mathrm{cm^{-2}\ K^{-1}}$) per unit surface area of the disk is

$$C = \int (3 - 2\rho^{-1}Td\rho/dT)\rho k/m_{\mathrm{H}}dz = (3\pi^{1/2}/2)\rho_c hk/m_{\mathrm{H}}\,, \tag{34}$$

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assuming that the surface density remains constant as the disk responds to slight temperature changes. Thus, the rate at which the internal temperature is changing is

$$\partial\epsilon_i/\partial t = F_i/CT = t_i^{-1}(\epsilon_s - \epsilon_i)\,, \tag{35a}$$

$$t_i = (9\pi^{1/2}/32)(k/m_H\sigma)\kappa\rho_c^2h^2T^{-3} = (2.9 \times 10^8 \text{ s})\alpha^{-3}\dot{M}_0{}^{*12/7}L^{*-20/7}r^{*-12/7}\,. \tag{35b}$$

We used for the opacity κ the central value of the Rosseland mean free-free opacity

$$\kappa = (6.4 \times 10^{22} \text{ g}^{-1} \text{ cm}^2)(\rho_c/\text{g cm}^{-3})T_K{}^{-7/2}\,. \tag{36}$$

The disk thickness and, therefore, the angle of the returning radiation and the surface temperature are determined by the internal temperature. Equations (14), (22), and (34c) give

$$\epsilon_s = \tfrac{1}{8}\epsilon_i + (7/16)r^*\partial\epsilon_i/\partial r^*\,. \tag{37}$$

The general solution of equations (35a) and (37) is

$$\epsilon_i = f[r^{*-12/7}(4/3 - t/t_i)]r^{*2}\,, \tag{38a}$$

where $f(x)$ is an arbitrary wave form. This solution represents damped, ingoing waves of velocity

$$v = \tfrac{3}{4}(dt_i/dr)^{-1} = -(7/16)(r/t_i) \tag{38b}$$

and damping time

$$t_d = -\epsilon_i(d\epsilon_i/dt)^{-1} = (8/7)t_i\,. \tag{38c}$$

Since these waves are damped, the rippling of the reheated disk arising from small local inhomogeneities dies out, and the disk is stable in the reheated region.

IV. THE OBSERVED SPECTRUM OF THE DISK

Returning radiation modifies the observed spectrum of the disk at both high and low energies: The returning radiation which is scattered by the disk adds a high-energy component to the spectrum, while the returning radiation which is absorbed changes the structure and the spectrum of the outer disk.

Consider the appearance of the outer disk. In the previous section we stated that the energy which the outer disk absorbs from the returning radiation will be reradiated thermally. Therefore, we assume that the outer disk radiates like a blackbody.

Assume that all the photons from a portion of the disk surface are radiated with an energy $E_o = 3kT$, which is approximately the energy for which the specific intensity of blackbody radiation is greatest. Then the specific luminosity L_{E_0} (ergs s^{-1} eV^{-1}) of the disk is, by equations (25a) and (27a),

$$E_oL_{E_0} = L_{\text{std}}(E_o/E_{\text{std}})^{4/3} \qquad \text{for } E_o/E_{\text{std}} > L^{*2/3}\,, \tag{39a}$$

$$E_oL_{E_0} = (7/4)L^{*4/3}L_{\text{std}}(E_0/E_{\text{std}})^{-2/3} \qquad \text{for } E_o/E_{\text{std}} < L^{*2/3}\,, \tag{39b}$$

$$E_{\text{std}} = (28 \text{ eV})\dot{M}_0{}^{*1/3}M^{*-2/3}\,, \tag{39c}$$

$$L_{\text{std}} = (1.0 \times 10^{35} \text{ ergs s}^{-1})\dot{M}_0{}^{*10/9}M^{*-2/9}\,, \tag{39d}$$

for radiation from the unheated and optically thick reheated outer regions. The spectra of the radiation from the two regions are quite different from each other, since the radial variation of temperature is different in the two regions.

Near the outer edge of the disk, the viscous interaction between the incoming gas stream and the disk will be an important heating mechanism, and the gas will also radiate energy it carries from the primary star. Thus, equation (39b) is very likely an underestimate of the specific luminosity for photon energies near the energy E_{out}, which is typical of radiation from the outer edge of the disk. If we use equation (31) for the value of the outer radius, then

$$E_{\text{out}} = 3 \times 10^{-2}\dot{M}_0{}^{*-7/21}M^{*11/21}L^{*2/7}E_{\text{std}}\,. \tag{40}$$

At lower energies, $E \ll E_{\text{out}}$, there will be a tail to the spectrum: $L_{E_0} \propto E_o{}^2$ for radiation from the disk, but radiation from the incoming gas stream will probably also be important.

Figure 1 shows the low-energy spectra for disks around holes of $a/M = 0$, 0.9, and 0.9999, calculated from equation (42). The upturn in the spectra at the energy $E_o = L^{2/3}E_{\text{std}}$, which corresponds to the transition between the unheated and reheated regions, is a pronounced feature, and is sensitive to the value of a/M of the hole.

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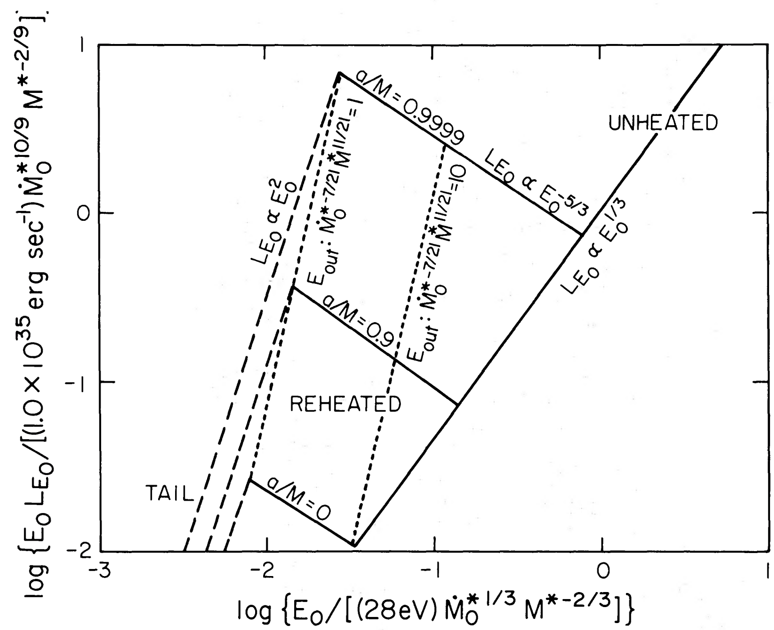

FIG. 1.—The observed, low-energy spectrum of an accretion disk, assuming that 1/3 of the returning flux is absorbed and reradiated thermally. Radiation from the unheated region of the disk, where the returning radiation does not strongly influence the surface temperature, has a different spectrum than does radiation from the reheated region, where returning radiation determines the temperature. The energy of the transition between the "unheated" and the "reheated" spectra depends upon the value of a/M. The low-energy tail of the spectrum lies at energies below the thermal energy $E_{\rm out}$ characteristic of the outer edge of the disk, and is produced by radiation from very near the outer edge.

Thus, observations at low energies (1–10 eV) of the outer disk might provide information about the hole. Unfortunately, in a binary system the primary star will radiate strongly at these energies; it may be very difficult to discern the disk. The value of the energy $E_{\rm out}$ at which the spectra tail off depends upon M^* and M_o^*; two choices for these parameters are shown in Figure 1.

In addition to radiating thermally, the outer disk scatters an appreciable fraction ($\sim\frac{2}{3}$) of the returning radiation. The luminosity $L_{\rm scat}$ of the scattered radiation is proportional to the solid angle subtended by the outer disk, seen from the inner disk, and this solid angle depends upon the thickness of the disk at its outer edge. If we use equation (31) for the outer radius,

$$L_{\rm scat} = (4 \times 10^{36}\ {\rm ergs\ s^{-1}})\dot{M}_0{}^{*8/7}M^{*-4/7}L^{*8/7}\,. \tag{41}$$

This luminosity is only a small fraction of the total luminosity of the disk (5×10^{-2} for $a/M = 0.9999$, 2×10^{-3} for $a/M = 0$, with $M^* = \dot{M}_0{}^* = 1$).

In the inner disk, where the flux of returning radiation is largest, electron scattering is the primary opacity mechanism, and most of the returning radiation is scattered, rather than absorbed. The luminosity of the scattered returning radiation from the inner disk is always much greater than that from the outer disk, and the spectra of the two types of radiation are similar. Therefore, the radiation scattered in the outer disk will be unobservable.

We have not presented the full machinery to properly calculate the spectrum of the returning radiation scattered in the inner disk, as seen by a distant observer. For such a calculation, we would need to know the spectrum of the returning radiation seen by a local observer, not just its total flux, as a function of position on the surface of the disk. However, we may approximate the observed spectrum in a simple manner.

Suppose we observe the disk from its axis. Then most of the observed radiation comes from $r \gtrsim 4M$. Radiation which comes directly to us from smaller radii is very redshifted and contributes little flux (see Paper I). However, radiation from the innermost disk may return to the disk at $r \gtrsim 4M$ and there be scattered to us. Thus, the observed spectrum will have two components at high energies: Radiation propagating directly to us will produce a component which dies at an energy characteristic of the surface temperature of the disk at $r \approx 4M$; scattered radiation will produce a component which peaks at a somewhat higher energy, characteristic of the innermost disk.

We have seen that most of the radiation which returns and is scattered does so in the inner disk due to gravitational focusing. If we assume that this scattered radiation and the thermal emission come off the disk with about the same limb darkening, then the ratio of the observed fluxes in the two components of the spectrum equals the

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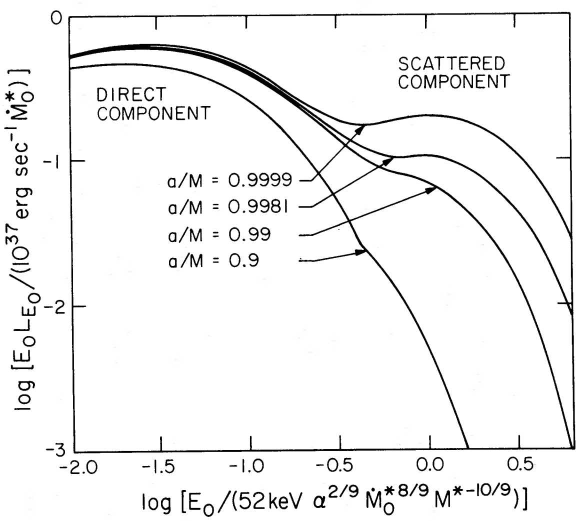

FIG. 2.—The observed, high-energy spectrum of an accretion disk which scatters the returning radiation. We assume that the radiation is emitted with the spectrum of equation (42), which is appropriate for a radiation-pressure-dominated disk with an electron-scattering atmosphere, and that the disk is seen from the polar axis. Curves are labeled with the value of a/M of the hole. The "direct-component" of the spectrum originates at $r \gtrsim 4M$ and propagates directly to the observer. The "scattered component" is radiation which originates at very small radii, returns, and is scattered on its way to the observer.

ratio of fluxes of the returning radiation and the locally generated energy in the inner disk at $r \geqslant 4M$. Table 5 gives this ratio; it is 0.4 for $a/M = 0.9999$; 0.06 for $a/M = 0.9$.

Both the total flux and the spectrum of the direct component may be calculated by the method of Paper I. We assume that the spectrum of the scattered component is similar to that of the radiation seen by a distant observer in the plane of the disk, which may also be calculated by Paper I. Since both gravitational focusing and Doppler blueshifts enhance the radiation that reaches the plane from the innermost disk, the spectrum of the scattered component is harder at high energies than a Newtonian spectrum of the disk (see Paper I).

We assume that, in the region of the disk which is producing most of the radiation, the disk is optically thick, radiation pressure dominates gas pressure, and electron scattering is the dominant opacity mechanism. For a homogeneous atmosphere with density roughly equal to that in the central regions of the disk, Novikov and Thorne (1973) calculate the specific intensity I_E (ergs s^{-1} cm^{-2} sr^{-1} eV^{-1}) of the emitted radiation and the surface temperature T_s:

$$I_E \propto x^{3/2}e^{-x/2}(e^x - 1)^{-1/2}\,, \tag{42a}$$

$$x = E/kT_s\,, \tag{42b}$$

$$T_s = (6 \times 10^8\ \mathrm{K})(\alpha^{2/9}M^{*-10/9}\dot{M}_0{}^{*8/9}r^{*-17/9}) \times (\text{relativistic corrections})\,, \tag{42c}$$

where E is the energy of the emitted photons. The corrections in equation (42c) are such that

$$T_s \approx (3 \times 10^8\ \mathrm{K})(\alpha^{2/9}M^{*-10/9}\dot{M}_0{}^{*8/9})r^{*-1/2} \qquad \text{for } a/M \approx 1\,,\quad r^* \leqslant 4\,. \tag{42d}$$

We have neglected the secular instability of the optically thick, radiation-pressure-dominated inner disk to the formation of overdense clumps and under-dense bubbles (Lightman and Eardley 1974). This instability might force part of the inner disk to become optically thin, in which case the spectrum of the emitted radiation will be different from that given here: The two-temperature model of Eardley, Lightman, and Shapiro (1975) for such an optically thin inner disk predicts an electron temperature which is higher ($T \sim 10^9$ K) and an emitted spectrum which is correspondingly more energetic than for our optically thick model. Therefore, results based on equation (42) must be interpreted with caution.

The corresponding observed spectra, seen on the polar axis, for the four values of a/M, are shown in Figure 2. The superposition of the two components of the spectrum results in a notch at the energy at which the lower-energy direct component begins to fall sharply; this energy is, roughly,

$$E_{\mathrm{notch}} \approx (20\ \mathrm{keV})(\alpha^{2/9}M^{*-10/9}\dot{M}_0{}^{8/9})\,, \tag{43}$$

for this model of the inner disk.

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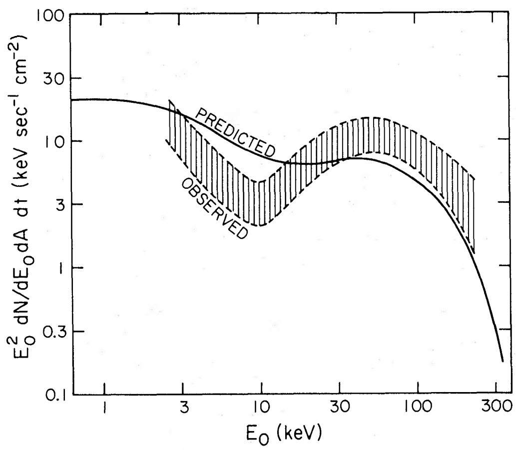

FIG. 3.—The "pre-transition" satellite and balloon observations of Cygnus X-1, compared to a theoretical spectrum. Parameters for the theoretical spectrum are taken from observational data, when possible (see text). The agreement of the theoretical spectrum with the observations is fairly good, considering the crude model of the disk on which it is based.

This notch is somewhat similar to, but not as deep as, that observed in the "pre-transition" spectrum of Cyg X-1. In spite of the crude model used for the inner disk and the approximate method used to generate the theoretical observed spectra, a fairly good fit to the actual pre-transition observations of Cyg X-1 can be obtained for reasonable values of mass and accretion rate. Figure 3 compares the observed spectrum of Cyg X-1 (from Thorne and Price 1975) with a theoretical spectrum, for which we assume that Cyg X-1 is at a distance of 2.5 kpc (Bregman *et al.* 1973; Margon *et al.* 1973), with a mass $M = 8\ M_\odot$ (Paczynski 1974), and an inclination angle $i = 0$. (From the absence of X-ray eclipses and limits given by Paczynski 1974 one can conclude that $i < 60°$. For all i in this range the spectrum is qualitatively the same as for $i = 0$.) We also assume that the accretion rate is $\dot{M}_0 = 4 \times 10^{17}$ g s^{-1}, the viscosity parameter is $\alpha = 0.1$ (which are reasonable values for a disk in a close binary system [Shakura and Sunyaev 1973; Novikov and Thorne 1973]), and that $a/M = 0.9999$. (This value of a/M is substantially greater than the steady-state value for disk accretion, $a/M = 0.998$ [Thorne 1975]. However, for the low rate of accretion we assume, the time scale for evolution of the hole is long, in excess of 10^8 years.) The fit would be even better for a larger value of a/M, since then the notch would be deeper. The notch would also be deeper if the form of the emitted spectrum were harder, the decrease in surface temperature with radius were steeper, or the scattered flux were larger due to the surface of the disk swelling out of the equatorial plane.

Note that the observations *can* be fitted moderately well by an optically thick, geometrically thin disk model. Thorne and Price (1975) and Lightman and Shapiro (1975) state that the disk must be optically thin in its innermost regions to produce the observed $E \geq 100$ keV X-rays. Indeed, were it not for relativistic effects, one would conclude that the surface temperature given by equation (42) is too low to explain, simultaneously, the high energies and low luminosity of Cyg X-1. However, the scattered component of the spectrum will be blueshifted by almost a factor of 2 and the direct component will be considerably attenuated if $a/M \approx 1$ and the disk is observed from near its polar axis (see Paper I). Thus, the actual luminosity of Cyg X-1 might be higher, and surface temperatures lower, than a Newtonian analysis of the observations would predict.

We conclude that returning radiation will be an important phenomenon in accretion disks around rotating holes. However, before accurate, detailed calculations of its effects on the appearance of the disk can be performed, one must have an accurate model for the structure of the inner disk to determine properties of the disk atmosphere and to give the thickness of the disk. If the inner disk is not geometrically thin, the results given here will need modification.

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APPENDIX

THE NET FLUX OF LOCALLY GENERATED ENERGY

The net flux, like other features of the radial structure of the disk, is governed by three conservation laws: conservation at rest mass, of angular momentum, and of energy. Page and Thorne (1974) present these laws in a convenient form, and their work needs little modification to include the transport of angular momentum and energy by returning radiation.

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For a thin, axisymmetric, steady-state disk in the equatorial plane of a Kerr hole, in which gas moves generally along circular geodesics, the conservation laws given by Page and Thorne become:

Conservation of angular momentum,

$$[-\dot{M}_0 u_\varphi + 2\pi r \int_{-h}^{+h} (t_\varphi{}^r + u_\varphi q^r + q_\varphi u^r) dz]_{,r} + [2\pi r(t_\varphi{}^z + u_\varphi q^z)]_{-h}^{+h} = 0\,; \tag{A1a}$$

Conservation of energy,

$$[-\dot{M}_0 u_t + 2\pi r \int_{-h}^{+h} (t_t{}^r + u_t q^r + q_t u^r) dz]_{,r} + [2\pi r(t_t{}^z + u_t q^z)]_{-h}^{+h} = 0\,. \tag{A1b}$$

The conservation of rest mass has been invoked to bring the conservation laws into this form. We use the notation of Page and Thorne: $\dot{M}_0$ is the accretion rate; $\boldsymbol{u}$ is the 4-velocity of the gas; $\boldsymbol{t}$ is the stress tensor in the local rest frame of the gas, and is orthogonal to $\boldsymbol{u}$; $\boldsymbol{q}$ is the energy flux in the local rest frame of the gas, and is orthogonal to $\boldsymbol{u}$; z measures vertical distance in the local rest frame; and the disk surfaces are at $z = \pm h$.

We assume that the returning radiation is absorbed on the disk surface, so that in the interior of the disk energy is transported only in the z-direction: $q^r = q_\varphi = q_t = 0$. This eliminates several terms in equations (A1). Since $\boldsymbol{t}$ is orthogonal to $\boldsymbol{u}$, we have

$$t_t{}^\alpha = -(u^\varphi/u^t) t_\varphi{}^\alpha \equiv -\Omega t_\varphi{}^\alpha\,, \tag{A2}$$

where Ω is the coordinate angular velocity of the gas. Equations (A1) may be written

$$(-\dot{M}_0 u_\varphi + 2\pi r W)_{,r} + 4\pi r(S + u_\varphi F) = 0\,, \tag{A3a}$$

$$(-\dot{M}_0 u_t - 2\pi r \Omega W)_{,r} + 4\pi r(-\Omega S + u_t F) = 0\,. \tag{A3b}$$

where

$$W = \int_{-h}^{+h} t_\varphi{}^r dz\,, \tag{A3c}$$

$$S = t_\varphi{}^z(z = h)\,, \tag{A3d}$$

$$F = q^z(z = h)\,. \tag{A3e}$$

Note that F is the net flux at the disk surface; it is the difference between the flux emitted at the surface and the returning flux.

We assume that the radiation emitted by the disk comes off vertically on the average, so that only the returning radiation contributes to S. Equation (5) relates the component $T^{(\varphi)(z)}$ of the stress tensor of the returning radiation to the intensity and direction of the returning radiation, as observed in the rest frame. The indices $(\varphi)(z)$ of this component refer to the orthonormal basis of the rest frame, $\boldsymbol{e}_{(t)}$, $\boldsymbol{e}_{(\varphi)}$, $\boldsymbol{e}_{(r)}$, $\boldsymbol{e}_{(z)}$, defined in the obvious manner. (For a description of this basis, see Novikov and Thorne 1973.) Since $\boldsymbol{e}_{(\varphi)}$ and $\boldsymbol{e}_{(z)}$ are orthogonal to the 4-velocity $\boldsymbol{u}$ $(= \boldsymbol{e}_{(t)})$, and since $\boldsymbol{e}_{(\varphi)}$ is the only spacelike basis vector with a component in the $\partial/\partial\phi$ direction,

$$t^{(\varphi)(z)}(z = h) = T^{(\varphi)(z)}\,; \tag{A4a}$$

$$S = e_{(\varphi)\varphi} T^{(\varphi)(z)}\,. \tag{A4b}$$

Following the procedure of Page and Thorne, equations (A3a) and (A3b) are solved simultaneously to give

$$F = \tfrac{1}{2}\Omega_{,r}(u_t + \Omega u_\varphi)^{-1} W\,; \tag{A5a}$$

$$F = \Omega_{,r}(u_t + \Omega u_\varphi)^{-2} r^{-1} \int (u_t + \Omega u_\varphi)(\dot{M}_0 u_{\varphi,r} - rS) dr\,. \tag{A5b}$$

Note that F may be expressed as the sum of the flux F_0 calculated with $S = 0$ (Page and Thorne express this quantity analytically) and a term which depends upon the returning radiation:

$$F(r) = F_0(r) - F_0(r_{\rm in}) - \Omega_{,r}(u_t + \Omega u_\varphi)^{-2} r^{-1} \int_{r_{\rm in}}^{r} (u_t + \Omega u_\varphi) S r dr\,, \tag{A6}$$

where the constant of integration in equation (A5b) was chosen so that F and W vanish at the inner edge of the disk $r_{\rm in}$.

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The quantities in equation (A6) which refer to the motion of the gas may be expressed conveniently in terms of the dimensionless functions introduced by Novikov and Thorne (1973):

$$\Omega = M^{1/2}r^{-3/2}\mathscr{B}^{-1}\,, \tag{A7a}$$

$$\Omega_{,r} = -(3/2)M^{1/2}r^{-5/2}\mathscr{B}^{-2}\,, \tag{A7b}$$

$$(u_t + \Omega u_\varphi) = -\mathscr{B}^{-1}\mathscr{C}^{1/2}\,, \tag{A7c}$$

$$e_{(\varphi)\varphi} = r\mathscr{B}\mathscr{C}^{-1/2}\mathscr{D}^{1/2}\,, \tag{A7d}$$

$$\mathscr{B} = 1 + aM^{1/2}r^{-3/2}\,, \tag{A7e}$$

$$\mathscr{C} = 1 - 3Mr^{-1} + 2aM^{1/2}r^{-3/2}\,, \tag{A7f}$$

$$\mathscr{D} = 1 - 2Mr^{-1} + a^2r^{-2}\,. \tag{A7g}$$

So, substituting these quantities in equation (A6), we find that

$$F(r) = F_0(r) - F_0(r_{\rm in}) - (3/2)M^{1/2}r^{-7/2}\mathscr{C}^{-1}\int_{r_{\rm in}}^{r} \mathscr{D}^{1/2}T^{(\varphi)(z)}r^2dr\,. \tag{A8}$$

The steady-state assumption requires that the net flux F equal the locally generated flux F_g (see eq. [3]).

The stress W must always oppose the shearing flow of the gas; hence, the positivity of $\Omega_{,r}$ means that W cannot go negative. As a consequence, at the inner edge of the disk, where the gas begins to plunge into the hole and the stress W nearly vanishes, we must also have $W_{,r} \approx 0$. At the inner edge, equation (A3b) becomes

$$u_{t,r} = -4\pi r\Omega S/\dot{M}_0 \qquad (\text{at } r = r_{\rm in})\,. \tag{A9}$$

This equation determines the radius of the inner edge. Expand $u_{t,r}$ near $r_{\rm ms}$, the radius of marginal stability:

$$u_{t,r} \approx u_{t,rr}(r - r_{\rm ms})\,, \tag{A10a}$$

$$u_{t,rr} = -Mr^{-3}\mathscr{C}^{-1/2} \qquad (\text{at } r = r_{\rm ms})\,. \tag{A10b}$$

A first-order expansion is justified since we find that the difference between $r_{\rm ms}$ and $r_{\rm in}$ is always extremely slight. Thus, by equation (A9),

$$r_{\rm in} \approx r_{\rm ms} + 4\pi\dot{M}_0{}^{-1}M^{-1/2}r^{7/2}\mathscr{D}^{1/2}T^{(\varphi)(z)}|_{r_{\rm ms}}\,. \tag{A11}$$

We have the interesting results that (for $T^{(\varphi)(z)} > 0$) the returning radiation will decrease the net flux everywhere and will destroy the nearly circular flow of the disk at some radius outside the radius of marginal stability. Because of the large orbital velocity of the gas, most of the returning radiation would be seen by a local observer to come from the direction of motion. The torque of the returning radiation, thus, tends to slow down the rotation of the disk and thereby forces the gas to spiral into the hole faster than it would otherwise.

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