ORBITAL RESONANCES IN THE SOLAR SYSTEM

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1 INTRODUCTION

There are numerous examples of orbital resonances in the solar system, by which we mean any system of two satellites orbiting the same primary whose orbital mean motions are in the ratio of small whole numbers. The mean motions in such a case are said to be commensurate, and the resonance is often called a commensurability. The term "satellites" here includes the planets as satellites of the Sun. The most famous of these resonances is also a special case since it involves three instead of two satellites. If we designate the mean motions of the first three Galilean satellites of Jupiter (Io, Europa, Ganymede) by n_1, n_2, and n_3, the relation $n_1 - 3n_2 + 2n_3 = 0$ is satisfied to nine significant figures. The stability of this resonance was known to Laplace (1829), and the subject of orbital commensurabilities has been a lively topic of celestial mechanics from that period to the present. Much of this interest was derived from the fact that reasonably accurate masses of the satellites involved in resonances could be derived.

Often an orbital resonance is characterized by the absence of objects in an otherwise crowded region of the solar system. The major gaps in the rings of Saturn occur at distances from Saturn corresponding to orbital periods near $\frac{1}{2}$ and $\frac{1}{3}$ the orbital period of the satellite Mimas. The Kirkwood gaps in the spatial distribution of the asteroids correspond to orbital periods that are integral fractions of Jupiter's period. At the same time, there are some asteroids in these gaps whose orbital periods are commensurate with those of Jupiter and others (Hilda group) that preferentially occupy resonant orbits.

In Table 1 we list and describe known orbital resonances. Also given is a reference where a particular resonance is discussed in some detail. Some of the parameters given in Table 1 are defined in later sections. The Titius-Bode law is not included in Table 1 since most formulations of this law of planetary distances from the Sun do not imply commensurate mean motions for the planets (Nieto 1972). There have been attempts to formulate a precise resonant structure for the solar system (e.g. Molchanov 1968), but the deviations of the planet's mean motions from precise commensurability are sufficiently large that the probability of finding the planets where they are is no different from that of a random distribution (Backus 1969, Henon 1969). For this reason the 5:2, 2:1, and 3:1 near commensurabilities for Jupiter-Saturn, Uranus-Neptune, and Saturn-Uranus, respectively, are also omitted.

The stability of orbital resonances has been understood for some time in terms of the libration of a pendulum-like system (e.g. Brown & Shook 1933, Chap. 8). In Section 4 we display the disturbing function, characterizing the perturbations of one satellite due to the presence of another, as an infinite series of simply periodic terms. If the ratio of the orbital periods of the satellites is near that of two *small* integers, the frequencies of a few of these terms with large coefficients approach zero and lead to large amplitude perturbations. Often a single term is so dominant that all the remaining terms in the disturbing function may be neglected, and a pendulum equation for the angular argument of the dominant term results with stability about either 0 or π depending on the sign of the coefficient. In Section 2 we give a heuristic explanation of one example of this stability, as well as for other phenomena exhibited by stable resonances.

Table 1 Commensurabilities of mean motions for solar-system members

System	Resonance variable ϕ	Libration center	Reference
Jupiter			
Io-Europa-Ganymede	$\lambda_I - 3\lambda_E + 2\lambda_G$	180°	Sinclair (1975)
Io-Europa	$\lambda_I - 2\lambda_E + (\varpi_E \text{ or } \varpi_I)$	0° or 180°	Sinclair (1975)
Europa-Ganymede	$\lambda_E - 2\lambda_G + \varpi_E$	0°	Sinclair (1975)
Saturn			
Mimas-Tethys	$2\lambda_M - 4\lambda_T + \Omega_M + \Omega_T$	0°	Greenberg (1975)
Enceladus-Dione	$\lambda_E - 2\lambda_D + \varpi_E$	0°	Sinclair (1972)
Titan-Hyperion	$3\lambda_T - 4\lambda_H + \varpi_H$	180°	Greenberg (1973a)
			Colombo et al. (1974)
Ring Gaps-Mimas	$\lambda_R - 2\lambda_M + \varpi_R$	—	Franklin et al. (1971)
Asteroids—Jupiter			
Trojans	$\lambda_T - \lambda_J$	$\lambda_J \pm 60°$	Brown & Shook (1933)
Thule	$3\lambda_T - 4\lambda_J + \varpi_T$	0°	Takenouchi (1962),
			Marsden (1970)
Hilda	$2\lambda_H - 3\lambda_J + \varpi_H$	0°	Schubart (1968)
Griqua	$\lambda_G - 2\lambda_J + \varpi_G$	0°	Sinclair (1969)
			Franklin et al. (1975)
Eight faint asteroids 2:1	$\lambda_A - 2\lambda_J + \varpi_A$	180°	Franklin et al. (1975)
Alinda 3:1	$\lambda_A - 3\lambda_J + 2\varpi_A$	0° & 180°	Sinclair (1969)
Kirkwood gaps	3:1; 5:2; 7:3; 2:1	—	Lecar & Franklin (1973)
			Sinclair (1969)
Planets			
Neptune–Pluto	$2\lambda_N - 3\lambda_P + \varpi_P$	180°	Cohen & Hubbard (1965)
	$(4\lambda_N - 6\lambda_P + 2\Omega_P)$?	180°	Williams & Benson (1971)

Although stability is easily understood, existence of so many objects in commensurate orbits and, at the same time, the absence of asteroids and Saturn's ring particles near some of the commensurate orbits are not so easily explained. Recent research has been concentrated on this problem of origin and considerable progress has been made. Generally, the origin of the orbital commensurabilities can be attributed either to dissipative processes in the form of collisions at the time of the origin of the solar system or to the slow differential increases in the semimajor axes of the satellite orbits from tidal transfer of angular momentum from the primary (Goldreich 1965). It is almost certainly true that we are observing the consequences of both processes among the observed commensurabilities.

One infers from the existence of the asteroid belt, from Saturn's rings, and indirectly from theories of the origin of the solar system that interplanetary space was much more crowded with asteroid-like objects than we see it today. It has been shown by Lecar & Franklin (1973), however, that the orbits of such bodies between Mars and Saturn are unstable (with a few exceptions) on a very short time scale except for those in the current asteroid belt. On longer time scales, the solar system could be swept as clean as we find it today due to the cumulative perturbations of the massive planets. The early distribution of debris over much of the solar system would have included many objects in stable commensurate orbits. Mutual collisions would repeatedly scatter objects out of a commensurability, but others would be scattered in. As the population of unstable objects was diminished by collision with the planets or, more likely, by ejection from the solar system, those within the potential wells of stable commensurabilities would remain behind. The Trojan asteroids and other asteroids in resonant orbits are most likely remnants of such a sequence of events. Pluto has avoided a close encounter with Neptune in its crossing orbit only because it was left in the stable 3:2 resonance, which keeps the conjunctions near Pluto's aphelion (Cohen & Hubbard 1965) and near 90° from the mutual orbit node (Williams & Benson 1971). Most of its companions were ejected long ago, although a few may survive undetected in similar resonances or in more distant orbits. The Titan-Hyperion system is another possible candidate for such a primordial commensurability (Goldreich 1965, Sinclair 1972), although there is not universal agreement on this point (Colombo et al. 1974).

Examples of orbital commensurabilities that may have been established by the tidal evolution of initially nonresonant orbits are necessarily confined to the satellites of the major planets. Only for these satellites have the tidal effects been possibly of sufficient magnitude to cause significant orbital evolution. The motivation for a tidal origin of the satellite resonances comes from a determination by Roy & Ovenden (1954, 1955) that the number of commensurate orbits among the satellites of the major planets is far more than could be accounted for in a random distribution of orbits. Since there is no obvious reason for the formation of satellites in resonant orbits, the current nonrandom distribution implies an evolutionary change in the original configuration, and the tidal transfer of angular momentum to the satellites from the spinning primary is a reasonable way to effect this change (T. Gold, private communication, 1962).

Goldreich (1965) proved the existing resonances among the satellites of Jupiter and

Saturn to be stable under continuing attempts by the tides to force the orbits to expand at different rates. He also showed that all the bodies involved in stable commensurabilities probably experience significant tidal evolution in the age of the solar system except for Titan. If the Titan-Hyperion resonance was the result of a primordial chance configuration, the observations are completely consistent with a tidal origin of these commensurabilities. Of course, "significant" tidal evolution of the satellite orbits is completely dependent on the unknown value of Q for the major planets, where Q, the dissipation factor, is 2π times the maximum energy stored in a tidal oscillation divided by the energy dissipated over a cycle. Goldreich (1965) determined a lower bound for Q of 1.5×10^5 for Jupiter and 6×10^4 for Saturn, based on the current distance of the innermost satellites. These lower bounds were used to estimate the orbital evolution of all the satellites. Dermott (1968, 1971) derived a frequency and amplitude dependence for Q, which would tend to disrupt the existing resonances. He concluded therefore that tidal evolution has been negligible ($Q \gg$ Goldreich's lower bounds), but he did not satisfactorily account for the large number of resonances at the time of satellite formation.

The tidal hypothesis for the origin of the orbital resonances among the satellites of Jupiter and Saturn gains considerable credibility with the development of a rather complete and reasonably clean and simple theory of the tidal evolution of a satellite pair from a nonresonant configuration, through transition into a stable commensurability with subsequent evolution within the commensurability. Allan (1969) followed the evolution of the Mimas-Tethys pair within the resonance and was able to estimate an age of the resonance based on Goldreich's lower bounds on Q. Sinclair (1972) gave the details of the capture of Mimas-Tethys and Enceladus-Dione from a nonresonant to a resonant configuration including a numerical evaluation of the probability of capture of Mimas-Tethys in the current resonance, a probability of escape of this pair from a nearby resonance encountered first, the probability of escape of Enceladus-Dione from a similar nearby resonance, and the certain capture of this latter pair in its current commensurability. Greenberg et al. (1972) and Greenberg (1973a) demonstrated the certain capture of a Titan-Hyperion–type pair of satellites under tidal evolution and gave a clear physical picture of the transition from the nonresonant to resonant configuration. Yoder (1973; 1976, to be published) developed a completely analytic theory applicable to the establishment and evolution of any two-body commensurability experiencing tidal evolution (under some constraints, which are satisfied by the satellite resonances). The nature of the capture can be followed in great detail in this latter theory as a function of one parameter whose application agrees with the numerical calculations of Sinclair. The consistency of the current libration amplitudes of the resonances among Saturn's satellites with these theoretical developments makes the tidal hypothesis of origin easily accepted—even to the point of trying to accommodate the Titan-Hyperion resonance (Colombo et al. 1974).

Because of the importance of these theoretical developments to our recent understanding of the origin of satellite commensurabilities, the theory is outlined in Section 4 following a heuristic discussion of stability, capture, and other properties of a resonance in Section 2 and a summary of current ideas on the formation of the Kirkwood gaps and the gaps in Saturn's rings in Section 3. In Section 5 we point out explicit successes and possible failures of the tidal hypothesis for the origin of orbital resonances as applied to the satellites of Saturn, and we end with a short discussion of an outstanding unsolved problem.

2 PHYSICAL DESCRIPTION OF THE RESONANCE PHENOMENON

Orbital resonances among the satellites of Saturn and between Jupiter and the asteroids except for the Trojans all depend on a nonzero eccentricity or inclination in the coefficient of their pendulum-like restoring accelerations. We describe below a simplified model of the simplest kind of eccentricity-type resonance. This model shows all the essential features of such a commensurability while eliminating some of the complicating detail. The model actually very closely resembles the Titan-Hyperion case (e.g. Goldreich 1965, Greenberg 1973a).

Consider two satellites of masses $m \gg m'$ in coplanar orbits about a primary. The inner satellite m is assumed to be in a circular orbit and is so much more massive than m' that perturbations by the latter can be ignored. The mean motions are assumed to be nearly commensurate, and m' is in an eccentric orbit. The orbits are shown schematically in Figure 1, where ϖ' is the longitude of the pericenter of the

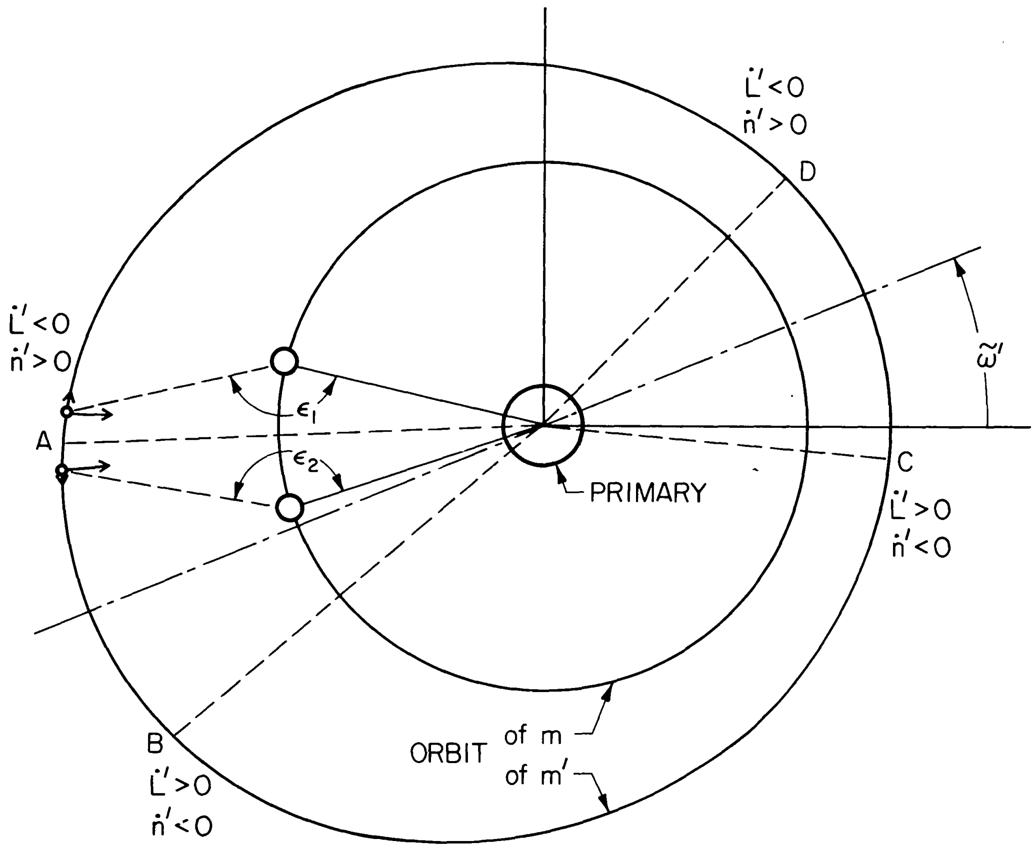

Figure 1 Large-eccentricity stability mechanism. Arbitrary positions of repetitive conjunctions are at points A, B, C, D. L and n are angular momentum and mean motions, respectively.

outer orbit. Four arbitrary positions of repetitive conjunctions are indicated by dashed lines, and the relative positions of each satellite just before and just after a conjunction are also shown along with the radial and tangential components of the perturbing force by m on m'. A dot over a symbol indicates time differentiation.

During the period from opposition to conjunction, m removes angular momentum from m' via the tangential component of the perturbing force, and from conjunction to the next opposition it adds angular momentum. If the conjunction occurs exactly at pericenter or exactly at apocenter, the effects of the tangential forces integrate to zero, and there is no net transfer. Repetitive conjunctions at any other point destroy this symmetry. If we assume that the line of apsides is fixed and also assume precise commensurability of the mean motions, a conjunction at point A in Figure 1, for example, would be followed by successive conjunctions at the same point for non-interacting satellites. However, the tangential component of the perturbing force is larger prior to conjunction than after (for $\varepsilon_1 = \varepsilon_2$ in Figure 1) since the orbits are diverging. In addition, the angular velocity of m' is closer to that of m prior to conjunction, as m' is slowing down as it approaches apocenter. This means m catches up with m' more slowly than it recedes after conjunction, so the larger tangential force opposing the motion of m' is also applied for a longer time than the smaller tangential force in the opposite sense after conjunction. Hence, a conjunction at A leads to a net loss of angular momentum by m' over an entire synodic period. The resulting increase in the mean orbital angular velocity n' means that the next conjunction is closer to apocenter.

Similarly, a conjunction after apocenter (point B in Figure 1) results in a net gain of angular momentum by m', a reduction of n', and a tendency for the next conjunction to be again closer to apocenter. The conjunctions thus librate stably about the apocenter of m', preserving the commensurability. Allowing a secular variation of ϖ' does not change this conclusion as the ratio n/n' is adjusted such that conjunctions still librate about the apocenter.

The same arguments applied to a conjunction at points C or D near pericenter show that conjunctions are again driven toward apocenter. The pericenter conjunctions thus correspond to an unstable equilibrium configuration like that of a pendulum near the top of its support. The stable point of the analogous pendulum corresponds to the apocenter conjunction.

Now suppose conjunctions occur repetitively at apocenter with no libration and that the inner orbit is being expanded by tidal interactions with the primary. The orbital period of m will increase, and successive conjunctions will occur slightly after apocenter on the average. Angular momentum will thus be secularly transferred in just the right amount to preserve the commensurability against the tendency of the tide to disrupt it (Goldreich 1965).

Two other characteristics of a stable commensurability can now be understood. First, if conjunctions always occur at apocenter of the outer satellite in this example, the radial force of m on m' accelerates m' toward the primary, and m' follows a trajectory slightly inside the trajectory it would have followed if m were not there. This means m' will reach its closest point to the primary slightly sooner than normal and the line of apsides will have rotated in a *retrograde* sense. If m is sufficiently massive, this regression of the line of apsides due to the resonant perturbation (conjunction always at apocenter) can dominate the normal prograde motion due to the oblateness of the primary and the secular perturbations from other satellites. This surprising result is actually realized in the Titan-Hyperion resonance where the line of apsides of Hyperion's orbit regresses about 19° yr^{-1}.

The second characteristic is the secular increase of the eccentricity in this type of orbital resonance. Recall that a tidal expansion of the inner orbit causes the conjunctions of the stable resonance to occur slightly after apocenter. A radial impulse force anywhere between apocenter and pericenter causes the orbiting body to fall closer to the primary, thereby increasing the eccentricity e' of its orbit. With conjunctions now occurring slightly after apocenter, the maximum of the radial perturbative force tends to increase e' secularly. These effects of the radial perturbing force lead us into the discussion of a second stability mechanism for an eccentricity-type resonance.

In the above example we could allow a secular variation in ϖ', but any effects on ϖ' due to the resonance were assumed small. This is a reasonable assumption as long as the eccentricity of the outer orbit is sufficiently large. For a small eccentricity, the maximum asymmetry of the tangential component of the perturbing acceleration becomes small, and m is much less able to alter the mean motion of m' by a transfer of angular momentum. The stability mechanism by which the conjunctions librate about the apocenter is thereby seriously weakened. On the other hand, for small eccentricity the radial perturbing force (which causes the regression of Hyperion's line of apsides) is much more effective in changing the position of the pericenter. This high mobility of the line of apsides coupled with the variation of e', also caused by the radial perturbation, leads to the situation where the apocenter *or* the pericenter librates stably about the conjunction (Greenberg 1974).

From the earlier discussion of the effect of the radial perturbations on the pericenter and eccentricity, we can deduce that a radial impulsive force toward the primary induces a regression of the line of apsides if applied within 90° of apocenter, and a positive precession if applied within 90° of pericenter. The radial impulse decreases e if applied when m' has passed pericenter but before it reaches apocenter, and it increases e' if applied while m' is on the remaining half of its orbit. Since the radial perturbation of m' by m has a relatively sharp maximum at conjunction, we can assume the radial force to be applied at conjunction and infer the effects on e' and ϖ' by the location of the conjunction relative to ϖ'. Again, there is little change in the mean motion for this mechanism since the tangential accelerations of m' are nearly balanced for small e regardless of the location of the conjunction.

Figure 2 describes the same configuration as Figure 1 except that the eccentricity is now very small. The mean motions may now be relatively far from commensurability, but the mean value of $\dot{\varpi}' = \langle\dot{\varpi}'\rangle$ is sufficiently large in magnitude that the line of apsides maintains its relative position with respect to the conjunctions of m and m' no matter where these conjunctions occur relative to inertial space. We assume the radial perturbation dominates the variation of ϖ', so that for the conjunction near apocenter $\langle\dot{\varpi}'\rangle$ is retrograde, and the conjunctions occur at successively decreasing longitudes. If conjunctions are repeatedly at nearly constant separation

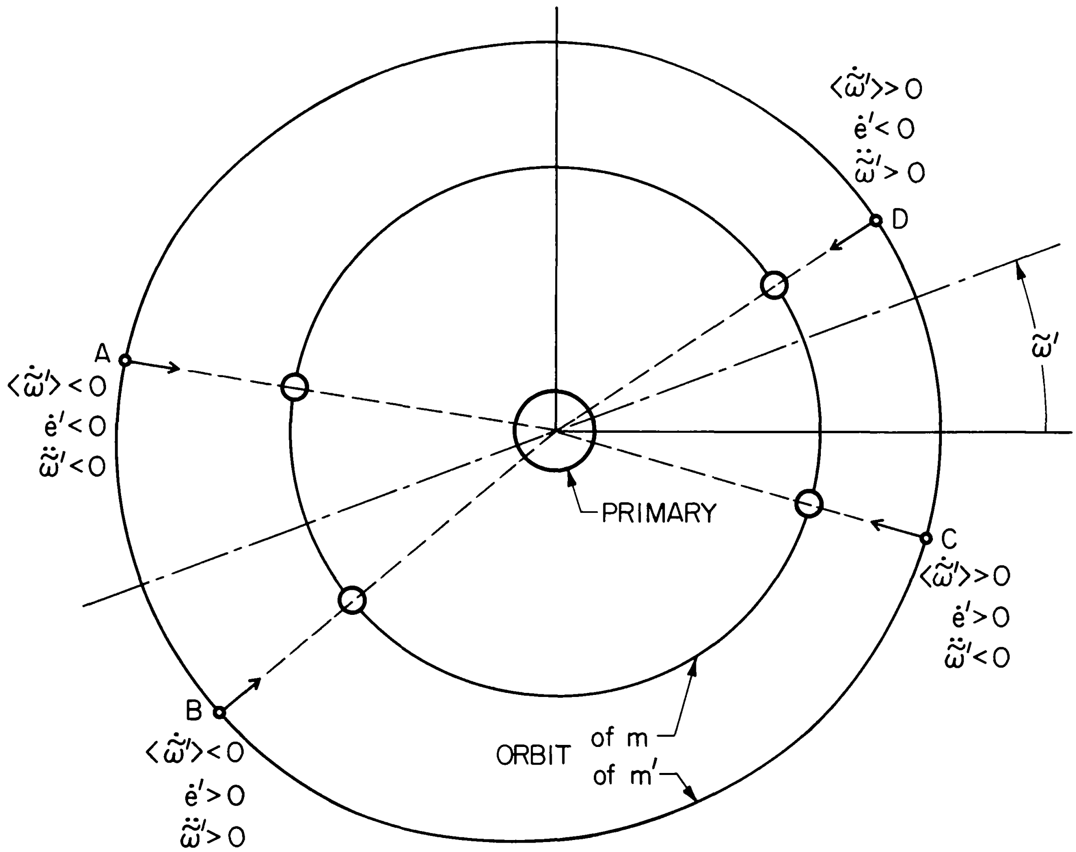

Figure 2 Small-eccentricity stability mechanism. Arbitrary positions of repetitive conjunctions are at points *A, B, C, D*. *e* and ϖ are eccentricity and longitude of pericenter, respectively. See Equation (37).

from apocenter as at point *A* in Figure 2, the eccentricity tends to decrease, which will accelerate the retrograde motion of the line of apsides [see Equation (37)]. The line of apsides then tends to catch up with the conjunctions and will be less distant at the next conjunction. On the other hand, if conjunctions are initially repeatedly separated from apocenter, as at point *B* where they are slightly ahead, e' is increased, and the retrograde motion of ϖ' is decelerated. The apocenter falls behind in its retrograde motion and is closer to the next conjunction. The apocenter thus librates stably about the conjunctions for small e'.

Unlike the large-eccentricity case where libration about the pericenter was not possible, the small eccentricity case allows stabilization of conjunctions near pericenter, as well as apocenter. Repetitive conjunctions near the pericenter induce a prograde motion in ϖ', so successive conjunctions must occur at ever increasing longitudes to maintain the configuration. But the radial perturbation accelerates the pericenter toward conjunction (conjunctions at *C* or *D* relative to ϖ' in Figure 2) by the same arguments leading to apocenter libration. The stable libration at either pericenter or apocenter is similar to the equivalent pendulum librating either above or below its support. This inverted libration was known to Brown & Shook (1933, Chap. VIII), but was only recently clarified for the satellites (Sinclair 1972, Greenberg et al. 1972, Greenberg 1973a, Yoder 1973).

If we now allow the inner orbit to expand due to tidal transfer of angular momentum from the primary, its period increases, and conjunctions for a libration in apocentric stability now tend to fall slightly past apocenter. The eccentricity and hence the restoring acceleration or stability of the resonance increase, $|\langle\dot{\varpi}'\rangle|$ decreases, and the commensurability of the mean motions becomes more exact. This latter conclusion follows since conjunctions are maintained near apocenter in spite of the reduction in $|\langle\dot{\varpi}'\rangle|$. At the same time, as the eccentricity grows the first stability mechanism involving angular momentum transfer becomes dominant. Tidal evolution in this case increases the stability and leads to secular transfer of angular momentum, which maintains the commensurability.

Something very different happens to a st^ble pericentric libration as the inner orbit expands. Conjunctions fall slightly past the pericenter leading to a *reduction* in the mean value of e. The pericentric libration is therefore eventually destroyed by the tidal expansion of the inner orbit.

The low-eccentricity stability mechanism also allows us to describe a method by which the tides carry a noncommensurate pair of satellites into a stable libration (Greenberg 1974). Assume that the inner orbit is being expanded by the tides as before, but that the outer satellite is too small for significant tidal evolution. Also assume that e' is not very large. The orbits tend to approach each other, and the mean motions approach the commensurability. The conjunctions progress in a retrograde sense at an ever decreasing rate as the resonance is neared. This means there are an increasing number of radial "impulses" increasing e' before conjunctions drift into the region where e' decreases and, likewise, an increasing number of radial impulses decreasing e'. The fluctuation in e' thus increases as resonance is approached, and if the initial e is not too large, e' can approach zero on one of its swings. But notice that conjunctions will be in the region of point A in Figure 2 while e' is decreasing, and if e' gets very small, the retrograde motion of ϖ gets very large, and the apocenter will attempt to overtake the conjunction [see Equation (37)]. If it fails to do so and the apocenter falls more than 90° behind, the conjunctions progress on around more slowly, inducing an even larger fluctuation, until eventually apocenter does overtake the conjunctions in the region of point A. The librations are thereafter stabilized by the small-eccentricity mechanism, and the commensurability evolves as described above with e' secularly increasing and the mean motions approaching ever closer to the exact commensurability. Notice that only capture into librations about apocenter is possible and that the capture is certain (Sinclair 1972, Greenberg et al. 1972, Greenberg 1973a, Yoder 1973). If e'_0 is too large, the maximum fluctuation may not reduce its value close enough to zero to allow a large $|\dot{\varpi}'|$. Capture is no longer certain but depends on the values of the various parameters when the first libration occurs. Probabilities of capture as a function of the value of e' far from resonance can be determined either numerically (Sinclair 1972, 1974) or analytically (Yoder 1973, 1974).

Similar heuristic descriptions of stability and evolution are possible for other resonance configurations. The inclination-type resonance of Mimas-Tethys is somewhat more subtle, however, and the reader is referred to Greenberg (1973b) for a lucid physical description of this resonance. Sinclair (1972) considered the opposite direction of approach to a resonance and showed that no capture is possible and no resonance is stable unless the inner satellite of any pair has a sufficiently dominant rate of tidal evolution (see Section 4).

Of course the above physical pictures were deduced from rather extensive mathematical developments, one of which is outlined in Section 4. Before doing so, however, current ideas on the formation of the gaps in Saturn's rings and the asteroid belt are described in the above physical terms.

3 SATURN'S RINGS AND THE ASTEROIDS

Gravitational theories for the origin of the Kirkwood gaps in the asteroid belt (Brouwer 1963, Message 1966) have not withstood subsequent criticism. A mathematical singularity was overlooked in Brouwer's theory (Jefferys 1967). Message depleted the gap region by a phase mixing where the asteroids associated with the gap spent most of their time at the extremes of librations and at the apocenter of highly eccentric orbits. Subsequent orbit determinations have revealed only three confirmed librators in the 2:1 resonance, although several more objects are also likely trapped at this commensurability (Franklin et al. 1975). Among the brighter asteroids, at least the gaps are real and not just a result of most of the gap asteroids being out of phase with the position of the gap at any one instant. Jefferys (1967) pointed out that nongravitational forces in the form of collisions must have been the important mechanism for creating the gaps. Since this idea has an appealing consistency, we use the concepts from Section 2 to describe how it works both for the asteroids and for Saturn's rings.

We have seen how a near commensurability of mean motions generates large fluctuations in the orbital eccentricity as the resonance variable either librates or circulates very slowly. These fluctuations are much smaller for those nearby orbits whose mean motions are not commensurate. If these latter orbits are highly populated, any object librating about a commensurability is likely to suffer a collision when near the apocenter or pericenter of an orbit whose eccentricity is near the extreme of its fluctuation. These collisions depopulate those orbits at the commensurability.

The asteroid belt presents a remaining problem, however. Nearly all the asteroids (about 30 are known) near the 3:2 commensurability with Jupiter (Hilda group) are in fact trapped in libration as is the asteroid Thule in the 4:3 resonance (Schubart 1968). These asteroids would all make close approaches to Jupiter were it not for the fact that the resonance causes the conjunctions to librate about the *pericenter* of the inner orbit. Like the apocentric stability discussed in Section 2, the resonance maximizes the distance of closest approach. Thule and the Hilda group, like the Trojans and Pluto, owe their survival to the orbital commensurability. Sinclair (1969) noted that those orbits near the 4:3 and 3:2 commensurabilities would be depopulated very rapidly by the close encounters with Jupiter (see Lecar & Franklin 1973). The eccentricities of the resonant orbits still fluctuate greatly, but with nearby orbits depopulated the asteroids occupying these resonant orbits suffer no collisions. Those orbits on either side of the 2:1 or 3:1 commensurability are far enough away from Jupiter to be completely stable over the age of the solar system in spite of their nonresonant mean motion, and collisions with these objects can account for the depopulation of the Kirkwood gaps.

Other types of orbital resonance in the asteroid belt were mapped by J. G. Williams (1969; 1973a,b). These "secular resonances" result when the longitude of the node or perihelion has a period matching that of one of the long-period oscillations in the planetary system. Like the resonances between the mean motions, the secular resonances lead to large eccentricities and inclinations, and we find gaps in the distributions of asteroid orbits corresponding to the resonant node and perihelion motions. It is tempting to attribute these gaps to collisions as well. However, Williams (private communication, 1975) noted the wide dispersal of orbital eccentricities and inclinations in the current asteroid belt and pointed out that a sufficient number of collisions to depopulate either the Kirkwood gaps or the secular resonances should have also flattened the distribution of asteroids and circularized their orbits. A possible way out of this dilemma is for perturbations over the age of the solar system to redisperse the eccentricities and inclinations, but he believes that the mechanism for the creation of the gaps is still uncertain.

The collision argument has often been used to qualitatively explain the Cassini division and other gaps in the distribution of Saturn's ring particles, where Williams's comments do not apply. Franklin & Colombo (1970) and Franklin et al. (1971) have refined the arguments to explain some of the details of the ring-particle distribution. Collisions will depopulate the region near the 2:1 and 3:1 commensurabilities with Mimas, and the collisions will persist until noncolliding orbits are obtained. In Section 2, we note that a secular motion of the pericenter causes the ratio n'/n to differ from an exact commensurability. For ring particles, Saturn's oblateness will dominate the pericenter motion, and $\dot{\varpi}$ will be positive. The relation $n-2n'+\dot{\varpi}=0$ for a stable resonance means n will be somewhat smaller than the value for an exact 2:1 commensurability with Mimas, and we indeed find the gap displaced from the exact commensurability toward larger orbits. Franklin et al. (1971) found it necessary to invoke a high mass density for ring B($\gtrsim 0.1$ g cm^{-3}) further increasing $\dot{\varpi}$ in order to account for the entire observed displacement of the Cassini gap from the exact 2:1 commensurability of mean motions. However, Greenberg (private communication, 1975) has found an error in their analysis for the oblateness contribution to $\dot{\varpi}$ of such a magnitude that ring B need contribute little or nothing to $\dot{\varpi}$ for consistency with the observations.

Collisions have thus been able to provide the necessary nongravitational forces to depopulate gaps in regions of a high density of orbiting objects. The collision hypothesis is consistent with the existence of the Trojans, Hilda and Thule asteroids, and with Pluto since nonresonant orbits in these regions were quickly eliminated, while the resonant objects were preserved by the resonance itself. The existence of so many commensurate orbits among the satellites of the major planets, however, has never been accounted for satisfactorily by any collision or other dissipative process during a formation phase. For this reason we turn to tidal friction as a means of altering the original distribution of satellite orbits toward the establishment of stable orbital resonances (Goldreich 1965). The successful analysis that allows us to follow this process of evolution from nonresonant to resonant orbits is discussed in the next section. The internal consistency of the tidal hypothesis supports this means of establishing most if not all of the satellite commensurabilities.

4 ANALYTICAL DEVELOPMENT

The usual approach to the analysis of orbital resonances has been that developed by H. Struve (Tisserand 1896, pp. 125–138), where time derivatives of the orbital elements $a, e, i, \omega, \Omega, \varepsilon$ are expressed in terms of partial derivatives of a disturbing function with respect to these same elements (Lagrange planetary equations). For example, Goldreich (1965), Allan (1969), Sinclair (1972), and Greenberg (1973a) have all used some form of the Lagrange equations in their analyses. Others have used various forms of the restricted three-body problem applied to the Jupiter-asteroid commensurabilities where the perturbations of one of the resonance partners could be ignored (Message 1966, Schubart 1964). Canonical variables are usually used in these later approaches. Sinclair (1970) extended these analyses to the unrestricted problem where both masses of a resonance pair are small compared with the primary, but may be comparable to each other. A constant Hamiltonian was obtained from which the perturbations of both partners could be determined. This treatment was reduced to one degree of freedom in an application to the Enceladus-Dione resonance (Sinclair 1972). Yoder (1973) in a treatment similar in some ways to that of Sinclair's (1972) analysis of Enceladus-Dione starts with modified Delaunay canonical variables and maintains the canonical form of the equations of variation through a series of transformations, which ultimately reduce the problem to one degree of freedom. This last development is the most versatile and has the simplest mathematical form. All two-body resonances satisfying the constraints of the approximation, including the three resonances among Saturn's satellites, can be studied in great detail with relative ease.

The advantage of using the Lagrange planetary equations is that the physical explanations of the resonance behavior, like those in Section 2, are more easily constructed. But they have the disadvantage of being difficult to extend to the most general case. The very great simplification of the mathematical description by Yoder makes the physical picture less immediate, but this is more than compensated by the power of the technique. Every kind of transition from a nonresonant to a resonant orbital motion due to tidal changes in the orbital parameters can be followed in terms of the rather simple mathematical model of a pendulum. Analytic expressions for the probability of capture into an orbital commensurability, which the tides cause a system to approach, are obtained in terms of the parameters defining the orbits and the dissipative properties of the primary. Because of this versatility, we shall describe Yoder's analysis in some detail.

In all cases with which we shall be concerned, the satellites whose orbital periods may be nearly commensurable are orbiting a primary body whose mass M is very much larger than either of the masses m or m' of the two satellites. The orbit of each satellite is therefore nearly an ellipse with the mutual interactions between the satellites resulting in small perturbations of the orbital elements. A Hamilton-Jacobi transformation of the canonical equations of motion of one satellite yields the canonical Delaunay variables [Plummer 1918 (1960), pp. 142–144] which can be modified to (Brown & Shook 1933, p. 132):

$$L = \sqrt{\mu_0 a}, \qquad \lambda$$
$$\Gamma = L\left[\sqrt{1-e^2}-1\right], \qquad \varpi \tag{1}$$
$$Z = L\sqrt{1-e^2}(\cos I - 1), \qquad \Omega$$

where a, e, I are the semimajor axis, eccentricity, and inclination of the orbital ellipse, respectively; Ω is the longitude of the ascending node measured from an inertial line in the primary equator plane; $\varpi = \omega + \Omega$ is the longitude of pericenter with ω being the angle in the orbit plane between the pericenter and ascending node; and $\lambda = \int n\,dt + \varepsilon$ is the mean longitude, where n is the mean orbital angular velocity and ε is the longitude at epoch. Finally, $\mu_0 = G(M+m)$, where G is the gravitational constant.

The time derivatives of the above variables are given by

$$\frac{dL}{dt} = \frac{\partial H}{\partial \lambda} \qquad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial L}$$
$$\frac{d\Gamma}{dt} = \frac{\partial H}{\partial \varpi} \qquad \frac{d\varpi}{dt} = -\frac{\partial H}{\partial \Gamma} \tag{2}$$
$$\frac{dZ}{dt} = \frac{\partial H}{\partial \Omega} \qquad \frac{d\Omega}{dt} = -\frac{\partial H}{\partial Z}$$

where H is the Hamiltonian.

$$H = H_0 + R,$$
$$H_0 = -\tfrac{1}{2}v^2 + \frac{GM}{r} = \frac{\mu_0}{2a} = \frac{1}{2}\frac{\mu_0^2}{L^2},$$
$$R = Gm'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r'^3}\right). \tag{3}$$

The sign of H is consistent with common celestial-mechanics usage. In Equation (3) the zero-order Hamiltonian H_0 is that for two orbiting point masses. The disturbing function R is restricted to that part due to the presence of the second satellite of mass m', where the two terms in R are called the direct and indirect terms, respectively. The position vectors $\mathbf{r}$ and $\mathbf{r}'$ locate m and m' relative to the primary center of mass, and v is the velocity of m. If $R = 0$, then all of the variables are constant except λ where $d\lambda/dt = -\partial H_0/\partial L = n$.

The disturbing function R for the satellite m can be expanded in an infinite series of the form (Allan 1969)

$$R = \sum \frac{Gm'}{a_>} C I^{|l-m-2p|} I'^{|l-m-2p'|} e^{|q|} e'^{|q'|} \cos \phi_{lmpp'qq'} \tag{4}$$

with a similar expansion for the disturbing function for m' due to the presence of m. In Equation (4)

$$\begin{aligned} \phi_{lmpp'qq'} &= (l-2p+q)\lambda - (l-2p'+q')\lambda' - (l-m-2p)\Omega \\ &\quad + (l-m'-2p')\Omega' - q\varpi + q'\varpi', \end{aligned} \tag{5}$$

where $\lambda - \varpi$ has replaced the mean anomaly. In Equation (4) $a_>$ is the larger of the two semimajor axes, and C is a series in $\alpha = a_</a_>$, e^2, e'^2, I^2, I'^2, whose lowest-order term is $O(a_</a_>)^l$ (i.e. the lowest-order term in C does not contain e or I). The summation indices are the integers $2 \leqq l < \infty$, $0 \leqq p \leqq l$, $0 \leqq m \leqq l$, $-\infty < q < \infty$.

An important characteristic of the terms in the series in Equation (4) is the equality of the magnitude of the coefficient of ϖ with the lowest power of e in the coefficient of the cosine and the magnitude of the coefficient of Ω with the lowest power of I (see Brown & Shook 1933, p. 141). Also notice that when R is expressed in terms of the canonical variables, L, Γ, Z are confined to the coefficients of the cosines, and λ, ϖ, Ω are confined to the arguments. Rotational invariance requires the sum of the coefficients of the angle variables in each argument to be zero.

Near a low-order commensurability of the mean motions of the two satellites, the frequency of some terms becomes very small. Often the combination of a large coefficient and small frequency makes the perturbations due to a single term completely dominant. If this is the case, we can write

$$H(J, \omega, J', \omega') = A_0(J, J') + A_1(J, J') \cos \phi$$

and (6)

$$H'(J', \omega', J, \omega) = A'_0(J', J) + A'_1(J', J) \cos \phi$$

for the respective Hamiltonians governing the motions of the two partners. The canonical variables defined in Equation (1) are represented by J, ω; and ϕ is the slowly varying resonance variable now written as

$$\phi = j\lambda + j'\lambda' + k\varpi + k'\varpi' + i\Omega + i'\Omega', \tag{7}$$

where j, k, i are identified with the appropriate expansion indices in Equation (5).

The identification of a resonance by a single resonance variable ϕ allows a convenient classification based on the form of ϕ and the associated leading term in the coefficient of the cosine. These classifications are given in Table 2.

The Hamiltonians in Equations (6) can be reduced to one-dimensional form by using the fact $(dL/dt)/j = (d\Gamma/dt)/k = dZ/dt/i = dx/dt$, so that

$$L = jx + L_0, \qquad \Gamma = kx + \Gamma_0, \qquad Z = ix + Z_0. \tag{8}$$

With $y = j\lambda + k\varpi + i\Omega$, Hamilton's equations reduce to

Table 2 Classification of resonances

ϕ	Type	Coefficient	Example (Table 1)
$j(\lambda'-\lambda)$	Synodic	O(1)	Trojan asteroids
$j\lambda+j'\lambda'+k\varpi$	Simple e	$O(e^{\|k\|})$	Titan-Hyperion
$j\lambda+j'\lambda'+i\Omega$	Simple I	$O(I^{\|i\|})$	
$j\lambda+j'\lambda'+i\Omega+i'\Omega'$	Mixed I	$O(I^{\|i\|}I'^{\|i'\|})$	Mimas-Tethys
$j\lambda+j'\lambda'+k\varpi+k'\varpi'$	Mixed e	$O(e^{\|k\|}e'^{\|k\|})$	

$$\frac{dx}{dt}=\frac{\partial H}{\partial y}(x,y,x',y') \qquad \frac{dx'}{dt}=\frac{\partial H'}{\partial y'}(x',y',x,y)$$
$$\frac{dy}{dt}=\frac{\partial H}{\partial x}(x,y,x',y') \qquad \frac{dy'}{dt}=-\frac{\partial H'}{\partial x'}(x',y',x,y) \tag{9}$$

where $\phi = y+y'$, and x is seen to be the fluctuation in the action variables.

From Equation (4), R' differs from R only in the factors C' and m, so $A'_1(x',x) = (m/m')(C'_1/C_1)A_1(x,x') = K(m/m')A_1(x,x')$. If we can ignore the variation of K, $dx'/dt = K(m/m')dx/dt$ or $x' = K(m/m')x$ from Equations (6) and (9), which leads to

$$\frac{dx}{dt}=\frac{\partial\tilde{H}}{\partial\phi}(x,\phi); \qquad \frac{d\phi}{dt}=-\frac{\partial\tilde{H}(x,\phi)}{\partial x}, \tag{10}$$

where

$$\tilde{H}=s(x)+\frac{m'}{Km}s'(x)+v(x)+A_1(x)\cos\phi \tag{11}$$

is a constant Hamiltonian with one degree of freedom. We have written $A_0(x,x') = s(x)+v(x,x')$ and $A'_0(x',x) = s'(x')+v'(x,x')$, where $s(x)$ is the secular part of H, which is independent of m', and $v(x, x')$ is the secular part of H, which depends on m'. In Equation (11), $K(m/m')x$ has replaced x' wherever it appears in $s'(x')$, $v(x,x')$ and $A_1(x,x')$. If the resonant term has any contribution from the indirect term in R [Equation (3)], then $C_0/C'_0 \neq C_1/C'_1$ and the canonical form of Equations (10) and (11) can only be written if we can neglect $v(x)$ (which we usually can). For the 2:1 simple e-type resonance, which does have a contribution from the indirect term, it happens that $C_1 = C'_1$ and Equations (10) and (11) apply with $K = 1$, which is the case in all resonances where the indirect term in R is not involved. Finally, we justify keeping C and C' constant by noting that the lowest-order term in C does not contain e or I, and the next higher-order terms are $O(e^2)$ or $O(I^2)$, which we can neglect for small e and I. We can thus consider C an expansion in $\alpha = a_</a_>$, which generally can be expressed in terms of Laplace coefficients (e.g. Brouwer & Clemence 1961, p. 490). The fractional fluctuation in C is comparable to that in $a_<$ or $a_>$, which in turn is like that in L or L'. From Equation (8) $\delta C/C_0 = \delta L/L_0 = j\delta x/L_0 = j(x_{max}-x_{min})/L_0$. But $A_1(x,x')$ also contains at least one factor of e or I from

Equation (4), and for small e or I, $\Gamma \approx -Le^2/2$ and $Z \approx -LI^2/2$, which leads to $\Delta e/e_0 \approx (k/je_0^2 - \frac{1}{2})j\delta x/L_0$. So the fractional fluctuation in e for small e is a factor e_0^{-2} larger than the fluctuation in C, and the variation in the coefficient of $\cos\phi$ is almost entirely determined by the x dependence of e. The same argument holds for factors of I, and keeping C constant is always a good approximation for small e and I.

The last simplification of the mathematical description of orbital resonances involves the expansion of the secular part of $\tilde{H}$ in a rapidly converging series in x to $O(x^2)$. The zero-order term in $\tilde{H}$ is $\mu_0^2/\{2[L(x)]^2\} + (m'/Km)\mu_0'^2/\{2[L(x)]^2\}$, which is clearly expandable since $\delta x/L_0 \ll 1$. The remaining secular terms like that due to the oblateness of the primary or to another satellite also have no leading factors of e or I and are thus similarly expandable. These secular terms are vital for separating the frequencies of nearly resonant periodic terms (e.g. see Allan 1969), but for transition into and evolution within a single resonance they do nothing but shift the zero of the resonant frequency. Only the zero-order part of $\tilde{H}$ is therefore retained in the expansion to $O(x^2)$.

This expansion takes the form

$$\tilde{H}_0 = A_0 + xA_{0x} + \frac{x^2}{2}A_{0xx}, \tag{12}$$

where

$$A_0 = \frac{\mu_0}{2a_0} + \frac{m'}{Km}\frac{\mu_0'}{2a_0'} \tag{13}$$

$$A_{0x} = -jn_0 - j'n_0' \tag{14}$$

$$A_{0xx} = \frac{3j^2}{a_0^2} + 3K\frac{m}{m'}\frac{j'^2}{a_0'^2}. \tag{15}$$

Each subscript x indicates a differentiation with respect to x. Inclusion of the remaining secular terms would change the magnitude of the coefficient only slightly. If we complete the square for the terms in x in Equation (12), absorb the remaining terms not involving x into $\tilde{H}$, and define a new time $\bar{t} = A_{0xx}t$, then we can write

$$H_f(x,\phi) = \tfrac{1}{2}(x+c)^2 + b(x)\cos\phi \tag{16}$$

$$\frac{dx}{d\bar{t}} = \frac{\partial H_f}{\partial\phi}; \qquad \frac{d\phi}{d\bar{t}} = -\frac{\partial H_f}{\partial x} \tag{17}$$

where

$$c = A_{0x}/A_{0xx}, \qquad b(x) = A_1(x)/A_{0xx}, \qquad H_f(x,\phi) = H(x,\phi)/A_{0xx}. \tag{18}$$

The differential equations obtained by Allan (1969), Sinclair (1972), and Greenberg (1973a) follow directly from Equations (16) and (17). Hereinafter we shall drop the subscript on H.

The form of H in Equation (16) is identical to that of a pendulum with an applied constant torque if the x dependence of b is ignored. The analogy is established with

b corresponding to $-g/l$; $x+c(t)$, to $-\dot{\phi}$; and the torque, to $-dc/dt$, where g is the acceleration of gravity and l is the length of the pendulum. The introduction of the tides gives c its time dependence. This allows us to discuss the transition into the evolution within an orbital resonance in terms of a pendulum initially rotating over the top of its support but being slowed by a torque until libration is established with subsequent damping of the libration. Many of the important two-body orbital resonances can be handled with Equation (26), including those whose partners have comparable masses. The values of the parameters and the form of $b(x)$ [through $A_1(x)$] change from resonance to resonance. Resonances not involving e or I such as the Trojan asteroids cannot be handled with this formulation, but such resonances did not evolve through tidal evolution in any case. [See Brown & Shook (1933, Chap. 9) for a treatment of the Trojan asteroids.] Also, any case with two or more terms with nearly equal resonant frequencies and with comparable coefficients cannot be treated since we have assumed that only one term is dominant in R. Finally, we assumed the disturbing function was expandable in powers of α, e, and I and that perturbations were small. This eliminates crossing orbits or those situations where close approaches are possible. The three two-body resonances among Saturn's satellites are well described by Equation (16), and this procedure may provide a route to understanding the establishment and evolution of the three-body resonance among Jupiter's Galilean satellites as a combination of two stable two-body resonances (Sinclair 1975; Yoder 1976, to be published).

Stability of Librations

One can easily determine those values of ϕ and x about which stable librations may occur (Yoder 1973). At such a point (x_0, ϕ_0), $(\dot{x}_0, \dot{\phi}_0) = (0, 0)$, and the system remains close to this point if slightly perturbed. This latter condition is described by the existence of closed curves about (x_0, ϕ_0) generated by $H(x_0+\delta x, \phi_0+\delta\phi)$ in a region surrounding the test point. From Equation (17), $\dot{x} = 0$ at $\phi = \pi$ or 2π, and $\dot{\phi} = 0$ yields $(x_0, \pi) = (-c+db/dx, \pi)$, and $(x_0, 2\pi) = (-c-db/dx, 2\pi)$, which we shall call π and 2π libration centers, respectively, if stable librations exist. The expansion of $H(x_0+\delta x, \phi_0+\delta\phi)$ about an equilibrium point takes the form of a quadratic in δx and $\delta\phi$ by neglecting higher-order terms. This curve is an ellipse and librations therefore are stable if $H_{x\phi}^2 - H_{xx}H_{\phi\phi} < 0$, where again the subscripts indicate differentiation. This leads to stability criteria

$$\begin{aligned} &b(1-b_{xx}) > 0, \qquad \phi_0 = \pi \\ &b(1+b_{xx}) < 0, \qquad \phi_0 = 2\pi. \end{aligned} \tag{19}$$

If $|b_{xx}| < 1$ at both the π and 2π centers, $b(\pi) > 0$ and $b(2\pi) < 0$ for stability. This is similar to a simple pendulum (where $b_{xx} = 0$), which always has only one stable center. If $|b_{xx}(\pi)|$ and $|b_{xx}(2\pi)|$ are both > 1 and $b/b_{xx} < 0$, the system can librate about either the π or 2π centers, but if $b/b_{xx} > 0$, neither center is stable.

Recall the example in Section 2 where libration about both apocenter and pericenter was possible for small e. This was a simple e-type resonance where $b(x)$ has a leading factor $e^{|k|}$. From Equations (4) and (18) and the approximation $\Gamma \approx -Le^2/2$,

$$b=\frac{GmC'e_0'^{|k'|}}{a_> A_{0xx}}$$

$$b_x=\frac{GmC'e_0'^{(|k'|-2)}|k'|k}{a_> A_{0xx}L_0}$$

$$b_{xx}=\frac{GmC'e_0'^{(|k'|-4)}k'^2|k'|(|k'|-2)}{a_> A_{0xx}L_0^2}. \tag{20}$$

For a sufficiently small value of the eccentricity e_0 $|b_{xx}(0)|>1$ for $|k|<4$ but only for $|k|=1$ does b_{xx} have the opposite sign of b. This includes the Titan-Hyperion and Enceladus-Dione resonances. The maximum value of e_0 that will allow stability at the π libration center is determined by $|b_{xx}(\pi)|=1$, which corresponds, for example, to $e_{En}(\pi)<0.0062$ and $e_{Hy}(\pi)<0.025$. Currently the average values are $\langle e_{En}\rangle=0.0044$ and $\langle e_{Hy}\rangle=0.104$. Enceladus could librate stably in the inverted position, since $e(\pi)<\langle e\rangle$.

Tidal Evolution

The tides raised on a homogeneous, spherical primary by a satellite lead to (Goldreich 1965)

$$\frac{dn}{dt}=\frac{27}{4}\frac{m}{M}\frac{n^2}{Q}\left(\frac{a_e}{a}\right)^5 \tag{21}$$

and

$$\frac{dL}{dt}=\frac{9}{2}\frac{m}{M}\frac{n^2}{Q}\frac{a_e^5}{a^3} \tag{22}$$

where M is the primary mass and a_e the equatorial radius. The importance of tidal evolution to the formation of orbital resonances comes from the strong dependence of dn/dt on m and a. The differential rate of the orbital evolution means that satellites originally in nonresonant orbits can gradually approach a commensurability (Goldreich 1965). Within a resonance, the tides lead to a change in the coefficient of the restoring term and change the amplitude of libration (see Section 2).

The tidal effects are inserted into the Hamiltonian of Equation (16) as a perturbation by replacing n and L where they occur by

$$n_0+\int\frac{dn}{dt}dt; \qquad L_0+\int\frac{dL}{dt}dt.$$

However, near a resonance $A_{0x}=-jn-j'n'$ is very near zero, so the fractional change in this coefficient is much larger than the fractional change in A_0 or A_{0xx}. Hence, we can neglect the tidal change in these later coefficients and write

$$A_{0xx}^2c(t)=-A_{0xx}(jn_0+j'n_0')-\left(j\frac{dn}{dt_0}+j'\frac{dn'}{dt_0'}\right)(\bar{t}-\bar{t}_0) \tag{23}$$

where $\bar{t}=A_{0xx}t$ as before. There is also an x dependence in the perturbation, which can be retained by expressing Equations (21) and (22) in terms of $L(x)$ and expanding to first order in x. The partial derivative in the canonical equations does not act on this x dependence in dL/dt, however. We generally omit the x dependence in Equation (23), but should note that it becomes important in the evolution of a resonance when the coefficient of $\bar{t}-\bar{t}_0$ is very small, as may happen in the Mimas-Tethys case (Allan 1969). Finally, the tidal variation of the coefficient of the cosine has little effect on the approach to and transition into a stable commensurability, although there is a secular change in this coefficient within the resonance (e.g. Allan 1969).

From Equations (16) and (17)

$$\frac{d\phi}{dt} = -x - c - b_x(x)\cos\phi, \tag{24}$$

so that far away from the resonance where $(d\phi/dt)/A_{0xx} = d\phi/d\bar{t} = (jn+j'n'+k\dot{\varpi})/A_{0xx}$ (for example) is not near zero, $d\phi/dt \approx -c$ and the x and cos ϕ terms in Equation (24) yield only small fluctuations. It is always possible to choose the signs so that $d\phi/dt > 0$ far away from resonance, then $c < 0$; and the only way the tides can bring a system to a resonant configuration is for the *magnitude* of c to decrease. Hence,

$$A_{0xx}^2 \frac{dc}{dt} \approx -j\frac{dn}{dt_0} - j'\frac{dn'}{dt_0} \tag{25}$$

must be positive.

The description of the orbital resonance has been reduced to that of a pendulum, which is rotating over the top of its support ($d\phi/dt > 0$, corresponding to nonresonant motion or circulation of the resonance variable) and being slowed by a torque (dc/dt due to the tides) toward a state where $d\phi/dt$ vanishes at least momentarily. The latter situation can lead to the stable libration exhibited by several satellite pairs in the solar system. Stability in the presence of the tidal torques is maintained by the dominance of the maximum restoring torque (Section 2). Many similarities to the description of spin-orbit coupling (Goldreich & Peale 1966, 1968) are evident. However, for spin-orbit coupling b was constant or time dependent instead of x dependent. This x dependence of b provides a much greater variety of evolutionary tracks toward the establishment of an orbital resonance compared with the spin resonance. Since the form of $b(x)$ is peculiar to each type of resonance, each presents its own mathematical problems.

Transition

By transition we mean either the capture from a nonresonant rotating state into a stable libration or the passage through and subsequent escape from such a resonance. In spin-orbit coupling this transition was handled by following the variation of ϕ directly (Goldreich & Peale 1966, 1968). The tides gradually decrease $\dot{\phi}$ such that a plot of $\dot{\phi}^2$ vs ϕ appears schematically as in Figure 3, where $\dot{\phi}$ reverses sign at $\dot{\phi} = 0$ and ϕ subsequently decreases. Capture into a librating state is possible if the variation of $\dot{\phi}^2$ is asymmetric about $\dot{\phi} = 0$, as shown in Figure 3. A second zero

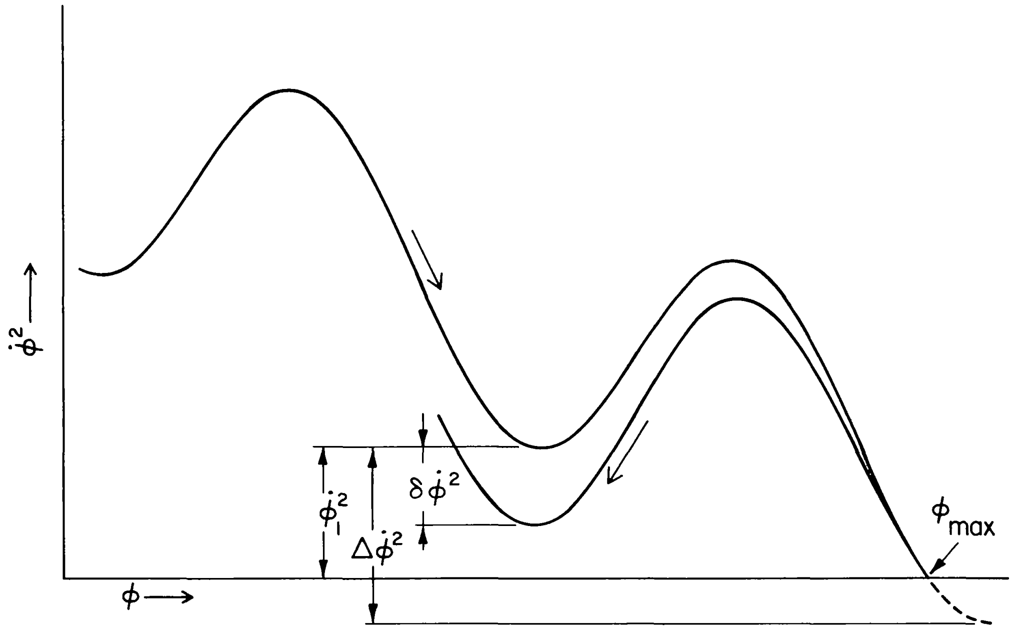

Figure 3 Reduction of $\dot{\phi}^2$ due to tides as a function of the resonance variable ϕ, showing asymmetry about transition. Capture probability is $\delta\dot{\phi}^2/\Delta\dot{\phi}^2$.

in $\dot{\phi}$ is then possible. If the value of $\dot{\phi}^2$ as the equivalent pendulum passes over the top the last time before $\dot{\phi} = 0$ is equally likely to have any value between 0 and $\Delta\dot{\phi}^2$ indicated in Figure 3, then the probability of capture into the resonance is

$$P = \frac{\delta\dot{\phi}^2}{\Delta\dot{\phi}^2}, \tag{26}$$

where $\delta\dot{\phi}^2$ is also defined in Figure 3. This definition has been adopted by Sinclair (1972, 1974) and Yoder (1973).

The necessary asymmetry in $\dot{\phi}^2$ about $\dot{\phi} = 0$ in the spin-orbit case is provided by a substantial decrease in the tidal torque as the resonance is traversed or alternatively by a core-mantle interaction, which cannot apply here. However, for a resonance between two orbital motions, the effect of the tides given by dc/dt is nearly constant during the transition phase and yields a negligibly small capture probability. It is the x dependence of the coefficient b that provides the asymmetry in $\dot{\phi}^2$ for the orbital resonance. This x dependence destroys the simplicity with which P is calculated for spin-orbit capture. Although the definition of P in Equation (26) can still be applied, a different approach is expedient in determining $\delta\dot{\phi}^2$ and $\Delta\dot{\phi}$ (Yoder 1973).

From Equations (16) and (17)

$$\frac{dx}{dt} = -b\sin\phi, \tag{27}$$

where we can eliminate $\sin\phi$ using Equation (16) to yield

$$t - t_0 = \int \frac{\text{sign}(-b\sin\phi)dx}{\sqrt{D(x)}}, \tag{28}$$

where

$$D(x) = b_{(x)}^2 - [H - \tfrac{1}{2}(x+c)^2]^2.$$

The variation in x is confined between two real roots of $D(x)$. From Equation (27) the roots bounding x correspond to $\phi = 2n\pi$ or $(2n+1)\pi$, where n is an integer. One can label a root with the value of ϕ and the sign of $\dot{\phi}$ when x assumes the root value. Hence, it is always possible to uniquely specify the nature of the motion—positive or negative circulation or libration—by the labels on the bounding roots. Evolution of a system due to tides can then be ascertained by following the motion of the roots and by keeping track of their labels in the complex plane. Although this process is often nontrivial (Yoder 1973), one is able to define analytically all the important parameters governing the evolution, including analytic expressions for the probability of capture at transition. This often avoids the necessity of numerical calculations for each specific resonance (e.g. Sinclair 1972, 1974).

Transition from positive circulation into negative circulation (escape) or positive circulation into libration (capture) begins with the coincidence of two roots on the real axis or one root reaching a value such that $b(x)$ evaluated at that root vanishes. The coincidence of two real roots with their subsequent motion off the real axis in the complex plane is associated with a probabilistic capture. The quantities $\delta\dot{\phi}^2$ and $\Delta\dot{\phi}^2$ in Equation (26) are determined from time integrals of $d(\mathrm{Im}x_\pi^2)/dt$ where $\mathrm{Im}x_\pi$ is the imaginary part of a ($\phi = \pi$) root, which left the real axis after coincidence with another ($\phi = \pi$) root. The vanishing of $b(x_\pi)$ is associated with certain capture, which is described in Section 2. The condition separating a certain capture with a probabilistic one is simultaneous coincidence of three ($\phi = \pi$) roots, two of which were previously complex.

Since various evolutionary tracks are defined by specific associations and configurations of the roots of $D(z)$, it is possible to completely determine the values of all the parameters in the problem separating the various possibilities and defining the capture probabilities. This determination requires using the adiabatic invariance of the action of an oscillating or rotating system

$$J = \oint x d\dot{\phi} \tag{29}$$

when the parameters are varied slowly compared with the oscillation period (Kulsrud 1957, Gardner 1959, Lenard 1959). The integral is over one complete cycle. Since the tidal changes are very slow indeed, the zero value of J far from transition [x represents positive and negative fluctuations of $\dot{\phi}$ about its mean value $(-c)$, see (24)] means that $J = 0$ at transition and (29) represents the necessary additional relation to determine the values of all the parameters at transition. Once in libration, J is also conserved adiabatically, and its value calculated at transition (time integration being over a libration instead of a rotation) can be used to follow the tidal evolution within a resonance.

We present the results of these exercises for the simple e-type resonance, of which the Titan-Hyperion and Enceladus-Dione commensurabilities are examples. Because of its formal identity, the simple I type is also included here. It is convenient to define dimensionless variables by $x' = x/(-\Gamma_0), c' = c/(-\Gamma_0), H' = H/\Gamma_0^2, b'(x) = b(x)/\Gamma_0^2, \bar{t}' = (-\Gamma_0 \bar{t})$. Dropping all the primes and the bar on t, we can write

$$b(x) = \beta(-kx+1)^{1/2} \tag{30}$$

$$\beta = \frac{4Gm'}{\mu_0}\frac{Ce_0^{(|k|-4)}}{A_{0xx}a_> a_0} \tag{31}$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial \phi} = -\beta(-kx+1)^{|k|/2}\sin\phi \tag{32}$$

$$\frac{d\phi}{dt} = -\frac{\partial H}{\partial x} = -x-c+\frac{|k|k}{2}\beta(-kx+1)^{\left(\frac{|k|}{2}-1\right)}\cos\phi \tag{33}$$

where $-\Gamma \approx e^2L/2 = -kx-\Gamma_0$ from Equation (8) has been used.

Both positive and negative values of k are retained, although the reason may not be immediately obvious. A resonance variable may be defined by $\phi = \lambda - 2\lambda' + \varpi$ for $k = +1$. But changing the sign of k only changes the sign of ϕ since the sum of the coefficients must vanish. The assumed sign of ϕ certainly can have no effect on the analysis of the resonance. Still, a negative k has a very important distinction. Recall that $\dot{\phi} = jn + j'n' + k\dot{\varpi}$ was chosen positive far from resonance, and it was assumed that tides were driving the system toward resonance ($dc/dt > 0$). For $k = 1$, $\dot{\phi} = n - 2n' + \dot{\varpi} > 0$ indicates that the inner satellite is orbiting too fast for the resonance, and the only way the resonance can be approached is for the inner orbit to expand sufficiently fast such that the rate of decrease of n is more than twice the rate of decrease of n'. For $k = -1$, $\dot{\phi} = -n + 2n' - \dot{\varpi} > 0$ means that the inner satellite is orbiting too slow, and the orbits must expand such that the rate of decrease of n is less than twice the rate of decrease of n'.

Hence, $k = +1$ means that the approach to a resonance is dominated by expansion of the inner satellite orbit (successive conjunctions circulate in retrograde sense), whereas $k = -1$ means an approach to the resonance from the other direction (successive conjunctions circulating in the prograde sense). The importance of this distinction is that Yoder (1973) found that capture into a resonance for $k = -1$ or -2 is impossible. *Hence, the only way the tides can lead to stable commensurabilities is for the ratio of the periods to approach a commensurability* p/(p+q) *from smaller values.* Sinclair (1972) reached this conclusion in his careful analysis of Mimas-Tethys and Enceladus-Dione. We see later that this fact may place some constraints on the frequency and amplitude dependence of Q for Saturn (Greenberg 1973c, Yoder 1973).

The condition that a system approaching a resonance enter stable libration with certainty is $|\beta| > \frac{1}{8}$ for $k = +2$ and $|\beta| > 0.272$ for $k = +1$. For values of $|\beta|$ less than these critical values, the probability of capture [Equation (16)] is

$$P = \frac{2}{1+\pi/(2\arcsin\delta)}, \tag{34}$$

where

$$\delta = \frac{2x_\pi - x_{2\pi-} - x_4}{x_4 - x_{2\pi-}} \leqq 1. \tag{35}$$

The roots of $D(x)$ in Equation (35) are labeled with the value of ϕ and the sign of $\dot{\phi}$ when x has the value of the root. The root x_4 is a 2π +root for $k = 2$ and also for $k = 1$ when $|\beta| < 0.21$. For $0.21 < |\beta| < 0.27$, x_4 is a π +root. The roots are evaluated at the time that two π roots coincide, marking the beginning of the transition. When $\delta = 1$, $|\beta| = 0.27$, above which value $P \equiv 1$. A simple e or I system with $k = 1$ can be trapped temporarily in the inverted libration mode for $0.21 < |\beta| < 0.27$. Such a libration state is unstable to further tidal evolution, however, and escape from the resonance to negative circulation always occurs (see Section 2).

In Figure 4, the capture probability is shown as a function of β/β_c where β_c is the critical value, with $|\beta| > |\beta_c|$ meaning certain capture. If $|\beta| \ll |\beta_c|$, the small value of the coefficient of cos ϕ means that the maximum fluctuations δx are small. Since $\beta \propto e_0^{|k|-4}$ or $I_0^{|k|-4}$, small fluctuations correspond to large values of e_0 or I_0 for $|k| < 4$. In the small fluctuation limit it is possible to derive the approximate expression (Yoder 1973),

$$P = \frac{2}{1+\pi|\beta|^{1/2}/[2b_x(0)]}. \tag{36}$$

As the derivative $b_x \propto \beta$, the second term in the denominator is very large, and $P \propto |\beta|^{1/2}$. The small-fluctuation approximation is seen to be valid for $P \lesssim 0.1$ for $k = 2$, and $P \lesssim 0.5$ for $k = 1$ in Figure 4. At transition, c can be evaluated along with all the other parameters with the aid of the action integral for positive rotation [Equation (29)]. The action integral is re-evaluated for libration at transition with the known parameters, where its value is thereafter adiabatically invariant as c continues to increase. The invariant action with a given value of β allows one to follow the decrease in the libration amplitude and the increase in the magnitude of

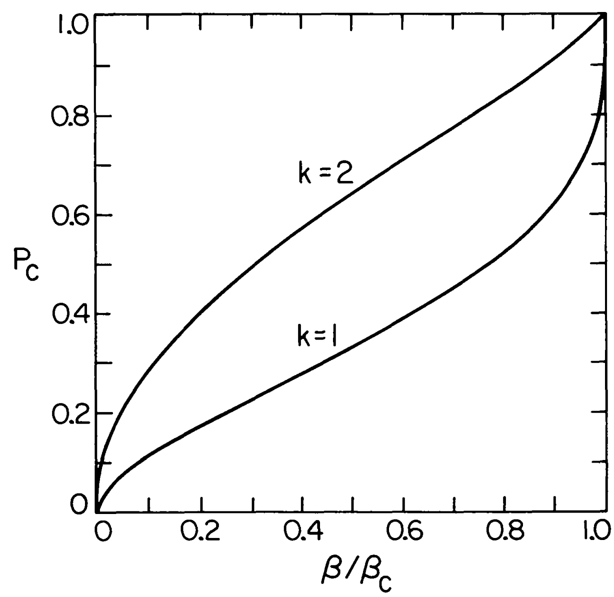

Figure 4 Capture probability for orbital resonances as a function of β/β_c. (After Yoder 1973.)

$b(x)$ within the resonance as a function of c. This is related to time through Equations (21) and (29), and one sees how the time scale for the evolution is directly related to the dissipation factor Q for the primary body. Alternatively, the current parameters for a commensurability can be integrated backward in time until the amplitude of libration reaches 180° or until $b(x)$ vanishes at one point in its fluctuation (e.g. Allan 1969). Given Q, this determines an age for the resonance.

In Figure 5, the value of c_0 at transition is shown as a function of $|\beta|$ for the simple resonance with $k = 1$. For large $|\beta|$, $c_0 \approx \beta/2$. The break at $|\beta| = |\beta_c| = 0.27$ results from the different mode of transition for $|\beta| >$ or $< |\beta_c|$. Subsequently larger values of c define new values of the parameters consistent with the invariant action integral. Figures 6 and 7 show the amplitude ϕ_m of libration as a function of $c(t)$ for two ranges of $|\beta|$. The particular function of $c(t)$ was chosen as abscissa in Figure 6 to emphasize the linear portion for large $|\beta|$, where $\sin \phi_m \approx 2c/\beta$ for $\beta \gg 1$ as long as $c/\beta \lesssim |\beta|^{-1/3}$ (Yoder 1973, 1974). A different set of coordinates was chosen for Figure 7 since the amplitude is reduced extremely slowly for small $|\beta|$ and the abscissa $\approx e/e_0$. The positive values of c implied in Figures 6 and 7 do not mean $-jn - j'n' > 0$ since $\dot{\phi}$ in Equation (33) contains x whose mean value becomes increasingly negative. We can use Equation (23) to convert the value of c into a time as long as $\left| j\frac{dn}{dt} - j'\frac{dn'}{dt} \right|$ does not change too much due to the secular transfer of angular momentum. Sinclair (1974) has shown this to be true until the mean motions are quite close to the commensurability.

For the simple e-type resonance with $k = 1$, $e = e_0(-x+1)^{1/2}$. At $\phi = \phi_m$, $\dot{\phi} = 0$ so Equation (33) yields the extreme values of x for a given ϕ_m and from Figures 6 and 7 for a given $c(t)$. With $\langle x \rangle = (x_{2\pi+} + x_{2\pi-})/2$ the average value of e is known as a function of time. Also from Equation (32) there is a secular change in $\langle x \rangle$ since

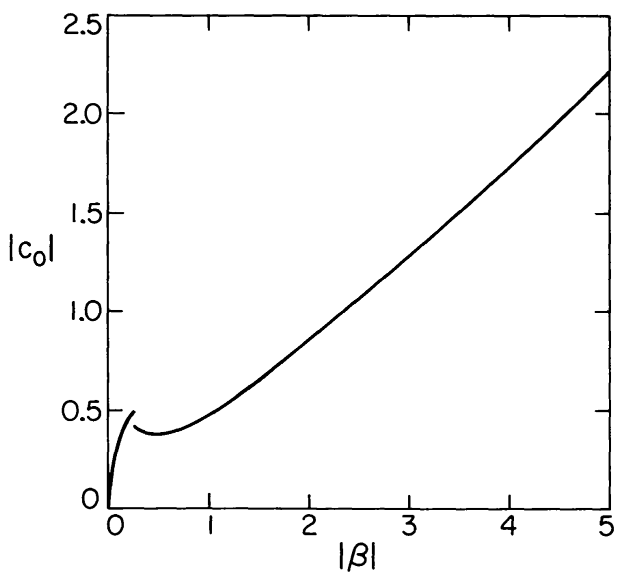

Figure 5 Value of parameter c at transition for use in determining amplitude evolution in Figures 6 and 7. (After Yoder 1973.)

the librations are *not* about $\phi = 0$ when the tides are included. For the normal librations considered here $\langle x \rangle$ increases its *negative* magnitude such that e secularly increases within the resonance. The complete time behavior of an orbital com-

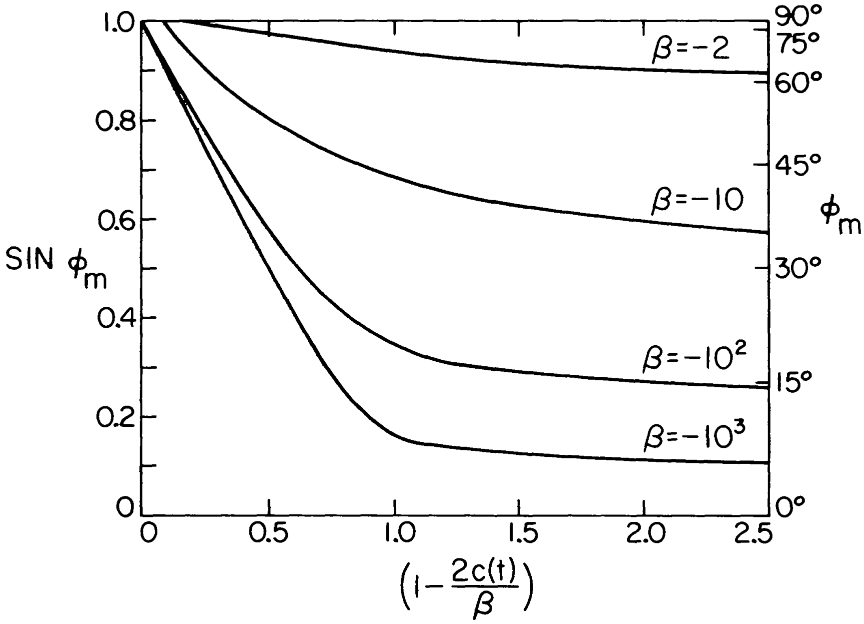

Figure 6 Libration amplitude as a function of c and β for large values of $|\beta|$. (After Yoder 1973.)

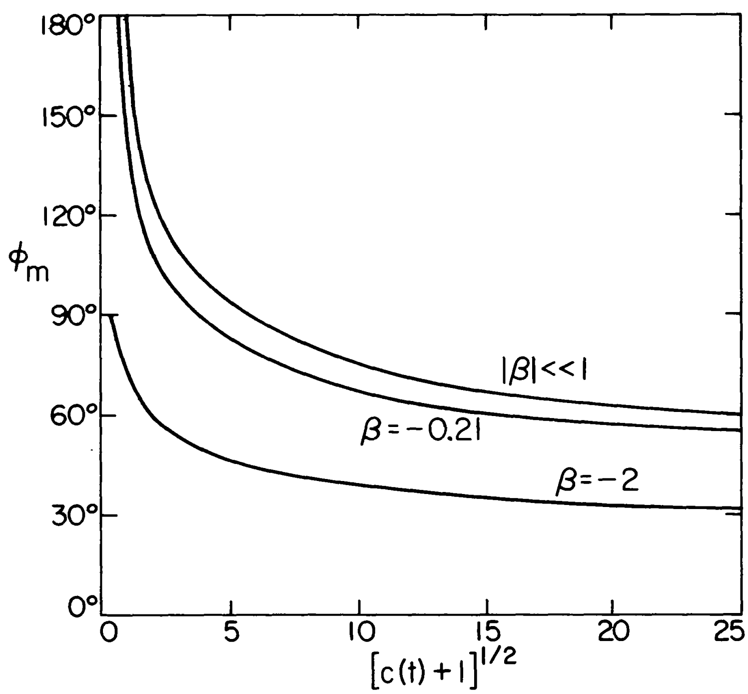

Figure 7 Libration amplitude as a function of c and β for small values of $|\beta|$. (After Yoder 1973.)

mensurability can thus be followed in detail. This process cannot be continued indefinitely, however, for eventually the magnitude of e becomes too large for the approximations to be valid. The success of the recent theoretical understanding of the tidal evolution of orbital commensurabilities has encouraged much discussion of their possible histories, which we follow in the next section.

5 DISCUSSION

Because of their relative simplicity, the resonances among the satellites of Saturn have received the most attention and are the best understood. We therefore consider the three two-body commensurabilities in this system. Table 3 lists additional data to which we shall refer.

All three resonances satisfy the restrictions that the tidal change in c [Equations (21) and (25)] is very slow, so that treating the tides as a perturbation is valid. This is especially so as $Q \gtrsim 6 \times 10^4$ in Equation (21), where the bound was established by the limited recession of Mimas from Saturn in 4.6×10^9 years (Goldreich 1965, Goldreich & Soter 1966).

The Mimas-Tethys resonance is a 2:1 mixed I type, where the conjunctions librate about the midpoint of the two nodes of the orbit planes on Saturn's equator plane. We have explicitly assumed in Section 4 that all but one term in the disturbing function can be neglected, but it is not a priori obvious that this assumption is valid for a given resonance. For example, near the 2:1 commensurability the arguments $(2\lambda-4\lambda'+2\Omega), (2\lambda-4\lambda'+\Omega+\Omega'), (2\lambda-4\lambda'+2\Omega'), (\lambda-2\lambda'+\varpi'), (\lambda-2\lambda'+\varpi)$ are all slowly varying. (The prime always refers to the outer satellite.) However, the time variations of node and pericenter due to the oblateness of Saturn are sufficiently rapid that frequencies of the above arguments are actually well separated, as can be vertified from Table 3 (Allan 1969). Compare the 78.8-year libration period with $\dot{\Omega}-\dot{\Omega}' = -293° \text{ yr}^{-1}$.

The Neptune-Pluto resonance is apparently a case where at least two resonance variables are simultaneously involved. Not only do conjunctions librate about the apocenter of Pluto's orbit (Cohen & Hubbard 1965), corresponding to a simple e-type resonance, but the apocenter and hence the conjunctions also librate about a point 90° from the node of Pluto's orbit plane on Neptune's (Williams & Benson 1971). There has been no published analytical analysis of this dual libration, although Greenberg & Franklin (1975) have successfully explained coupled modes in 2:1 librating asteroids.

For Mimas-Tethys, $-dn/dt+2dn'/dt \approx 2 \times 10^{-23}$ rad sec^{-2} [Equation (21)] is positive, and the system evolves through the resonances in the order in which they are listed above. The commensurability ratio $\frac{1}{2}$ is approached from smaller values, so this system has the possibility of being trapped at any one of them with subsequent damping of libration. The positive value of dc/dt deduced here depends on assigning the same value of Q to the tidal dissipation from both satellites, which will not be the case if Q is frequency or amplitude dependent. The relative tidal acceleration of the two satellites almost matches the commensurability relation such that $(-dn/dt+2dn'/dt)/(dn/dt) \approx 0.1$. This means the Mimas-Tethys system evolves very

Table 3 Orbital data for satellites of Saturn

Satellites	Semi-major axis (10^3 km)	Period (days)	Eccen-tricity	Incli-nation	Mass (m/M)	$(d\varpi/dt)_{sec}$ (°/yr)	$(d\varpi/dt)_{res}$ (°/yr)	$(d\varpi/dt)$ (°/yr)	$(d\Omega/dt)$ (°/yr)	C	Libra-tion ampli-tude	Libra-tion period (years)
Mimas	186	0.942	0.0201	1°.517	6.7×10^{-8}	365.60	0.0	365.60	−365.23	−0.4086	97°	70.78
Tethys	295	1.888	0.0000	1°.093	1.14×10^{-6}	~72.0	~0.0	~72.0	−72.2			
Enceladus	238	1.370	0.0044	1′.4	1.27×10^{-7}	152.4	−29.3	123.1	~−152	−1.19	1.5°	12
Dione	377	2.737	0.0022	1′.4	1.8×10^{-6}	30.7	0.0	30.7	~−30.7			
Titan	1222	15.95	0.0289	0°.3	2.4×10^{-4}	0.50	0.0	0.50	−0.49	3.26	36°	2
Hyperion	1481	21.28	0.1042	0°.5	2×10^{-7}	3.49	−22.14	−18.65	−3.20			

slowly, and the x dependence of c (corresponding to the changing dn/dt due to the tides) becomes important in the evolution within the resonance (Allan 1969, Yoder 1973).

The order in which the resonances are encountered was deduced from the current values of $\dot{\varpi}$ and $\dot{\Omega}$ in Table 3, which are dominated by the oblateness of Saturn. However, from Equations (1), (2), (3), and (4), an approach to a resonance induces the variations

$$\left(\frac{d\varpi}{dt}\right)_{\text{res}} = n_0|k|\frac{m'}{M}\frac{a_0}{a_>}Ce_0^{(|k|-2)}; \quad \left(\frac{d\Omega'}{dt}\right)_{\text{res}} = n_0\frac{m'}{M}\frac{a_0}{a_>}C\frac{I'}{I_0} \tag{37}$$

for the simple e and lowest-order mixed I type, respectively. For $k = 1$, $d\varpi/dt$ will be large and negative ($C < 0$) for small e_0. The node can have a large negative motion if $I_0 \ll I'$, but the presence of I' in the numerator means the node motion will nearly always be dominated by the nonresonant secular perturbations. It is much more likely that a small value of e_0 (or e'_0) can induce such a large retrograde motion in ω (or ω') that a simple e-type resonance can be encountered before any of the inclination resonances and can automatically enter libration (see Section 2). The first encountered e-type resonance would have been that involving e' (Tethys), and since presently $e' \ll 1$ this would make ϖ' most likely to have the large retrograde motion as the system approached the 2:1 commensurability. From Equation (37), $e'_0 < 1.3 \times 10^{-6}$ in order for $|(d\varpi'/dt)_{\text{res}}| > |d\Omega/dt|$. Since such a small value of e'_0 is extremely unlikely, it appears safe to assume that the Mimas-Tethys system first encountered the well-separated inclination resonances in the order given (Yoder 1973, 1974). We need but account for its avoidance of the first I-type resonance and capture into the second.

Since $m'/m \approx 17$ for this resonance, we can neglect the variation in Tethys's parameters and approximate the resonance as a simple I type. Then $\beta = 4m'CI'_0/(MA_{0xx}a_0a'_0I^3)$ and $A_{0xx} = 12/a_0^2 + 48m/(m'a'^2)$ where again the prime refers to the outer satellite. The current values of the parameters yield $\beta_{\text{now}} = -9.2 \times 10^{-5}$. This is so much smaller in magnitude than the critical value $\beta_c = -0.27$ that the small fluctuation limit $|\beta| \ll 1$ should apply throughout the evolution. In this limit, $\langle x \rangle \approx -c(t)$ in libration, and $c(t) \approx 0$ at transition (Yoder 1973) (see Figure 5). From Table 3 and Figure 7 the current amplitude of libration is 97°, corresponding to $[c(t)+1]^{1/2} \approx 4$. But $b(\langle x \rangle) = \beta(-\langle x \rangle + 1)^{1/2} = \beta[c(t)+1]^{1/2} = b(0)[c(t)+1]^{1/2}$, where $b(0)$ is the value of b at transition. Thus $b_{\text{now}}/b_{(0)} = I_{\text{now}}/I_0 \approx 4$, where I_0 corresponds to the value of I when the libration amplitude is 180°. However, we pointed out earlier that the x dependence of c was important in this particular resonance because of the near cancellation of the tidal decelerations in dc/dt. When the x dependence is included, a numerical integration is sometimes necessary, and the amplitude is 180° when $I_0 = 0\overset{\circ}{.}4156$ (Allan 1969). This corresponds to a value of β at transition of -4.5×10^{-3}, which verifies the small fluctuation limit throughout the evolution. The probability of capture is then [Equation (36)] $P \approx 4b_x(0)/\pi|\beta|^{1/2} \approx 4.3\%$, which was first obtained numerically by Sinclair (1972).

The inclination of $0\overset{\circ}{.}42$ now applies as the Mimas-Tethys system passed through the first-encountered, simple I-type resonance. In this case $\beta = 8.33 \times 10^{-4}$ and $P = 7.3\%$ by Equation (36), which was also numerically verified by Sinclair (1972). We can thus account for Mimas-Tethys skipping the first resonance encountered and stopping in the second because $|\beta| \ll |\beta_c|$ and the captures are probabilistic.

The Enceladus-Dione pair is a 2:1 simple e type with $\beta = 4m'C/(MA_{0xx}\, aa'e_0^3)$ with $A_{0xx} = 3/a^2 + 12m/(m'a'^2)$. From Table 3, the current values of the parameters lead to $\beta_{\text{now}} = -19$. Since $|\beta_{\text{now}}| \gg |\beta_c|$ and must have been larger in the past, the large-fluctuation limit applies throughout the history of this resonance. The system entered libration with certainty (Sinclair 1972, Yoder 1973). However, Sinclair (1974) noted that the existing resonance should be the last one to be encountered of the five associated with the 2:1 commensurability. Also, the values of both orbital inclinations are so small that $|\beta| > |\beta_c|$ for the inclination resonance and should have been captured there. These I-type resonances could have been avoided only if the inclinations were larger in the past to make the captures probabilistic. However, this problem vanishes if we perform the same exercise on this pair as we did earlier for Mimas-Tethys.

For $|\beta| \gg 1$, $\sin \phi_m = 2c(t)/\beta$, and it can also be shown that $\langle e \rangle / \langle e_0 \rangle = \beta/2c(t)$ in this limit (Yoder 1973, 1974). Thus, $\sin \phi_m = \langle e_0 \rangle / \langle e \rangle$, where e_0 is the mean value of e in the circulating phase before transition. Two values of ϕ_m are quoted in the literature: $\phi_m = 1\overset{\circ}{.}5$ (Sinclair 1972), for which $e_0 = 1.15 \times 10^{-4}$ and $\phi_m = 20\overset{\prime}{.}0$ (Goldreich 1965), for which $e_0 = 2.6 \times 10^{-5}$. For the larger value of e_0, $(d\varpi/dt)_{\text{res}} + (d\varpi/dt)_{\text{sec}} = -952^\circ\ \text{yr}^{-1} \ll d\Omega/dt = -152^\circ\ \text{yr}^{-1}$, where Equation (37) was used. If Dione's eccentricity was no smaller during positive circulation of the resonance variables than it is now, $(d\varpi'/dt)_{\text{res}} + (d\varpi'/dt)_{\text{sec}} = 27.4^\circ\ \text{day}^{-1}$. Hence, the simple e-type resonance in which we find Enceladus and Dione was encountered long before any of the other resonances, and we need not require special circumstances to account for their avoidance. Again the tidal hypothesis is consistent with the state of Enceladus and Dione, since the resonance in which we find them, and the one into which they were captured with certainty, is the first one encountered as the orbits expand. Also, the small amplitude of libration is accounted for by the rapid damping for $|\beta| \gg 1$ (see Figure 6).

The remaining commensurability, Titan-Hyperion, is a 4:3 simple e type whose properties fall midway between the extremes of the other two pairs. As $C > 0$, the conjunctions of this system librate about $\phi = \pi$ or the apocenter of Hyperion's orbit. The value of $|\beta_{\text{now}}| = 4mC/(MA_{0xx}\, a'^2 e'^3)$, with $A_{0xx} = 48/a'^2 + 27(m'/m)/a^2$, is 0.058, and it is apparent that neither approximation $|\beta| \gg |\beta_c|$ or $|\beta| \ll |\beta_c|$ is valid. The general procedure employing the action integral J [Equation (29)] must be used to determine the history of this resonance. At the present time we can evaluate $\dot{\phi}$, H, and e' at the extreme of a libration $\phi = \phi_m = 36^\circ$. These relations along with $\beta = \beta_{\text{now}}(e/e_0)^3$ allow us to express H and hence J in terms of e_0. This latter integral is evaluated numerically as a function of e_0', and e_0 is determined by the condition that J equal its value at transition. The important parameters thus deduced are $e_0 = 0.022$, $|\beta| = 5.85$, $c_0 = -2.62$, $c_{\text{now}} = 20.1$, where c_0 is determined from the action integral evaluated at transition (Yoder 1973, 1974). Automatic capture is assured (e.g. Greenberg et al. 1972, Greenberg 1973a).

From Table 3 we see that even today, with Hyperion's large eccentricity, the retrograde motion of the apsides exceeds that of the node. From Equation (37), this retrograde motion was considerably higher at transition when $e_0 = 0.022$, and the first resonance encountered by the Titan-Hyperion system is where we find it today. From Figure 7 we see that for the reasonably small value of $|\beta|$ appropriate to the Titan-Hyperion case, the amplitude of libration decreases very slowly after its initial drop from 90°. The value of ϕ_m after a time is more indicative of the value of β (i.e. the mean eccentricity) before transition than of the age of the resonance with larger $|\beta|$ (smaller e_0) leading to smaller amplitudes. This was shown by extensive numerical integrations of the Titan-Hyperion system (Colombo et al. 1974). These authors also found that e_0 near 0.02 leads to asymptotic amplitudes near the present 36°, which is also consistent with the above analytical results. Thus, Titan and Hyperion seem to fit very well into the pattern of tidal origin of commensurabilities. However, we have so far implicitly assumed that the tidal evolution of the orbits has been of sufficient magnitude to accomplish the observed evolution. That this is not necessarily so is shown by the ages of the resonances for the given value of Q for Saturn.

First, we can show that the inner two resonances could have evolved sufficiently for the tides to have accounted for their existence. For Mimas-Tethys, the current 97° amplitude and $c(t) \approx 0$ at transition leads to $\Delta c \approx 15$ from Figure 7. Using Equations (21) and (25), with $Q = 6.5 \times 10^4$, we find $-dn/dt + 2\,dn'/dt = 2 \times 10^{-23}$ rad sec^{-2} and a time since transition τ of about 6×10^8 years. Allan (1969) finds $\tau = 2.4 \times 10^8$ years by a numerical integration of a more accurate approximation. This is much younger than the solar system, and the tides are thus capable of forming the commensurability.

For Enceladus-Dione, $\beta = 1.06 \times 10^6$ and $\Delta c = 5.17 \times 10^5$ with $\phi_m = 1.^\circ5$. With $Q = 6.5 \times 10^4$ and Equations (21) and (25), $-dn/dt + 2\,dn'/dt = 1.1 \times 10^{-23}$ rad sec^{-2}, and the age $\tau \approx 1.7 \times 10^9$ years. The tides are again adequate.

For Titan $-3dn/dt + 4dn'/dt \approx 4.6 \times 10^{-25}$ rad sec^{-2}, and the time required for the evolution of this system is about 6×10^{10} years by the above procedure. This exceeds the age of the solar system by more than an order of magnitude and led Goldreich (1965) and Sinclair (1972) to abandon the tidal hypothesis for this particular resonance. Hence, the Titan-Hyperion example may have resulted from a primordial chance distribution of objects around Saturn, and it exists today because the resonance has prevented close encounters with Titan (see also Colombo et al. 1974). On the other hand, all the other properties of this resonance are consistent with a tidal origin, and it is tempting to try to salvage the hypothesis.

The lower bound on Q for Saturn was determined by the limited evolution of Mimas. However, if the Q appropriate to the tide from Titan were 2–4×10^3, the evolution time scale is again within the age of the solar system (Colombo et al. 1974). One could have a large Q for the Mimas tide and a small Q for the Titan tide with the proper assignment of amplitude and frequency dependence. However, such an exercise is very arbitrary, and one must be careful not to destroy the tidal hypothesis for the other two resonances in the process. The sign of dc/dt for these resonances could be reversed for some assignments of amplitude and frequency dependence (Greenberg 1973c; Yoder 1973, 1974). The small libration amplitude of the Enceladus-Dione resonance is perhaps the most convincing support of the tidal hypothesis, as it is very improbable to have resulted from random processes.

There is one possible observational test for the origin of the Titan-Hyperion resonance. The rotation rate of Hyperion is unknown, but the current eccentricity of its orbit leads to a capture probability of about 0.24 for trapping in the Mercury-like $\frac{3}{2}$ *spin*-orbit resonance (Peale 1974). If Hyperion is found in this *spin* resonance, e would have to have been large when tides had slowed it to this value, which was $\lesssim 10^9$ years from the date of formation. The current large eccentricity would have been primordial, and the resonance would have been due to chance events. Finding Hyperion synchronously rotating does *not* prove the converse, however.

The remarkable success of the tidal hypothesis for the origin of at least two of the orbital commensurabilities among Saturn's satellites has not been transferred to the three-body commensurability among Jupiter's satellites. Sinclair (1975), in addition to the Laplace relation, found three two-body resonance variables which are also librating. Although this suggests a tidal origin (Goldreich 1965), no consistent evolutionary scheme has explained the extremely small amplitude of libration of the three-body commensurability (Sinclair 1975). Yoder (1976, to be published) noted that the two-body dc/dt must be positive for stability of the Laplace resonance, and suggests (Yoder, private communication, 1975) that some of the difficulties in the evolutionary scheme might be resolved by allowing for eccentricity changes due to tidal dissipation in the satellites themselves. There also exists a Laplace relation for the satellites of Uranus (Greenberg 1975), but the resonance variable is slowly circulating rather than librating. It is curious that there are also no two-body commensurabilities among Uranus's satellites (Greenberg 1975), and a comparative study of Uranus and Jupiter systems may lead to our understanding of the details of origin and damping of the well-known Laplace relation at Jupiter. Surely this is the outstanding problem in the study of orbital resonances.

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