THE ASTROPHYSICAL JOURNAL, 206:499–508, 1976 June 1

AN ANALYSIS OF THE VARIABLE RADIAL VELOCITY OF ALPHA CYGNI

L. B. LUCY
Department of Astronomy, Columbia University
Received 1975 September 25

ABSTRACT

On the basis of 447 radial velocities obtained at the Lick Observatory by Paddock in the years 1927–1935, an attempt is made to discover the nature of the semiregular variability of α Cygni (A2 Ia). Harmonic analysis of the 144 velocities obtained in 1931 suggests that this variability is due to the simultaneous excitation of many discrete pulsation modes. The amplitudes and periods of these modes are then determined by least-squares fitting to all the data, and a final solution is obtained that comprises 16 terms with periods from 6.9 to 100.8 days. All terms are found to have highly significant amplitudes, and most terms also pass a test of the stability of their amplitudes and phases. Reasons are given for believing that most terms represent nonradial oscillations, and this leads to the suggestion that the resulting surface motions are to be identified with macroturbulence. An argument is also given for believing that the pulsational instability persists down to periods at which atmospheric oscillations become progressive, and this leads to the suggestion that such waves are observed as microturbulence and give rise to the observed mass loss. The importance of further monitoring of the variability of supergiants is stressed.

Subject headings: stars: binaries — stars: individual — stars: mass loss — stars: semiregular variables — turbulence

I. INTRODUCTION

Following up clues in the earlier literature, Abt (1957) demonstrated that probably all Ia supergiants of early and intermediate type are small-amplitude velocity-variables. He also noted that a few of them are known to be small-amplitude light-variables. The typical ranges are ~ 10 km s^{-1} in velocity and $\lesssim 0.1$ mag in light; the typical time scale of the semiregular variations is ~ 10–100 days.

Although the variability of a few of these stars has been known for several decades, its character remains unknown. All that can be said with certainty is that the variability is *not* simply periodic and that its time scale is too short for orbital motion. We may also say with some confidence that the variability is basically a pulsation phenomenon since the typical time scales are comparable to the expected periods of radial pulsation in the fundamental mode (Abt 1957). The possibility that the variations are due to oscillations confined more or less to these stars' atmospheres is therefore ruled out.

Because of the limited duration (~ 30 days) of his observing program, Abt could neither exclude nor investigate the possibility that the "semiregular" variability of these supergiants is due to two or more simultaneously excited pulsation modes. There is, therefore, no observational basis for believing that the variation is in any degree stochastic. Such a variation might be expected, however, if, following Underhill (1960), we were to associate the variable radial velocity with the large-scale atmospheric motions inferred from the line profiles of these same supergiants since the term "macroturbulence" used to describe these motions seems to imply that they are stochastic. In fact, of course, the evidence for these large-scale motions, though strong, does not in any way determine their statistical character. Indeed, if we accept that the same motions cause both the line broadening and the line shifts, then the argument identifying the velocity variability with pulsations should lead us to entertain the hypothesis that macroturbulence is itself a pulsation phenomenon, being related perhaps to the line broadening observed at certain phases for the β Cephei stars. We may recall that some aspects of the variable velocity and line broadening of the β Cephei stars can be understood by supposing that their pulsations are nonradial (Ledoux 1951; Osaki 1971).

With the generality of the variability of the Ia supergiants established, the logical next step would be to secure extensive observations of one such star, so that the hypothesis of multiple periodicity could be tested. Such observations are in fact available already for α Cyg (A2 Ia), whose variable radial velocity was the subject of an astonishingly extensive investigation by Paddock (1935) at the Lick Observatory. Paddock's observations span the years 1927–1933, during which time 399 radial velocities were determined from 794 plates all taken with the Mills three-prism spectrograph (11 Å mm^{-1}) attached to the 36 inch refractor. Moreover, all the α Cygni plates were measured twice by Paddock himself on a Hartmann Spectro-Comparator against one of the plates as a standard. Because of this, the *changes* in velocity are followed with high precision, and the loss of accuracy that results from combining the work of different measurers is avoided. It is rare indeed to work with such homogeneous data.

TABLE 1

ADDITIONAL MILLS VELOCITIES

JD (2,428,000+)	Velocity (km s^{-1})	JD (2,428,000+)	Velocity (km s^{-1})	JD (2,428,000+)	Velocity (km s^{-1})
2.668	−4.1	2.897	−5.6	6.775	−5.5
2.692	−4.5	2.911	−4.0	6.789	−5.2
2.703	−5.2	2.925	−5.6	6.803	−5.6
2.717	−4.6	2.938	−6.4	6.817	−5.4
2.731	−5.2	2.952	−4.8	6.831	−6.3
2.745	−4.8	2.966	−6.1	6.844	−5.7
2.758	−5.8	2.980	−5.6	6.858	−5.1
2.772	−5.4	2.994	−5.3	6.872	−5.7
2.786	−4.9	6.664	−1.8*	6.886	−5.5
2.800	−4.9	6.678	−6.1	6.900	−5.5
2.814	−6.0	6.692	−4.1	6.914	−5.4
2.828	−5.0	6.706	−4.0	6.928	−5.8
2.841	−5.9	6.720	−4.7	6.941	−4.0
2.855	−5.1	6.734	−4.6	6.955	−4.8
2.869	−6.9	6.748	−5.4	6.969	−5.5
2.883	−5.2	6.761	−5.5	6.983	−4.8

* This observation is omitted from all calculations.

Although these data are now over 40 years old, their quality should not be doubted. Under the direction of Campbell and then Moore, the radial velocity program at Lick in those years yielded accurate velocities free from fluctuating systematic errors. This may be demonstrated using the list of spectroscopic binaries whose orbits were recomputed by Lucy and Sweeney (1971). In this list there are 25 systems for which the velocity residuals give $\sigma \leq 1.5$ km s^{-1}, where σ is the standard error of a radial velocity of average weight, and for 13 of these the velocities were obtained with the Mills spectrograph. Since these binaries were usually observed over several seasons, this result confirms both the precision of the Mills velocities and the stability of the velocity system.

Our purpose in this paper, then, is to analyze Paddock's superb data with the hope of discovering the character of the semiregular variability of these supergiants.

II. SHORT-PERIOD OSCILLATIONS?

Although Paddock's main concern was the night-to-night velocity variations of α Cyg, he also came to believe that significant variations occur within a single night, and he sought support for this belief by taking 18 or 19 consecutive plates on each of four nights. In his paper, Paddock plots the data for three of these nights and draws freehand curves through the points to suggest the possibility of an oscillation with period $\sim$5 hr and amplitude $\sim$0.5–1.2 km s^{-1}. Subsequently, Paddock and Moore pursued this question further by taking 24 plates on each of the nights 1935 July 19 and 23. These velocities, kindly made available to me by G. H. Herbig, are given in Table 1.

In addition to giving this direct evidence for short-period variations, Paddock argues that such variations must exist because his measuring error (determined from his duplicate measurements of each plate) is markedly less than the external error (determined from measurements of closely consecutive plates). However, with the possible exception of early-type stars with broad, diffuse lines, this is always found to be so and is regarded as evidence of additional sources of error and not of true velocity changes (Petrie 1962). Since some of these additional sources of error, including certainly those related to the photographic plate, may afflict even Mills velocities, we may fairly conclude that this indirect argument of Paddock's is less than compelling. The significance of the effect must therefore be decided from the variations observed on the six nights during which many plates were taken.

However, before the significance of the short-period oscillation can be tested, the alternative hypothesis that the variations are due to additional sources of error must be considered. Adopting this hypothesis, we may use the residuals from a least-squares straight line fitted to the n_j velocities obtained on the jth night to derive an estimate s_j of σ, the external standard error of the radial velocity of α Cyg determined from one Mills spectrogram. This has been done for all the nights for which five or more velocities are available, and the results are given in Table 2. From these independent estimates s_j, the precision of Paddock's velocities is seen to have remained remarkably uniform over a 6-year interval; consequently, the combined estimate

$$s = \left(\sum \nu_j s_j^2 \Big/ \sum \nu_j\right)^{1/2} = 0.60 \text{ km s}^{-1}, \qquad (1)$$

TABLE 2

INDEPENDENT ERROR ESTIMATES

Date	n_j	s_j (km s^{-1})
1929 Aug. 3	18	0.56
1931 Oct. 24	7	0.46
1931 Oct. 25	5	0.61
1932 Aug. 2	18	0.46
1932 Aug. 3	19	0.56
1932 Aug. 4	19	0.67
1935 Jul. 19	24	0.65
1935 Jul. 23	23	0.64

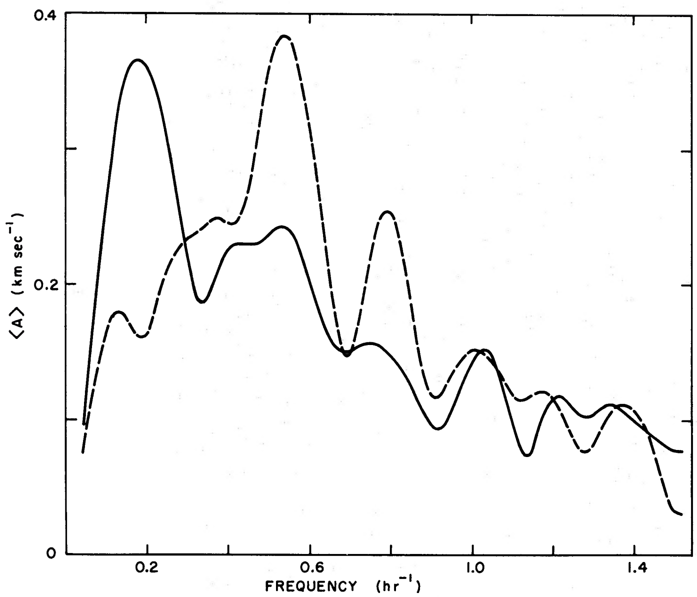

FIG. 1.—Weighted mean amplitude spectrum for short-period oscillations. Solid line is the spectrum of the actual observations; dashed line is a "noise" spectrum.

where $\nu_j = n_j - 2$ is the number of degrees of freedom associated with s_j, provides an unusually reliable estimate of the external standard of error of a velocity of unit weight (i.e., from one plate). The high precision of these Mills velocities may be appreciated by comparing this estimate with the external standard error of 2.1 km s^{-1} that Petrie (1962) reports for the velocities of A-type stars obtained with the Victoria two-prism spectrograph (11 Å mm^{-1}).

We now return to the question of the significance of the short-period variations. For each of the six nights for which 18 or more velocities are available, we have calculated the amplitude spectrum,

$$A_j(f) = \frac{2}{T_j}\left|\int_0^{T_j} \Delta_j v e^{-2\pi i f t} dt\right| , \qquad (2)$$

where $\Delta_j v$ is the velocity variation remaining after subtraction of the least-squares straight line fitted to the data obtained in the interval $(0, T_j)$ on the jth night. ($\Delta_j v$ is made into a continuous function by connecting consecutive residuals with straight lines.) Then, in order to optimize our ability to discover a persistent oscillation of small amplitude, these spectra are combined to obtain $\langle A(f)\rangle$, defined by

$$\langle A(f)\rangle = \left\{\sum \nu_j [A_j(f)]^2 \Big/ \sum \nu_j\right\}^{1/2} , \qquad (3)$$

and this function is plotted in Figure 1. [Note that $A(f)$ is related to the power spectrum $P(f)$ by the relation $A^2 = 4P/T$.] At first sight, Figure 1 appears to confirm Paddock's discovery since it shows $\langle A(f)\rangle$ to have a pronounced peak corresponding to an oscillation with period = 5.6 hr and amplitude = 0.36 km s^{-1}. Before this confirmation can be accepted, however, the significance of the peak must be tested. We may do this by replacing each velocity residual on each of the six nights by x_k, a random variable from a normal distribution of zero mean and standard deviation = 0.60 (eq. [1]), and then repeating the calculations leading to $\langle A(f)\rangle$. Five such "noise" spectra were calculated, and three—one of which is plotted in Figure 1—were found to contain peaks comparable to the one found for the real data. Accordingly, there is no basis for rejecting the hypothesis that the variations within one night are due solely to observational errors.

Since others might interpret Figure 1 as providing marginal evidence in support of the short-period oscillation, we now compare its period with the critical period below which standing atmospheric oscillations are impossible. For an isothermal atmosphere, this critical period is (Lamb 1945)

$$P_L = 4\pi c_s/\gamma g , \qquad (4)$$

where c_s is the velocity of sound, γ is the ratio of specific heats, and g is the gravitational acceleration. Now, from Groth's (1961) model atmosphere analysis of α Cyg, we have $T_{\rm eff} = 9170 \pm 500$ K and $\log g = 1.13 \pm 0.2$. Adopting these values, taking the temperature of the reversing layer = 0.8 $T_{\rm eff}$, setting $\gamma = 1$ (isothermal oscillations), and assuming the mean molecular weight = 0.62, we obtain $P_L = 8.4$ days. Accordingly, if the atmosphere of α Cyg were subject to an oscillating disturbance with 5.6 hr period, its response would be not a standing oscillation but a traveling wave with wavelength short compared to the scale height. Such waves would not cause significant fluctuations in radial velocity; instead, they would be detectable as *micro*turbulence. This argument strongly supports the conclusion that the short-period variations are due to observational errors.

TABLE 3
DISTRIBUTION OF VELOCITIES

Year	No. of Velocities	No. of Plates
1927	2	2
1928	28	53
1929	8	33
1930	11	21
1931	144	338
1932	60	208
1933	61	139
1934	0	0
1935	2	47

Having failed to confirm the short-period oscillation, we now average the velocities for each night on which many plates were taken, and we assign weight $w_j = n_j$ to this average. The data now have the form of a single velocity on each of 316 nights. The distribution of these in time is given in Table 3.

III. BINARY ORBIT?

If averages of Paddock's data are formed over intervals long compared with the time scale of the semiregular variations (i.e., $\gg 10$ days), it becomes obvious (see Fig. 2) that α Cyg also undergoes long-period variations with range ~ 6 km s^{-1}. The simplest explanation of this would be that α Cyg is a spectroscopic binary. Accordingly, a search was made for orbits with periods greater than 150 days. Of the orbits found, two are sufficiently superior in their fits to the data that the others may be discarded. For these two orbits, the orbital elements and their standard errors are given in Table 4, and the orbits are plotted in Figure 2. The elements were obtained by least-squares fitting to all 316 velocities, each assigned unit weight. The standard deviation of the residuals (σ) are also given in Table 4, and they show the two orbits to be essentially equally successful in fitting the data.

Because this long-period variation must be removed from the data before the semiregular variability can be properly analyzed, confirmation of the orbital motion is highly desirable, and this of course requires additional velocities. Accordingly, at G. W. Preston's suggestion, the author, using the Caltech Grant machine, measured ~ 150 coudé plates from the Mount Wilson and Victoria plate files, the plates

TABLE 4
ORBITAL ELEMENTS

Element	Orbit 1	Orbit 2
P (days)	846.8 ± 9.3	776.4 ± 10.7
T_0	$417^* \pm 19$	366 ± 17
γ (km s^{-1})	-2.81 ± 0.17	-3.24 ± 0.16
K (km s^{-1})	3.12 ± 0.56	2.61 ± 0.38
e	0.60 ± 0.10	0.72 ± 0.10
ω	$318.^\circ3 \pm 9.^\circ8$	$334.^\circ9 \pm 7.^\circ9$
σ (km s^{-1})	2.094	2.107

* The origin of time is JD 2,426,600.

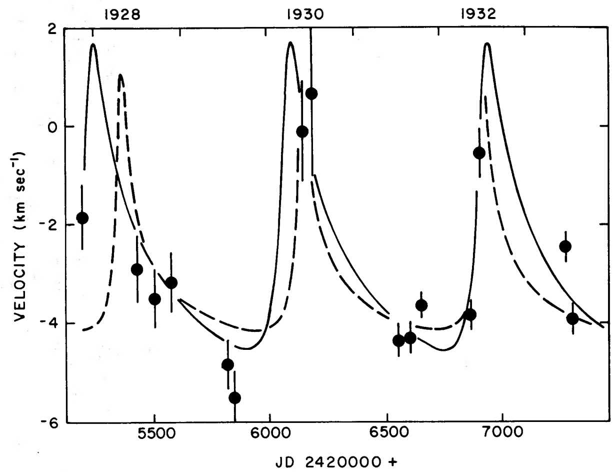

FIG. 2.—Long-period velocity variation. Group means and their standard errors are shown together with the orbital motions corresponding to the elements in Table 4.

having kindly been made available by H. W. Babcock and K. O. Wright, the respective directors. This measuring program yielded 113 apparently acceptable radial velocities for the years 1927–1957, most of them having *internal* standard errors less than 0.6 km s^{-1}. However, when these velocities were combined with the Lick data, there was a marked increase in σ for both orbits, and this seems to be primarily a consequence of fluctuating systematic errors of ~ 2–3 km s^{-1} in the additional velocities. This attempt to add to the observational data must therefore be judged a failure.

Because of this failure, the long-period variation has been expressed in the form

$$v = \gamma + \sum_{m=1}^{M} \left(A_m \cos \frac{2\pi mt}{T} + B_m \sin \frac{2\pi mt}{T} \right), \quad (5)$$

where T is the elapsed time between the first and last observations, and the coefficients have been determined by finding the least-squares solution subject to the constraint:

$$\sum_{m=1}^{M} (A_m{}^2 + B_m{}^2) = \alpha^2 \,. \quad (6)$$

This constraint is made necessary by the large gaps between the blocks of data. Unconstrained solutions take advantage of their freedom to vary wildly in these gaps and, by doing so, become suspect near the ends of the blocks of data. By choosing α to be as small as is possible without increasing the standard deviation of the residuals significantly above its minimum value (i.e., for the unconstrained solution), this problem is avoided. The adopted solution has $M = 7$ and $\alpha = 3.5$ km s^{-1}, and is a fit to the $J = 312$ velocities (again with unit weight) obtained in the years 1928–1933. When this representation of the long-period variation is subtracted from the data, all oscillations with periods longer than ~ 280 days have been removed. (The isolated observations of 1927 and 1935 are omitted since, in the absence of a physical model for the long-period variations, we cannot hope to separate the contributions of the long- and intermediate-period variations to these velocities.)

After elimination of the long-period variation, the standard deviation of the velocity variation is $\sigma = 2.06$ km s^{-1}. Observational errors alone would be expected to give $\sigma_E = 0.60/\sqrt{\bar{w}}$, where $\bar{w}$ is the mean weight defined by

$$\frac{1}{\bar{w}} = \frac{1}{J}\sum_j \frac{1}{w_j}\,. \tag{7}$$

Making this calculation, we find that $\bar{w} = 2.2$, so that $\sigma_E = 0.41$ km s^{-1}, and $\sigma/\sigma_E = 5.1$. Paddock's observations therefore provide a virtually error-free description of the semiregular variation; consequently, no miracles of data analysis are required to extract signal from noise.

IV. SEMIREGULAR VARIABILITY

Having shown that the short-period oscillation is not statistically significant and having failed to confirm the reality of orbital motion, we now discuss the variability at intermediate periods.

a) The 1931 Data

The obvious starting point for the analysis is the data for 1931 since in that year Paddock measured α Cygni's velocity on 144 nights (see Table 3) within an interval of 173.588 days. This block of data is such that $\bar{w} = 2.2$, so that we again have $\sigma_E = 0.41$ km s^{-1}.

The semiregular variability in 1931 is shown in Figure 3. In this diagram, successive observations are connected by straight lines, these lines being dashed when three or more nights elapsed without an observation. This diagram reveals that the range of the variation is 10.3 km s^{-1} ($\gg\sigma_E$) and that the typical time scale is $\sim$10 days.

The amplitude spectrum (eq. [2]) of the continuous velocity variation shown in Figure 3 is plotted in Figure 4. Also plotted in this diagram is the amplitude spectrum obtained when α Cygni's velocity v_j at time t_j and of weight w_j is replaced by $v_j' = 0.8 \cos 2\pi f t_j + 0.60 x_n/\sqrt{w_j}$, with $f = 0.0665$, and where x_n is a random variable from a normal distribution of zero mean and unit variance. This synthetic amplitude spectrum therefore approximates that of a star undergoing a pure sinusoidal oscillation of amplitude 0.8 km s^{-1} and period $\sim$15 days, and for which the observing program is identical to that for α Cyg in 1931. Inspection of this synthetic spectrum reveals that the peak due to the oscillation stands well clear of the noise but that the sidelobes do not. Moreover, from the general level of the noise, we see that additional oscillations with amplitudes greater than $\sim$0.25 km s^{-1} could be readily and reliably discovered.

[Note that a pure sinusoidal term of amplitude A and frequency f_0 appears in the amplitude spectrum as $A|\sin x/x|$, where $x = \pi(f - f_0)/T$, and T is the length of the record. The amplitudes of the first two sidelobes are 0.217 and 0.128 $\times$ A, respectively, and the full width at half-maximum is $= 1.207/T = 0.007$ day^{-1} for $T = 173.588$.]

Inspection of the two amplitude spectra for $f \geqslant 0.15$ day^{-1}, including frequencies not plotted in Figure 4, shows that the α Cyg data and the synthetic data yield comparable amplitudes. This result confirms that we do indeed know the level of observational error, and it also demonstrates that α Cyg is quiescent for periods shorter than $\sim$6.5 days. For $f \leqslant 0.15$, on the other hand, we see that α Cyg has many ($\sim$12) peaks standing far above the expected noise level and, moreover, that several of these (e.g., those at $f \approx 0.07$ and $f \approx 0.085$) have widths that exclude their interpretation as pure sinusoidal terms. Since these latter peaks do show incipient resolution, however, it is tempting to suppose that they represent two or more sinusoidal terms whose frequencies are too close for resolution when the length of the record is only 174

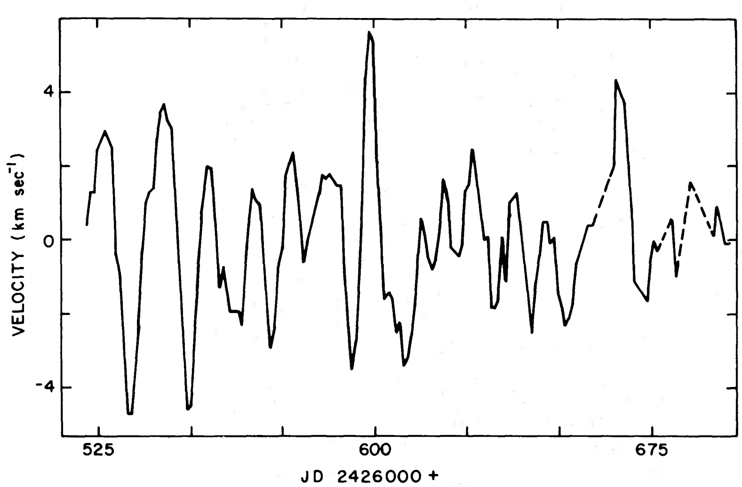

FIG. 3.—Velocity variation in 1931 after removal of long-period variation

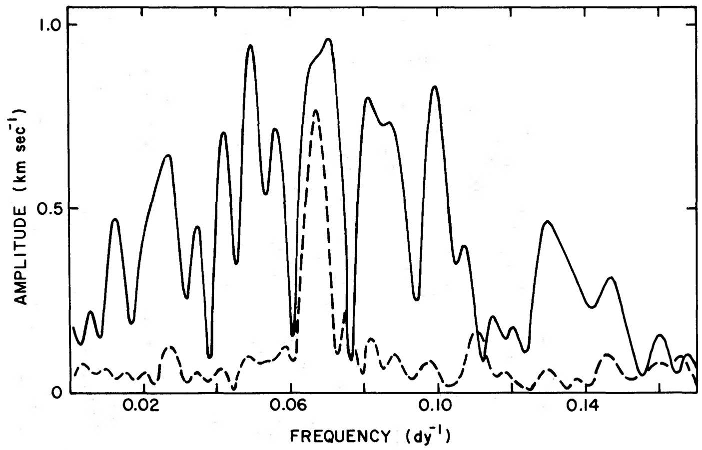

FIG. 4.—Amplitude spectrum of the 1931 velocity variation (Fig. 3). Solid line is the spectrum of the actual observations; dashed line is a "noise" spectrum with addition of pure sine term.

days. Thus the amplitude spectrum of α Cyg suggests the hypothesis that its semiregular variability is due to the simultaneous excitation of many discrete modes of oscillation.

b) *Search for Periods*

The above hypothesis is not proven by the amplitude spectrum since the duration of the 1931 observing run is not sufficient to demonstrate beyond question resolution into discrete frequencies. Paddock's data do, however, offer the possibility of demonstrating this hypothesis if the amplitudes and phases of the discrete modes of oscillation are stable in time, for then the variation during the entire period of observation (1928–1933) may be represented as

$$v = \sum_l (a_l \cos 2\pi f_l t + b_l \sin 2\pi f_l t)\,; \tag{8}$$

and we can determine the constant coefficients a_l, b_l, and f_l by minimizing

$$R = \sum_j w_j' u_j^2\,, \tag{9}$$

where u_j is the velocity residual at time t_j. (Because of the previous removal of the long-period variability, the systemic velocity is set equal to 0 in eq. [8].)

If we are unable to discover all the terms contributing to the variability, the residuals u_j will not be due solely to observational errors, and it is then quite inappropriate to assign weight w_j (= number of plates) to the individual velocities. Instead, therefore, we assign weight

$$w_j' = (\sigma^2 + \sigma_*^2)/(\sigma^2/w_j + \sigma_*^2)\,,$$

where $\sigma = 0.60$ km s^{-1} (eq. [1]) and σ_*^2 is our estimate of the variance of the contributions of true velocity variations to the residuals u_j. The contributions of observational errors to the residuals can of course be reliably estimated since we know σ and w_j. Accordingly, having computed a least-squares solution, we may subtract this contribution to obtain:

$$\sigma_*^2 = \left(\sum_j w_j u_j^2 - J\bar{\sigma}^2\right) \Big/ \sum_j w_j\,, \tag{11}$$

and this estimate of σ_*^2 may be used in deriving weights for the next least-squares solution. From equation (10), we see that $w_j' \approx 1$ when $\sigma_*^2 \gg \sigma^2$, and that $w_j' \approx w_j$ when $\sigma_*^2 \ll \sigma^2$. (In the final solution given below, $\sigma_* = 0.89$ km s^{-1}, so that undiscovered terms are still making a nonnegligible contribution to the residuals.)

The general procedure adopted in obtaining a least-squares fit to all the data was as follows: (1) Frequencies of a few ($\sim$2–3) major terms were estimated from the 1931 amplitude spectrum and a least-squares fit of these terms to the 1931 data was computed. (2) The amplitude spectrum of the residuals from the least-squares solution was then used to estimate frequencies of the next few terms, and these were then included in a new least-squares solution. (3) Step (2) was repeated until no significant terms remained. (4) The 1932 data were now added, and the resulting least-squares solution was subjected to the cycle-count test (see below) in order to discover errors due to the loss or gain of a cycle in the intervening winter. (5) Such errors having been eliminated, the 1933 data were added, and the resulting least-squares solution was subjected to the cycle-count test. (6) Such errors having been eliminated from this solution, the 1928–1930 data was added, and the resulting least-squares was solution subjected to the cycle-count test.

The final solution is given in Table 5. For the purpose of constructing this table, each term is written in the form $A \cos 2\pi f(t - t_0)$, and the quantities A, t_0, and f are tabulated, together with their standard errors. In addition, the period $1/f$ is given and two quantities, $\log p$ and d, that relate to tests discussed below.

TABLE 5

PULSATION PERIODS AND AMPLITUDES

Period (days) (1)	A (km s^{-1}) (2)	t_0 (days) (3)	$f \times 10^3$ (days^{-1}) (4)	log p (5)	d (6)
100.8	0.43 ± 0.08	−10.0 ± 3.2*	9.92 ± 0.07	−6.6	2.1
49.1	0.43 ± 0.10	36.1 ± 1.7	20.38 ± 0.08	−4.6	2.4
40.5	0.75 ± 0.11	23.3 ± 0.9	24.71 ± 0.05	−12.6	0.8
35.8	0.47 ± 0.10	−3.4 ± 1.2	27.93 ± 0.08	−6.0	3.2
24.2	0.57 ± 0.08	−3.1 ± 0.5	41.31 ± 0.06	−10.9	1.3
18.9	0.89 ± 0.09	2.1 ± 0.3	53.00 ± 0.04	−21.7	2.1
18.1	0.80 ± 0.08	13.2 ± 0.3	55.26 ± 0.05	−19.5	1.6
15.6	0.79 ± 0.09	2.8 ± 0.3	64.13 ± 0.04	−18.6	4.1
14.4	0.90 ± 0.09	−1.7 ± 0.2	69.32 ± 0.04	−24.4	4.0
12.4	0.74 ± 0.08	1.8 ± 0.2	80.88 ± 0.04	−17.1	1.9
11.4	1.02 ± 0.09	0.0 ± 0.2	87.53 ± 0.03	−28.1	2.6
10.0	0.82 ± 0.09	7.3 ± 0.2	99.72 ± 0.04	−18.6	1.1
9.5	0.63 ± 0.09	−0.8 ± 0.2	105.01 ± 0.06	−11.9	4.9
7.8	0.40 ± 0.09	−0.6 ± 0.3	128.96 ± 0.09	−5.0	3.4
7.5	0.51 ± 0.09	4.9 ± 0.2	133.54 ± 0.06	−7.8	4.9
6.9	0.29 ± 0.08	4.5 ± 0.3	145.72 ± 0.10	−2.8	1.3

* The origin of time is JD 2,426,600.

c) *Tests*

Before discussing the implications of this solution, we must investigate the statistical significance of the amplitudes of the individual terms, the reliability of their frequencies, and the stability of each term's amplitude and phase.

i) *Significance of the Amplitudes*

An immediate impression of the significance of the amplitudes is available from their standard errors, but one must not overlook the fact that amplitudes, being nonnegative, do not follow a normal distribution even if they are due to observational errors that are normally distributed. In this circumstance, however, a_l and b_l are normally distributed; consequently, in the limit of large sample size, the probability that errors will give an amplitude $\sqrt{(a_l^2 + b_l^2)}$ exceeding A is $\exp(-A/2\sigma_A^2)$. According to this formula, the 5 percent level of significance occurs when $A/\sigma_A = 2.45$, and the 0.1 percent level when $A/\sigma_A = 3.72$. Fifteen of the 16 terms in Table 5 therefore have amplitudes that are significant at the 0.1 percent level.

Although the above discussion clearly establishes the high significance of all 16 terms, it is of interest to apply also a test that is not restricted to large sample size. Such a test is provided by the theory of multivariate linear hypotheses, which is applicable to this problem if we regard the frequencies as exactly known. In our case, we wish to test, for each term, the hypothesis that $a_l = 0$ and $b_l = 0$. If adopted, this hypothesis causes R (eq. [9]) to increase from R_0 to R_H, and the significance of this change may be evaluated from the ratio $\hat{F} = \frac{1}{2}(J - M)(R_H - R_0)/R_H$, where J is the number of observations and M is the number of parameters (32). Now, it may be shown (see, e.g., Hamilton 1964, chap. 4) that F is distributed as F_{ν_1,ν_2} with $\nu_1 = 2$ and $\nu_2 = J - M$, and from this one may prove (cf. Lucy and Sweeney 1971) that the probability of observational error giving $F > \hat{F}$ is

$$p = (R_0/R_H)^\beta , \tag{12}$$

where $\beta = \frac{1}{2}(J - M)$. Accordingly, if $p \ll 1$, we must reject the hypothesis and accept the amplitude as being significant.

The probabilities from equation (12) are given in column (5) of Table 5, and they clearly confirm the high significance of the amplitudes.

ii) *Cycle-Count Test*

Because of the absence of observations in the winter months, the least-squares solutions are vulnerable to errors in which one complete oscillation is added or lost during the winter. Accordingly, at several stages in the analysis, this possibility was tested for by changing a term's frequency f_l to $f_l \pm f_*$, where $f_* = 1/365.25$, and then computing the least-squares solution with all frequencies fixed. This was done for all the terms one at a time, so that, with 16 terms, the solution of 32 linear least-squares problems is required. In fact, this test was often carried out with f_l also changed to $f_l \pm f_*/2$ and to $f_l \pm f_*/3$. From the results of such calculations, the frequency f_l was regarded as suspect if one of the above changes led to less than a significant increase in R. For such a case, a full least-squares solution (i.e., with the frequencies also varying) was computed and the indicated change adopted if it gave a smaller R. The final solution is such that the changes $f_l \pm f_*/m$ for $m = 1, 2, 3$ give increases in R for all terms.

This test encourages belief in the reliability of the frequencies. However, when the frequencies of so many terms have to be determined from observing runs that are shorter than the beat periods of neighboring frequencies, the complexity of the possible errors is such that it is quite unlikely that all errors have been eliminated from the frequencies in Table 5.

iii) *Stability*

A basic assumption in our least-squares search for discrete pulsation modes is that their amplitudes and phases remained fixed from 1928 to 1933. We now test this assumption by dividing the 312 velocities chronologically into two equal groups and, for each group, solving by least-squares for the amplitudes (a_l, b_l) corresponding to the frequencies f_l given in Table 5, assuming these to be known exactly. For each term, therefore, this pair of least-squares solutions gives two independent points and their error ellipses in (a, b)-space. Obviously, the hypothesis of amplitude and phase stability must be rejected if the separation of these points is too large to be attributed to the uncertainties in the amplitudes.

Now, to a reasonable approximation, the error ellipses are circles, and the radii of the circles are equal for a given term. Accordingly, on the hypothesis of stability, the two points for each term may be regarded as independent random choices from the circular normal distribution function,

$$\phi(a, b) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(a - \bar{a})^2 + (b - \bar{b})^2}{2\sigma^2}\right), \quad (13)$$

where $\bar{a}$ and $\bar{b}$ are the true amplitudes. Adopting this function therefore as our model for the probability density function (pdf) of amplitude errors, we may draw pairs of points (a_x, b_x) and (a_y, b_y) randomly from this pdf and thereby compute the pdf of the statistic $d = D/\sigma$, where $D^2 = (a_x - a_y)^2 + (b_x - b_y)$. This calculation has been carried out using a random number generator, with the result (from 5000 pairs) that the expectation value of d is ~ 1.87, that $d > 3.55$ occurs with probability ~ 0.05, and that $d > 4.45$ occurs with probability ~ 0.01. Thus, if we adopt the 5 percent level of significance, the hypothesis of stability for a given term is rejected if $d > 3.55$.

We now illustrate the use of this test by giving the details for two terms. For the 40.5 day term, we find that $a_x = -0.82 \pm 0.16$, $b_x = -0.32 \pm 0.15$ for the first group of observations; and that $a_y = -0.89 \pm 0.14$, $b_y = -0.41 \pm 0.14$ for the second group. These then give $D = 0.114$ and $\sigma = 0.148$ (σ is taken to be the square root of the mean variance for the four amplitudes), so that $d = D/\sigma = 0.77$. The corresponding figures for the 9.5 day term are $a_x = 0.45 \pm 0.13$, $b_x = 0.07 + 0.13$; and $a_y = 0.81 \pm 0.13$, $b_y = -0.60 \pm 0.13$, which give $D = 0.641$ and $\sigma = 0.13$, so that $d = 4.93$.

The quantity d is given for each term in Table 5, and the distribution of these values is compared in Table 6 with the expected distribution on the hypothesis of stability. From the individual values, we see that this hypothesis is rejected for four terms at the 5 percent level of significance ($d > 3.55$), and for two terms at the 1 percent level ($d > 4.45$). In view of this modest rejection rate, it seems not unreasonable to believe that all terms are in fact stable; the rejected terms should therefore be interpreted as indicating that the final solution is not free from error.

TABLE 6

DISTRIBUTION OF d

Interval	Observed No.	Expected No.
0–1..........	1	3.2
1–2..........	5	6.3
2–3..........	4	4.5
3–4..........	3	1.6
4–5..........	3	0.3
>5..........	0	0.1

V. DISCUSSION

In view of the results of § IV, it is not unreasonable to conclude that the semiregular variability of early- and intermediate-type supergiants is indeed due to the simultaneous excitation of many discrete pulsation modes and to believe also that the amplitudes and phases of these oscillations are stable. The following questions then arise: Which modes are being excited? What is the mechanism of excitation? Are these oscillations related to other dynamical phenomena occurring at the surfaces of these supergiants? What are the implications of the multiple-periodicity of these stars? We therefore conclude this paper with a brief discussion of each of these questions.

a) *Identification of Modes*

As reasonable parameters for α Cyg, we adopt $M_{\text{bol}} = -8$, $\mathfrak{M} = 12\,\mathfrak{M}_{\odot}$, and $R = 140\,R_{\odot}$. Because of observational uncertainties, this choice may be described as being consistent with Blaauw's (1963) absolute magnitude calibration of the MK system ($M_v = -7.5$ at A2 Ia), with Groth's (1961) determination of α Cygni's atmospheric parameters ($T_{\text{eff}} = 9170 \pm 500$ K, $\log g = 1.13 \pm 0.2$), and with the predictions of stellar evolution theory under the assumption of negligible mass loss.

With these parameters, α Cygni's mean density $\bar{\rho} = 4.4 \times 10^{-6}\rho_{\odot}$, so that its pulsation period for the fundamental radial mode is $P_0 = 14.3$ days, if $Q_0 = 0.03$ days. This estimate of P_0 is comparable with the periods of the six terms of largest amplitude in Table 5, but it is markedly shorter than the periods of several other terms with highly significant amplitudes. Now, if we regard T_{eff} and Q_0 as known, then $P_0 \propto L^{0.25}g^{-0.5}$, so that large changes in L and g are required if P_0 is to exceed the periods of all observed terms, as it must if the latter are to be interpreted as radial pulsations. The necessity of such changes is avoided, however, if we contemplate the possibility that nonradial modes are excited, for there is then no upper limit to the spectrum of discrete eigenperiods. With this interpretation, the large-amplitude terms with periods $\sim P_0$ will be low-order p- and g-modes.

Support for the belief that nonradial oscillations dominate the list of observed terms comes from the tendency of these terms to occur in close pairs, which suggests the effect of rotation in lifting the degeneracy of the eigenfrequencies of such oscillations with respect to the order m of the surface harmonic $P_l^m(\theta)e^{\pm im\phi}$. The resulting frequency splitting is given by

$$\Delta f = (1 - C_l)mv_R/\pi R \tag{14}$$

(Ledoux 1951), where v_R is the star's rotational velocity and C_l is a constant depending on the eigenfunction for the undistorted star. Taking $l = m = 2$, $C_2 = 0.15$ (Ledoux 1951), and $\Delta f = 0.005$ day^{-1}, which approximates the observed splitting for four pairs of terms, we obtain $v_R = 10.4$ km s^{-1}. Since this is in the range expected for α Cyg on the assumption of conservation of angular momentum during its evolutionary expansion (see, e.g., Rosendhal 1970), it lends strong support to the hypothesis that nonradial pulsations dominate.

Further progress in identifying these modes clearly requires extensive calculations of the periods of nonradial, nonadiabatic oscillations for realistic models of α Cyg.

b) *Excitation*

For realistic models of the *envelope* of α Cyg, extensive nonadiabatic pulsation calculations have been carried out in a search for the source of excitation. The helium ionization zones were found to contribute to excitation but never by an amount sufficient to overcome dissipation in deeper layers. In fact, no self-excited nonradial or radial oscillations were discovered even when extreme variations of the parameters were explored.

This failure to discover the excitation mechanism might tempt one to question the interpretation of the semiregular variability as a pulsation phenomenon. There are, however, other pulsating variables (e.g., the β Cephei stars) for which the excitation mechanism has not yet been identified with certainty.

c) *Surface Motions*

In addition to the semiregular variability in velocity, spectroscopy provides the following further evidence of dynamical activity at the surfaces of these supergiants: From equivalent widths, curve-of-growth analyses yield rather large values ($\sim$10–20 km s^{-1}) for the microturbulence parameter (see, e.g., Wright 1957). From line profiles, on the other hand, even larger velocities are inferred; consequently, an additional large-scale velocity field must be postulated, and this is commonly referred to as macroturbulence. Finally, P Cygni profiles at Hα demonstrate the presence of radial outflows of matter from these stars.

If these various dynamical phenomena are related, we might expect this advance in our understanding of the semiregular variability to further our understanding of these other surface motions. An obvious possibility is to identify macroturbulence with the surface motions generated by the superposition of these numerous nonradial oscillations. A definitive test of this idea will be possible when the modes have been identified since the time-dependent line profiles could then be calculated and the macroturbulence predicted. In advance of this identification, numerical experiments of this kind have been carried out (Lucy 1976) with results that confirm the idea's plausibility.

The amplitude spectrum (Fig. 4) appears to indicate that the excitation mechanism no longer overcomes dissipation for periods shorter than $\sim$6.5 days. If this were true, however, the near equality of this period and the critical period (estimated to be 8.4 days in § II) below which atmospheric waves become progressive would be an accidental coincidence. Rather than accept this, it is surely preferable to suppose that oscillations continue to be excited at shorter periods but that they fail to produce detectable variations in radial velocity because of their progressive character. As mentioned earlier (§ II), such waves would contribute to microturbulence, and this leads therefore to the suggestion that the high microturbulence of these supergiants is due to progressive atmospheric waves excited by the same mechanism that excites the standing oscillations of longer period. It is of course also tempting to attribute the radial outflows to the effect of these progressive waves on the tenuous outer layers of these stars.

d) *Implications*

If α Cyg is typical of the early- and intermediate-type supergiants, then each of these highly luminous stars is such that the diligent application of a 50-year-old technology is capable of yielding a set of accurate pulsation periods whose number exceeds the number of parameters governing its structure. Such sets of periods would therefore provide an extraordinarily stringent test of stellar evolution theory. Moreover, since models of these stars are *not* homologous under variations of such parameters as mass, luminosity, and mean molecular weight, a fit to an observed set of periods will require not only that stellar evolution theory be correct but also that these parameters have been correctly chosen. There is, therefore, the possibility of using such periods for the theoretical determination of the distances, masses, and compositions of these supergiants.

The realization of these possibilities would bring about such a giant step forward in the precision of stellar astrophysics that thought should be given to the initiation of long-term observing programs to monitor the variable radial velocities of many such supergiants. From a good observing site (e.g., a space observatory), light variability could also be monitored with the necessary precision ($\sim$0.003 mag), and this would be of considerable interest since there may be many modes excited that produce negligible variations in radial velocity. Moreover, from a good site in the Southern Hemisphere, Magellanic Cloud supergiants could be included in such an observing program.

I would like to thank K. H. Prendergast and G. W. Preston for useful discussions and encouragement. This work has been supported by the National Science Foundation under grant GP-30927. A grant of computer time at the Goddard Institute for Space Studies, NASA, is also acknowledged.

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L. B. LUCY: Department of Astronomy, Columbia University, New York, NY 10027