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INTERNAL DUST IN GASEOUS NEBULAE

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ABSTRACT

A least-squares fit of observed ratios of hydrogen lines and radio fluxes in the planetary nebula NGC 7027 to the corresponding theoretical values, with a two-parameter reddening law, reveals the existence of a large amount of internal dust in this nebula. A similar analysis of observations in the Orion Nebula indicates an optical thickness for absorption between 2.5 and 9 at Hβ of a dust component in this nebula. The interpretation of the infrared fluxes observed in these two nebulae as the thermal radiation from dust grains within the gaseous sources is thus greatly strengthened.

Subject headings: infrared — interstellar reddening — Orion Nebula — planetary nebulae

I. INTRODUCTION

The deviation of observed intensity ratios of hydrogen lines in gaseous nebulae from the theoretical values, predicted by the recombination theory, is commonly interpreted as the reflection of interstellar reddening (Aller and Liller 1968; Kaler 1970). A few observed lines and the radio continuum flux are used to determine the parameter of the reddening law, which then serves in the correction of the observed intensities of all the other lines in the nebula. The reddening law usually used has the form

$$\log I_\lambda{}^t = \log I_\lambda{}^o + cf(\lambda)\,, \tag{1}$$

where $I_\lambda{}^t$ is the theoretical intensity and $I_\lambda{}^o$ is the observed one, $f(\lambda)$ is the reddening function, and c is a parameter characteristic to the nebula under consideration. Equation (1) is the logarithmic expression of the familiar extinction law

$$I_\lambda{}^o = I_\lambda{}^t \exp(-\tau_\lambda{}^y)\,, \tag{2}$$

where $\tau_\lambda{}^y$ is an optical depth, expressed in terms of the reddening function as

$$\tau_\lambda{}^y = yf(\lambda)\,. \tag{3}$$

The parameters y and c are related through $y = c \ln 10$.

The optical depth in equation (2) is that of an absorbing layer between the source and the observer. In gaseous nebulae, the absorbing matter—namely, the dust grains—could be present within the light source itself, and not necessarily only in the foreground. In a medium of gas well mixed with an absorbing dust, the intensity at the boundary is given by

$$I_\lambda{}^b = I_\lambda{}^t[1 - \exp(-\tau_\lambda{}^x)]/\tau_\lambda{}^x\,. \tag{4}$$

Here $I_\lambda{}^t$ is the expected intensity at the boundary, had no dust existed within the source, and $\tau_\lambda{}^x$ is the optical depth of the dust component of the medium. Equation (4) expresses the two main effects of internal dust on the Balmer decrement as found by Cox and Mathews (1969). It implies that for small optical depths the reddening law due to internal dust is roughly parallel to the curve of reddening by external dust and that there is a reduction in the intensity of all the Balmer lines emitted by the nebula. If $g(\lambda)$ is the reddening function of the internal dust in the nebula, we can write

$$\tau_\lambda{}^x = xg(\lambda)\,. \tag{5}$$

The presence of dust grains in the close neighborhood of gaseous nebulae has been suggested in the last few years as an interpretation to the large infrared fluxes observed in some planetary nebulae and H II regions (Gillett, Low, and Stein 1967; Krishna Swamy and O'Dell 1967; Harper and Low 1971; Gillett, Merrill, and Stein 1972). We do not anticipate that the dust near a gaseous nebula will be necessarily well mixed with the gas, but in a nebula with a large infrared excess some mixture may be expected. In § IV we shall show that the final results are rather insensitive to the exact functional dependence of the internal reddening on $\tau_\lambda{}^x$. Equation (4) is therefore probably a better approximation to the intensity of the radiation at the boundary of such nebula than just $I_\lambda{}^t$, the recombination-theory intensity. The general expression for an observed radiation from a gaseous nebula is therefore

$$I_\lambda{}^o = I_\lambda{}^t \frac{1 - \exp[-xg(\lambda)]}{xg(\lambda)} \exp\,[-yf(\lambda)]\,. \tag{6}$$

For the ratio of the two line intensities $S_{ij} = I_{\lambda_i}/I_{\lambda_j}$ we have

$$S_{ij}{}^o = S_{ij}{}^t D_{ij}(x, y)\,, \tag{7}$$

where

$$D_{ij}(x, y) = \frac{g(\lambda_j)\{1 - \exp\,[-xg(\lambda_i)]\}}{g(\lambda_i)\{1 - \exp\,[-xg(\lambda_j)]\}} \exp\,\{-y[f(\lambda_i) - f(\lambda_j)]\} \equiv G_{ij}(x)F_{ij}(y)\,. \tag{8}$$

Throughout this work we shall take for $S_\lambda{}^t$ the intensity ratios computed by the recombination theory for case B. This case is probably more accurate for the two nebulae that we shall mainly discuss (Miller and Mathews 1972; Batchelor and Brocklehurst 1972). If case A or an intermediate case prevailed, the numerical results would have been somewhat different but the main qualitative result would remain unchanged. Balmer self-absorption is also very unlikely in the two nebulae (Cox and Mathews 1969). Expression (4) with the recombination-theory intensities $I_\lambda{}^t$ does not take into account the indirect effect of the dust on the Balmer decrement, through extinction in the frequencies of the Lyman lines. This, however, affects the decrement mainly by reducing the effective optical thickness of the nebula in the Lyman lines. For nebulae with large optical depths in the Lyman lines, the weakening of case B will be small and in any case it will change only slightly the numerical results.

When intensities of several hydrogen lines are measured in a nebula, by using equation (7) we can find by a least-squares method the values of x and y that give the best fit of the observed ratios to the corresponding theoretical values. If the fit with $x \neq 0$ is significantly better than the fit with $x = 0$, we may infer the existence of dust within the gaseous nebula, independently of the evidence from observed infrared fluxes. The differences between internal and external reddening were discussed extensively by Mathis (1970). The analytic expression (7) and the least-squares procedure provide a method to utilize these differences quantitatively for the detection of internal dust in gaseous nebulae and, under favorable circumstances, perhaps even for an estimation of its amount.

II. THE REDDENING LAW

For the external reddening function $f(\lambda)$ we take the Whitford interstellar reddening law (Johnson 1968), normalized to $f(\infty) = 0$. If we assume that the dust grains within gaseous nebulae have the same extinction properties as interstellar grains in the range of the visible to the radio regions of the electromagnetic spectrum, then the optical depth of internal dust for extinction is $xf(\lambda)$. If a nebula has a large optical depth for scattering, $\tau_\lambda{}^x$ in equation (4) is the optical depth for true absorption since scattering events in the nebula on the average do not remove photons from the beam intercepted by the telescope. Thus if a_λ is the albedo of the internal dust grains, we have

$$\tau_\lambda{}^x = xf(\lambda)(1 - a_\lambda)\,.$$

The infrared fluxes observed in gaseous nebulae and their interpretation as thermal emission from dust grains heated by ultraviolet radiation imply that the albedo of the grains is smaller than 1 in the ultraviolet. For particle sizes of the order of 0.1 μ or smaller, the albedo is roughly proportional to $1/\lambda^3$ (Kerker 1969). Therefore, if internal grains are small, one expects a_λ in the visible and in the infrared regions of the spectrum to be of the order of 0.1 or smaller. If the albedo is indeed small for this or other reasons, or if the nebula is not optically thick for scattering, in which case $\tau_\lambda{}^x$ in equation (4) contains also a scattering component, we have

$$g(x) \simeq f(x)\,.$$

In § IV we shall show that one can relax these assumptions a great deal without affecting the results significantly. For $g(\lambda) = f(\lambda)$ the function $D(x, y)$ satisfies

$$\lim_{x \to 0} D(x, y) = D(0, y + x/2)\,. \tag{9}$$

Thus a small amount of internal dust is indistinguishable from external dust by this function. The exact minimal value of x that may still be detected depends strongly on the nature and the quality of the observed data. In order to determine the two parameters x and y, one needs at least two values of S^o. We consider the ratios of hydrogen lines or radio fluxes to the flux in Hβ. Let λ_1 and λ_2 be the wavelengths of the lines in the numerators of the two S^o ratios. The larger $|\lambda_1 - \lambda_2|$, the more pronounced is the difference between the fit with $D(0, y)$ and the fit with $D(x \neq 0, y)$. When this difference is larger than the average scattering of the observational points around the reddening curve, the presence of internal dust in the nebula may be ascertained.

The smallest observational errors claimed so far in measurements of line intensities in gaseous nebulae are those of Miller and Mathews (1972), who measured the decrement of the first four Balmer lines in NGC 7027 with errors not exceeding 5 percent. We found that with the three Balmer-line ratios alone, values of x as large as 2.6 are still indistinguishable from $x = 0$, even with this accuracy; the function $D(2.6, y_1)$ could still be fitted to $D(0, y_2)$ within the observational errors. In order to identify $x \neq 0$ in a nebula one needs a broader wavelength base. The best candidates for a search for internal dust are therefore nebulae with a large reddening parameter c, for which radio-flux or Paschen-line intensities, as well as Balmer-line intensities, are known with high accuracy.

III. RESULTS

The least-squares method was applied to a few planetary nebulae and to the Orion Nebula. Table 1 lists S^o, the observed ratio of fluxes in hydrogen lines or in the radio continuum to the flux in Hβ, for NGC 7027, and S^t, the corresponding theoretical values for an electron temperature $T_e = 20{,}000^\circ$ K. The observed Balmer-line ratios are the set B of Miller and Mathews (1972). The ratio of the radio flux to the flux in Hβ is the average between the value computed by Miller and Mathews and the value given by Cahn and Kaler (1971). The table also gives W, the relative observational error for each line ratio, estimated on the basis of Miller and Mathews's report. The

TABLE 1

RATIOS OF HYDROGEN LINES AND RADIO FLUXES TO THE FLUX IN Hβ FOR NGC 7027

Parameter	Hδ	Hγ	Hα	Radio
S^o	0.156	0.307	7.58	24.77
W	5	3	1	15
S^t	0.264	0.476	2.74	1.00

NOTE.—In tables 1 and 2, S^o = observed values, S^t = theoretical values, W = observational errors (in percent).

error in the radio datum was estimated from the difference between the value of Miller and Mathews and that of Cahn and Kaler. The absolute error in each value of S^o is $e = WS^o/100$.

Similar data for the Orion Nebula are given in table 2. The observed values are those of Peimbert and Costero (1969) at their point I, and S^t is given for $T_e = 10{,}000°$ K. The values of W were assigned according to the relative intensities of the lines. In both tables the theoretical values S^t are those of Brocklehurst (1971).

We define the function

$$\phi(x, y) = \sum_k \left[\frac{S_k{}^o - S_k{}^t D_k(x, y)}{e_k}\right]^2 , \tag{10}$$

where k runs over all the intensity ratios considered. We then find a minimum point of the function $\phi(x, y)$ by the maximum-neighborhood method, developed by Marquardt (1963).

With the Whitford reddening law $f_W(\lambda)$ for $f(\lambda)$ and $g(\lambda)$ we find for NGC 7027 a minimum value of $\phi(x, y)$ at $X_m = 2.6$, $y_m = 1.8$, corresponding to the values $\tau_m{}^x = 2.9$ and $\tau_m{}^y = 2.1$, where $\tau_m{}^x = x_m g(\lambda_\beta)$ and $\tau_m{}^y = y_m f(\lambda_\beta)$. Table 3 gives the predicted values S^p of the intensity ratios where $S^p = S^t D(x_m, y_m)$. The table also gives $r = (S^o - S^p)/e$, the error in the computed value, relative to the observational error, and also the value of ϕ at the minimum point.

Setting $x = 0$ constant and letting y vary, we obtain for NGC 7027 a minimum point at y_n corresponding to $\tau_n{}^y = 3.0$. Table 4 gives the predicted values of the intensity ratios when $D(0, y_n)$ is used, along with the corresponding values of r and the value of $\phi(0, y_n)$ for this nebula.

Figure 1 is a plot of the values of S^o/S^t for NGC 7027, and of the two reddening curves $D(x_m, y_m)$ and $D(0, y_n)$ versus the inverse wavelength $1/\lambda$.

For the Orion Nebula we take as $g(\lambda)$ the reddening law for the Orion Sword region, as given by Johnson (1968), normalized to $g(\infty) = 0$. With $f(\lambda) = g(\lambda)$ or with $f(\lambda) = f_W(\lambda)$ we find a minimum of $\phi(x, y)$ at x_m and y_m corresponding to $\tau_m{}^x = 6.0$, $\tau_m{}^y = 0.0$. With $x_n = 0$, a minimum point is found for y_n corresponding to $\tau_n{}^y = 1.2$. Tables 5 and 6 give for the Orion Nebula the same data that are given in tables 3 and 4

TABLE 2

RATIOS OF HYDROGEN LINES TO THE INTENSITY OF Hβ IN THE ORION NEBULA

Parameter	H9	Hδ	Hγ	Hα	P9	P7	P6
S^o	0.060	0.229	0.427	4.07	0.059	0.138	0.251
W	25	15	10	5	25	20	15
S^o	0.073	0.260	0.469	2.85	0.025	0.055	0.090

TABLE 3

PREDICTED RATIOS OF LINES AND RADIO FLUXES IN NGC 7027 WITH $\tau_m^x = 2.9$, $\tau_m^y = 2.1$

Parameter	Hδ	Hγ	Hα	Radio
S^p	0.151	0.309	7.576	25.127
r	0.58	−0.19	0.06	0.10
ϕ min............			0.387	

NOTE.—S^p = predicted values, r = errors in the computed ratios relative to the observational errors, and ϕ min = $\phi(x_m, y_m)$.

TABLE 4

PREDICTED RATIOS OF LINES AND RADIO FLUXES IN NGC 7027 WITH $\tau_n^x = 0.0$, $\tau_n^y = 3.0$

Parameter	Hδ	Hγ	Hα	Radio
S^p	0.148	0.303	7.581	20.515
r	1.078	0.424	0.01	1.145
ϕ min............			2.654	

NOTE.—S^p = predicted values, r = errors in the computed ratios relative to the observational errors, and ϕ min = $\phi(x_n, y_n)$.

TABLE 5

PREDICTED RATIOS OF HYDROGEN LINES IN THE ORION NEBULA WITH $\tau_m^x = 5.95$, $\tau_m^y = 0.0$

Parameter	H9	Hδ	Hγ	Hα	P9	P7	P6
S^p	0.064	0.234	0.436	4.149	0.057	0.138	0.241
r	−0.25	−0.14	−0.22	−0.39	0.12	0.007	0.27
ϕ min.........				0.371			

NOTE.—S^p = predicted values, r = errors in the computed ratios relative to the observational errors, and ϕ min = $\phi(x_m, y_m)$.

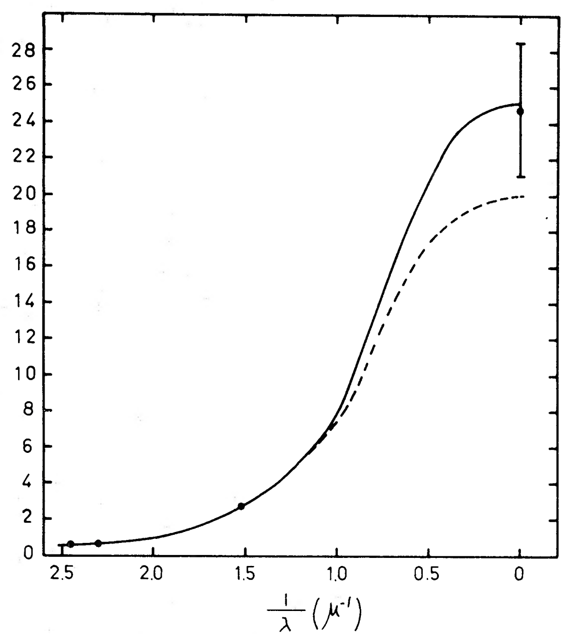

FIG. 1.—Reddening curve and the observed S^o/S^t in NGC 7027. *Solid line*, $D_{1/\lambda}(2.56, 1.84)$; *broken line*, $D_{1/\lambda}(0.0, 2.65)$; $f(\lambda)$ and $g(\lambda)$ are Whitford's reddening laws.

TABLE 6

PREDICTED RATIOS OF HYDROGEN LINES IN THE ORION NEBULA WITH $\tau_n{}^x = 0.0$, $\tau_n{}^y = 1.17$

Parameter	H9	Hδ	Hγ	Hα	P9	P7	P6
S^p	0.053	0.207	0.394	4.218	0.052	0.121	0.208
r.	0.49	0.64	0.77	−0.73	0.47	0.62	1.13
ϕ min.				3.662			

NOTE.—S^p = predicted values, r = errors in the computed ratios relative to the observational errors, and ϕ min = $\phi(x_n, y_n)$.

for NGC 7027. Figure 2 is a plot of the values of S^o/S^t for the Orion Nebula as well as of the reddening curve $D(x_m, 0)$. Two reddening curves $D(0, y_n)$ are also drawn in the figure, one with the Whitford function as the interstellar reddening law and the other with the interstellar reddening of the Orion Sword region.

The observed data in the Orion Nebula enable us to determine also confidence sets in the (x, y)-plane or the (τ^x, τ^y)-plane (ellipses in a linear approximation). The support lines of the set of a given confidence limit determine intervals on the x and y axes. We thus obtain that with a better than 90 percent confidence, if the internal reddening could be approximated by the Orion Sword reddening law, the true value of τ^x lies between 2.5 and 9.0 and the true value of τ^y is between 0 and 0.20.

IV. DISCUSSION

Tables 3 and 4 and figure 1 show that the two-parameter reddening law with $x \neq 0$ fits the observed lines and radio flux ratios in NGC 7027 to the theoretical ratios better than the one-parameter law.

The four intensity ratios used in the least-squares fit for NGC 7027 are too few for the establishment of meaningful confidence ranges for the values of τ^x and τ^y. We do find, however, that the pair of values $\tau_n{}^x = 0$ and $\tau_n{}^y = 3.0$, which are obtained from

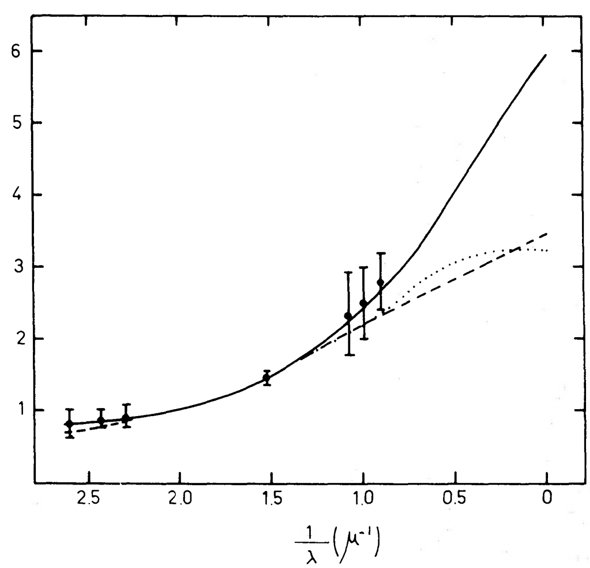

FIG. 2.—Reddening curves and the observed S^o/S^t in the Orion Nebula. *Solid line*, $D_{1/\lambda}(1.05, 0.0)$; $g(\lambda)$, the Orion Sword reddening. *Broken line*, $D_{1/\lambda}(0.0, 0.2)$; $f(\lambda)$, the Orion Sword reddening. *Dotted line*, $D_{1/\lambda}(0.0, 1.02)$; $f(\lambda)$, the Whitford reddening.

the conventional one-parameter reddening law, lie outside the 93 percent confidence set of the observations in the (τ^x, τ^y)-plane.

In the Orion Nebula, the seven observed line ratios are sufficient to determine 90 percent confidence intervals for τ^x and τ^y. Although the interval of τ^x is rather large, the value $\tau^x = 0$ lies safely outside it. Based on a slightly different statistical test we can conclude also that the values of τ^x and τ^y obtained from the one-parameter reddening law for the Orion Nebula, $\tau_n{}^x = 0.0$ and $\tau_n{}^y = 1.2$, lie outside the 99 percent confidence set in the (τ^x, τ^y)-plane, defined by the observations and the function $D(x, y)$.

Any function of the form $G(x)F(y)$ should give by a variational method a fit which is admittedly always no worse than the function $F(y)$ alone. The fit with the two parameters, however, is not necessarily better than that with one parameter, as in fact is the case for the observations in the planetaries NGC 7026 and 6572. The function $G(x)$ was not randomly chosen; it represents a possible physical process in a gaseous nebula. For the two nebulae NGC 7027 and the Orion Nebula the values obtained for $\tau_m{}^x$ and $\tau_m{}^y$ are both positive and within a reasonable range of values. Finally, the statistical analyses give us some measure of the confidence that we can have in the various numerical results.

All our statistical inferences are based on two fundamental assumptions: (1) The reddening of emission lines in gaseous nebulae as a function of τ^x and τ^y can indeed be represented by expression (7). In particular, the function $G(x)$ is a good approximation for internal reddening. (2) The functions $f(\lambda)$ and $g(\lambda)$ that we use represent correctly the reddening properties of dust grains.

Expression (4) is the exact solution of the radiative transfer equation in a medium of a gas well mixed with a nonscattering dust. We have found that the assumption of uniformity is not essential for our conclusion about the presence of internal dust in the two nebulae. If we assume, for example, that internal dust in the Orion Nebula is confined only to the outer 80 percent layer of the nebula, we still obtain $\tau_m{}^x = 4.2$ and $\tau_m{}^y = 0.0$. The fit of observed to theoretical line ratios is worse than in the uniform model, but it is statistically significantly better than the one-parameter fit (result A is by our definition statistically significantly better than result B if a confidence set of at least 90 percent around the point A does not include the point B). In a model for NGC 7027 with the central 50 percent of the nebula free of dust we can match the observations to the theory with $\tau_m{}^x = 3.7$, $\tau_m{}^y = 1.7$. The fit is slightly better than in the uniform model. These results may indicate that dust in the Orion Nebula is distributed throughout the emitting gas whereas in the planetary nebula the content of dust at the center is poorer than in the outer layers of the nebula. No statistical evaluation of the different distributions of internal dust is possible because the differences in the theoretical fits are statistically insignificant.

The function used for $g(\lambda)$ and $f(\lambda)$ are those derived from observed interstellar reddening. Both the Whitford law and the Orion Sword reddening (Johnson 1968) represent total extinction by interstellar grains. If the albedo of internal dust in the visible region of the spectrum is small, or if the scattering optical depth of the nebula is smaller than unity, one can use, to a good approximation, the extinction reddening law, in place of the true absorption law. We found that the assumption of small albedo is also not essential for our qualitative conclusions. In the Orion Nebula, if the internal dust grains had an albedo as large as 0.8 at $\lambda = 3800$ Å and if the albedo varied with λ even as strongly as $1/\lambda^3$, the fit of observations to theory with internal dust would still be significantly better than the fit with external dust alone.

For lack of any other information about the extinction properties of internal dust grains in nebulae, we have adopted for the internal dust the reddening laws of interstellar grains. The results, however, are only weakly sensitive to the exact nature of the internal reddening law. If we take for NGC 7027 the reddening function of Seaton

(1960) instead of Whitford's, we obtain essentially the same results. If we take for internal reddening in the Orion Nebula the Whitford curve with its total to relative extinction of 3, rather than the Orion Sword law with a total to relative extinction of 5, we obtain the best fit with $\tau_m{}^x = 4.8$ and $\tau_m{}^y = 0.0$. The $\tau_n{}^y = 0$ and $\tau_n{}^y = 1.2$ values obtained from the one-parameter fit lie outside the 92 percent confidence set on the (τ^x, τ^y)-plane.

We conclude that the hydrogen line intensities and the radio flux observed in NGC 7027 and in the Orion Nebula do indicate the presence of internal dust in the two nebulae.

On principle, the least-squares method may also be used to determine the electron temperature T_e of the region emitting the hydrogen radiation, when T_e as well is being varied as a free parameter. We find for NGC 7027 that when the S^t values for $T_e = 10{,}000°$ K are used instead of those for $T_e = 20{,}000°$ K, the fit of the observed data to the theoretical ratios is worsened, as is evident, for example, from the value of $\phi(x_m, y_m)$. For $T_e = 5000°$ K, the fit is worsened still more. Although in both cases the differences are statistically insignificant, the trend seems to indicate that given better observational data, we could narrow the range of possible electron temperatures in this nebula.

Accurate measurements of hydrogen line intensities in other H II regions in the Galaxy are rather scarce, and in no case are they numerous or accurate enough to determine the value of x. Even for planetary nebulae, the available observational data are insufficient for that purpose. The most extensive list of photoelectric measurements in planetary nebulae, which includes both Balmer lines in the blue and Hα and other infrared lines, is that of O'Dell (1963). When we combine these measurements with the radio data as compiled by Cahn and Kaler (1971), we find for NGC 7662 and IC 2149 a minimum point of the function $\phi(x, y)$ at $x > 0$ and $y = 0$. In both nebulae, however, the fit of the theoretical to observed line ratios with $x = 0$ and $y \neq 0$ is only slightly worse, and the difference is statistically insignificant. Observational data for NGC 7026, 6572, and IC 418 are slightly better fitted to theory with $x = 0$, $y \neq 0$ than with $x \neq 0$, but again the difference is statistically insignificant. Figure 3 is a plot of

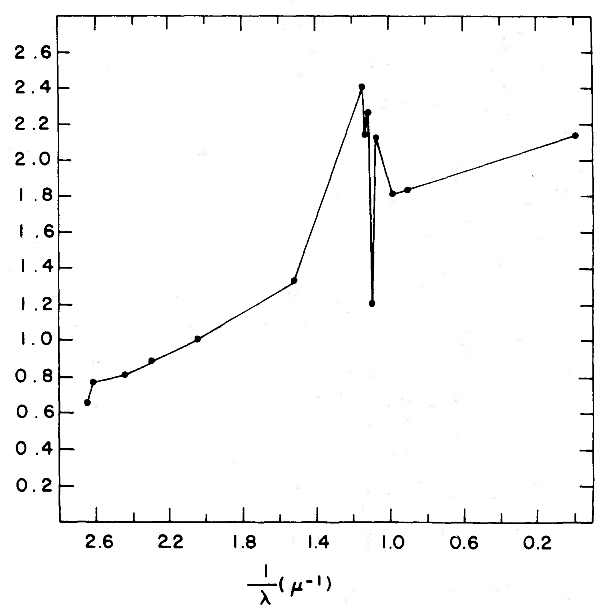

FIG. 3.—The observed S^o/S^t in IC 418

S^o/S^t versus $1/\lambda$ for IC 418. One can see that many smooth curves can be fitted to the observed points with the same level of accuracy.

Although our value of $\tau_m{}^y = 2.1$ for NGC 7027 is much smaller than the one-parameter value of $\tau_n{}^y = 3.0$, no revision is necessary in the estimate of the distance to the nebula (Cahn and Kaler 1971). The apparent surface brightness of the nebula should be revised downward by the factor $\exp(\tau_m{}^y - \tau_n{}^y)$, but the K parameter in Seaton's (1968) expression (4) should also be revised downward by the factor $\{[1 - \exp(-\tau_m{}^x)]/\tau_m{}^x\}^{1/5}$. Thus the value of the radius R should be multiplied by $\{[1 - \exp(-\tau_m{}^x)] \exp(\tau_n{}^y - \tau_m{}^y)/\tau_m{}^x\}^{1/5}$, which is very nearly 1.

We conclude by stressing the need for new measurements of hydrogen line intensities and radio continuum fluxes of very high accuracy in planetary nebulae and H II regions. In particular, these are desirable for nebulae which are strong infrared sources.

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