THE ASTROPHYSICAL JOURNAL, 185:277–292, 1973 October 1

NONRADIAL OSCILLATIONS OF COOLING WHITE DWARFS

YOJI OSAKI* AND CARL J. HANSEN
Joint Institute for Laboratory Astrophysics† and Department of Physics and Astrophysics, University of Colorado, Boulder
Received 1973 April 9

ABSTRACT

The equations for linear adiabatic nonradial oscillations have been solved for cooling white-dwarf models. Periods and eigenfunctions are obtained for the g_2, g_1, f, p_1, p_2 modes of nonradial quadrupole oscillations. Periods of the g_1-mode range from 50 to 200 seconds for typical white-dwarf stars with $M = 0.4 \sim 1.0\, M_\odot$ and $L = 10^{-4}\, L_\odot \sim 10^{-2}\, L_\odot$. It is shown that there exist period-luminosity relations for g-modes along the cooling white-dwarf sequence. The damping of nonradial oscillations due to radiative heat leakage, neutrino losses, and gravitational radiation is studied in the quasi-adiabatic approximation, and all models are found to be pulsationally stable. It is found that the emission of gravitational waves is the most efficient mechanism for the damping of f- and p-modes while radiative heat leakage is the most efficient for g-modes. Damping times range from a few tens of years to about 10^5 years—time scales much shorter than Kelvin or cooling times for white dwarfs. Results are compared with observations of ultrashort-period variables. It is suggested that light variations of HL Tau 76 ($P = 747$ s) and G44-32 ($P = 600$ s) may be associated with nonradial g_1 oscillations of *white dwarfs with convective envelopes*. The importance of gravitational radiation from nova outbursts is also discussed.

Subject headings: gravitation — interiors, stellar — pulsation — white dwarf stars

I. INTRODUCTION

In Van Horn, Richardson, and Hansen (1972, hereafter referred to as Paper I), detailed studies of the radial pulsations of representative cooling white-dwarf models were performed. In this paper, we study the nonradial oscillations of those same models. The importance of studying nonradial oscillations of white dwarfs is evident from recent observations by B. Warner and his co-workers. Warner and Robinson (1972) discovered five more ultrashort-period variables among dwarf novae, which increased the number of known white-dwarf variables to 10. Their discovery strongly suggests that pulsations in white dwarfs are intimately connected with the dwarf-nova (and possibly nova) phenomenon either because the oscillation itself is a direct cause of outbursts or because it is an indirect result of outbursts. They suggest that these oscillations are g-modes of nonradial oscillations in white dwarfs. In support of this suggestion is the work of Warner *et al.* (1972), who have demonstrated that the 71-s oscillation of the old nova DQ Her may be interpreted as a nonradial quadrupole mode from the observed phase variations of the oscillation during the eclipse of the star.

One characteristic of nonradial oscillations in white dwarfs which sets them apart from radial modes is the behavior of the periods of the g-modes. These periods differ from what would be predicted from the usual period-density relation (i.e., $P(\bar{\rho}/\bar{\rho}_\odot)^{1/2} = Q$ with $Q \sim 0^{\mathrm{d}}.04$) in that they are longer. This is one of the reasons to call upon nonradial oscillations to explain relatively long periods observed in some white dwarfs (e.g., $P = 747$ s for HL Tau 76 in Landolt 1968; $P = 212$, 273 for R548 in

* JILA Visiting Fellow (1972–1973) on leave from Department of Astronomy, University of Tokyo.

† Operated jointly by the National Bureau of Standards and the University of Colorado.

Lasker and Hesser 1971), because periods of radial pulsations range from a few seconds to tens of seconds. The nonradial quadrupole oscillations of white dwarfs are also interesting because they emit gravitational radiation, and the nova outbursts may well be possible galactic sources of gravitational waves (as will be discussed later).

Previous studies of nonradial oscillations in white dwarfs have mostly been limited to zero-temperature objects. Approximate periods of the g_1-mode for white dwarfs with finite internal temperatures have been estimated by Baglin and Schatzman (1969). The full fourth-order differential equation which describes (in a linear sense) nonradial oscillations has been solved for hot white dwarfs by Harper and Rose (1970), but their static models are far more luminous than ordinary white dwarfs. In this paper, that fourth-order differential equation system has been solved for moderately realistic white-dwarf models with luminosities ranging from $10^{-4}\,L_\odot$ to $10\,L_\odot$, and various damping mechanisms of oscillations have been studied in the quasi-adiabatic approximation. In § II (and Appendices A and B) we describe the computational procedures used; results and discussions are deferred to §§ III and IV.

II. BASIC EQUATIONS AND METHOD OF COMPUTATION

a) *Adiabatic Oscillations*

The basic equations governing the linear adiabatic nonradial oscillations of a gaseous star have been discussed by Ledoux and Walraven (1958). These are:

the equation of motion,

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\rho'}{\rho^2}\,\mathrm{grad}\,p - \frac{1}{\rho}\,\mathrm{grad}\,p' - \mathrm{grad}\,\Phi'\,; \tag{1}$$

the equation of continuity,

$$\frac{\partial\rho'}{\partial t} + \mathrm{div}\left(\rho\,\frac{d(\delta\boldsymbol{r})}{dt}\right) = 0\,; \tag{2}$$

the adiabatic condition,

$$\frac{\rho'}{\rho} = \frac{1}{\Gamma_1}\frac{p'}{p} - A\delta r\,; \tag{3}$$

and Poisson's equation,

$$\nabla^2\Phi' = 4\pi G\rho'\,. \tag{4}$$

Here the Eulerian and Lagrangian perturbations are represented by prime (′) and δ, respectively. The quantity A appearing in equation (3) represents the local dynamical stability (or the Schwarzschild criterion) and is given by

$$A = \frac{1}{\rho}\frac{d\rho}{dr} - \frac{1}{\Gamma_1}\frac{1}{p}\frac{dp}{dr} = \frac{\chi_T}{\chi_\rho}(\nabla - \nabla_{\mathrm{ad}})/H_p\,, \tag{5}$$

where $\chi_T = (\partial \ln p/\partial \ln T)_\rho$ and $\chi_\rho = (\partial \ln p/\partial \ln \rho)_T$, and other symbols have their usual meanings. For highly degenerate white dwarfs, $\chi_T \ll 1$ so that stratification is nearly neutral, which makes frequencies of g-modes very low.

We assume that the spatial and temporal behavior of the perturbations can be represented by

$$f'(r, \theta, \phi, t) = f'(r)(N_l^m)^{-1/2} P_l^m(\cos\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} e^{i\sigma t}$$

$$= f'(r)(N_l^m)^{-1/2} Y_l^m(\theta, \phi) e^{i\sigma t}\,, \qquad 0 \leq m\,(\text{integer}) \leq l\,, \tag{6}$$

where P_l^m is the associated Legendre function and N_l^m is a normalization factor for the spherical harmonics defined as

$$N_l^m \equiv \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |Y_l^m(\theta, \phi)|^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{(2l+1)}\frac{(l+m)!}{(l-m)!\,\epsilon_m}\,. \tag{7}$$

Here $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$ for $m \neq 0$. The system of equations (1)–(4) can be reduced to one fourth-order differential equation (Ledoux and Walraven 1958). In order to solve it numerically it is, however, more convenient to solve the equivalent four first-order differential equations with four variables. Following Dziembowski (1971), we choose these four variables as

$$y_1 = \frac{\delta r}{r}\,, \qquad y_2 = \frac{1}{gr}\left(\frac{p'}{\rho} + \Phi'\right)\,, \qquad y_3 = \frac{1}{gr}\Phi'\,,$$

and

$$y_4 = \frac{1}{g}\frac{d\Phi'}{dr}\,, \tag{8}$$

and we also introduce a dimensionless frequency ω by

$$\sigma^2 = \omega^2 GM/R^3\,. \tag{9}$$

The four basic linear differential equations and the appropriate boundary conditions are then essentially those given by Dziembowski (1971). (The complete set is reproduced in Appendix A.) This system of equations, along with two inner and two outer boundary conditions plus a normalization condition

$$\delta r/r = 1 \text{ at the outer boundary}\,, \tag{10}$$

forms a proper eigenvalue problem. We have used a Henyey-type relaxation method due to Baker and Lucy (Baker 1968) to solve for both eigenvalues and eigenfunctions. The convergence of this relaxation method depends on how good the initial guesses of eigenvalues and eigenfunctions are. Good guesses imply speedy convergence. However, since it is difficult to make a good guess, particularly for eigenvalues of g-modes in highly degenerate stars, we have used the following method, which is similar to that used by Castor (1971) for the radial pulsation problem. If we set aside one of the boundary conditions while keeping the normalization condition (10), we can solve this system of equations with an arbitrary value of ω^2. We then substitute this solution into the missing boundary condition. In general, that condition is not satisfied for the arbitrary ω^2, but the numerical value associated with the boundary condition serves as a discriminant for eigenvalues of the original system. Once we get

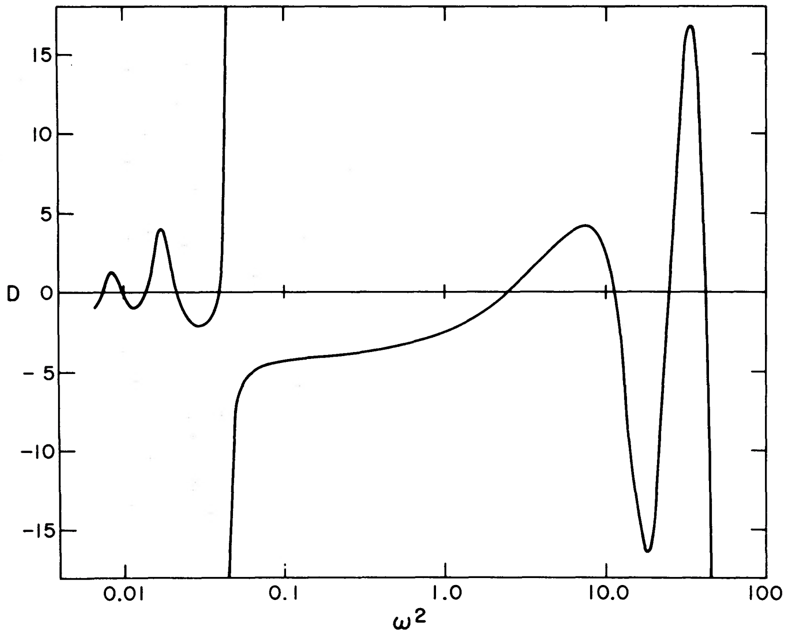

FIG. 1.—Behavior of the discriminant as a function of ω^2 for the 10N model of 0.398 $M_\odot$

a reasonably good initial guess of the eigenvalue and corresponding eigenfunctions, we can use the relaxation code to obtain solutions for the full system. The boundary condition which has been chosen as a discriminant is

$$D \equiv (l + 1)y_3 + y_4 \text{ at the outer boundary} . \tag{11}$$

An example of the behavior of D as a function of ω^2 is shown in figure 1. As can be seen in the figure, the g-branch and p-branch of nonradial oscillations are well separated in a degenerate star. This figure also assures us that we have not missed any eigenmodes between $\omega^2 = 0.04 \sim 2.5$.

b) Stability Integral in the Quasi-Adiabatic Approximation

In the quasi-adiabatic approximation the stability or instability of a star to pulsations is determined by the stability integral

$$W = \int \left(\frac{\delta T}{T}\right) \delta\left(\epsilon - \frac{1}{\rho} \nabla \cdot \boldsymbol{F}\right) dM_r , \tag{12}$$

wherein $W > 0$ for pulsational instability and $W < 0$ for pulsational stability. The quantities $\delta T/T$ and $\delta(\epsilon - \rho^{-1}\nabla \cdot \boldsymbol{F})$ represent the space parts of variations calculated from the adiabatic analysis. The quantity W is related to the damping rate of the oscillation ($\delta \boldsymbol{r} \propto e^{i\sigma t - \kappa t}$) by

$$\kappa = -W/4E_K , \tag{13}$$

where

$$E_K = \tfrac{1}{2}\sigma^2 \int |\delta \boldsymbol{r}|^2 dM_r \tag{14}$$

is the kinetic energy of oscillation. Since cooling white dwarfs are not in thermal equilibrium, the effect of thermal imbalance may affect the stability integral. However, this effect has been neglected in this study, and it will be discussed in a separate paper of this series for radial pulsations (Van Horn, Cox, and Hansen 1973). The detailed expression of the stability integral for nonradial oscillations will be given in Appendix B. One of its characteristics for nonradial oscillations is the existence of an extra dissipation term due to the horizontal energy exchange.

The quasi-adiabatic approximation is expected to be fairly good for the white-dwarf stars studied here because regions near the stellar surface that should be treated strictly nonadiabatically contain negligible mass and so have little effect on the overall stability.

c) *Gravitational Radiation From the Quadrupole Mode* ($l = 2$)

If a star undergoes a nonradial quadrupole oscillation, gravitational radiation is emitted. The energy loss by gravitational radiation is given in the weak-field limit to general relativity as (Landau and Lifshitz 1962)

$$-\frac{dE}{dt} = \frac{G}{45c^5}\left(\frac{d^3\mathscr{D}_{\alpha\beta}}{dt^3}\right)^2 , \tag{15}$$

where $\mathscr{D}_{\alpha\beta}$ is the mass quadrupole moment, defined as

$$\mathscr{D}_{\alpha\beta} = \int \rho(\boldsymbol{x})(3x_\alpha x_\beta - \delta_{\alpha\beta}x_\gamma^2)d\boldsymbol{x} . \tag{16}$$

Since the emission rate of gravitational radiation is the same for modes with different m belonging to the same $l = 2$ mode, we shall estimate it for the simplest case of $l = 2$ and $m = 0$.

If we write

$$\rho(\boldsymbol{x}) = \rho_0(r) + \rho'(r)(N_2{}^0)^{-1/2}Y_2{}^0(\theta, \phi)e^{i\sigma t} , \tag{17}$$

and recall that $\mathscr{D}_{11} = \mathscr{D}_{22} = -\frac{1}{2}\mathscr{D}_{33}$ with the off-diagonal elements vanishing for the axisymmetric mode, we obtain

$$\mathscr{L}_{\mathrm{GW}} \equiv -\langle dE/dt\rangle = \frac{GD_{33}{}^2}{60c^5}\sigma^6 , \tag{18}$$

where $\mathscr{L}_{\mathrm{GW}}$ represents the time-averaged emission rate of gravitational radiation and

$$D_{33} = 2(N_2{}^0)^{1/2}\int_0^R r^2\rho'(r)4\pi r^2 dr . \tag{19}$$

By using Poisson's equation

$$4\pi G\rho' = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi'}{dr}\right) - \frac{6}{r^2}\phi' , \tag{20}$$

we finally obtain

$$\mathscr{L}_{\mathrm{GW}} = \frac{GM^2R^4\sigma^6}{75c^5}d^2 . \tag{21}$$

Here

$$d \equiv \frac{1}{MR^2}\int_0^R r^2\rho'(r)4\pi r^2 dr$$

$$= \left(\frac{1}{g}\frac{d\phi'}{dr} - 2\frac{1}{gr}\phi'\right)_{r=R} = -5(y_3)_{r=R} \tag{22}$$

is the dimensionless mass quadrupole moment of oscillation.

III. RESULTS OF CALCULATIONS

a) Periods and Eigenfunctions

The static white-dwarf models used here are the same as those of Paper I. Their details (except for very slight differences in envelope structure) may be found in the original paper (Savedoff, Van Horn, and Vila 1969). Extensive calculations have been done for the 0.398 $M_\odot$ models with neutrino losses and a few supplementary calculations with models of different mass. Adiabatic eigenfrequencies and eigenfunctions have been computed for g_2, g_1, f, p_1, p_2 models of quadrupole oscillation, and results are shown in figure 2 and in tables 1 and 2. As already noted, pressure modes (p-branch) and gravity modes (g-branch) of nonradial oscillations are very clearly separated for white dwarfs. There is no difficulty in identifying the mode number for models given in tables 1 and 2 because p_n- and g_n-modes have n-nodes and the f-mode has no nodes in the variation of radial displacement. This is not always true when the central condensation of the static model is very high (see Robe 1968; Dziembowski 1971). It is found that the dimensionless frequency ω^2 and eigenfunctions of the f-mode and p-modes are essentially determined by the degree of central concentration of mass in the static models and they are quite similar to those for the corresponding

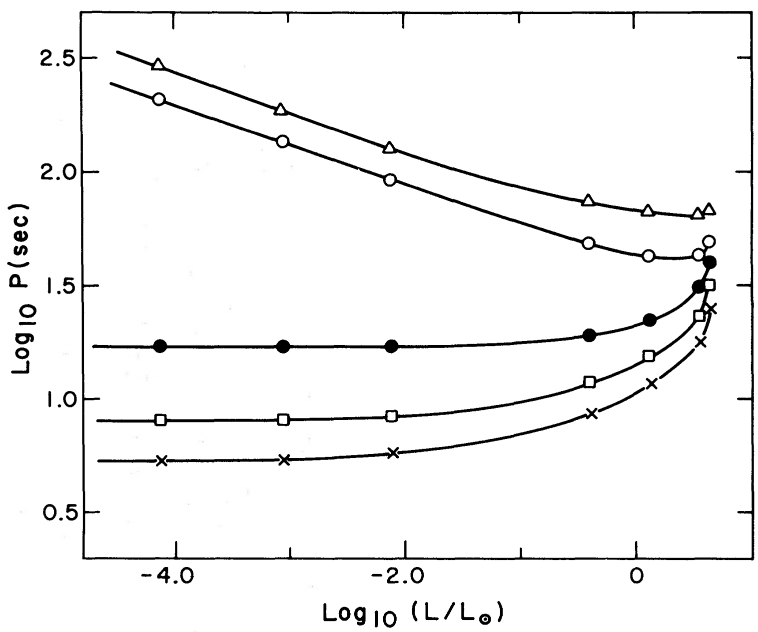

FIG. 2.—Periods in seconds of the (*top to bottom*) g_2, g_1, f, p_1, p_2 modes of nonradial quadrupole oscillations ($l = 2$) as functions of stellar luminosity for the 0.398 $M_\odot$ models.

TABLE 1

PERIODS OF NONRADIAL QUADRUPOLE OSCILLATIONS ($l = 2$) FOR COOLING WHITE DWARFS OF $M = 0.398\ M_\odot$ WITH NEUTRINOS

				PERIODS (seconds)				
MODEL NO.	$L/L_\odot$	$R/R_\odot$	$\rho_c/\bar{\rho}$	g_2	g_1	f	p_1	p_2
4N.......	4.44	4.14×10^{-2}	90.1	68.19	49.02	39.88	32.05	25.15
5N.......	3.55	3.25×10^{-2}	58.8	64.55	42.88	30.22	23.06	17.89
6N.......	1.32	2.40×10^{-2}	30.5	67.43	42.95	22.11	15.36	11.62
8N.......	0.406	1.98×10^{-2}	19.1	74.56	48.27	19.10	11.81	8.718
9N.......	7.65×10^{-3}	1.50×10^{-2}	9.19	127.5	91.94	17.10	8.292	5.829
10N.......	8.77×10^{-4}	1.43×10^{-2}	7.90	185.7	135.7	16.99	8.018	5.412
11N.......	7.38×10^{-5}	1.39×10^{-2}	7.28	291.1	209.8	16.97	7.957	5.299

polytropes having the same $\rho_c/\bar{\rho}$ with $\Gamma_1 = 5/3$. Eigenfunctions of radial displacement for the f-mode are shown in figure 3.

On the other hand, g-modes behave quite differently in cooling white dwarfs. When the star is fairly luminous, the weak degeneracy in the core is not strongly felt by the g-spectrum and the periods of g-modes decrease as the star contracts (from 4N to 5N of 0.398 $M_\odot$). However, when the degeneracy in the core becomes sufficiently high, the periods of g-modes tend to increase as the star continues to cool. Thus there exist minimum periods in cooling white dwarfs, which are $P_{\min} \sim 42$ s for the g_1-mode and $P_{\min} \sim 64$ s for the g_2-mode of the 0.398 $M_\odot$ models. We can infer from figure 2 that periods of the g_1-mode for models with white-dwarf luminosities are very well represented by a period-luminosity relation

$$\log_{10} P(\text{seconds}) = 1.587 - 0.178 \log_{10} (L/L_\odot)\,, \tag{23}$$

for

$$M = 0.398\ M_\odot \quad \text{and} \quad 10^{-4} L_\odot \lesssim L \lesssim 10^{-2} L_\odot\,.$$

A similar relation can be obtained for a 1 $M_\odot$ white dwarf with the same luminosity range, namely,

$$\log_{10} P(\text{seconds}) = 1.331 - 0.171 \log_{10} (L/L_\odot)\,. \tag{24}$$

Period-luminosity relations given as equations (23) and (24) can be easily understood. As Cowling (1941) first pointed out, the square of the frequency of the g-mode is proportional to the quantity A of equation (5), i.e. (see also Ledoux and Walraven 1958; Chanmugam 1972),

$$\sigma_g^2 \propto -A\,, \tag{25}$$

TABLE 2

PERIODS OF NONRADIAL QUADRUPOLE OSCILLATIONS ($l = 2$) FOR 1 $M_\odot$ MODELS

				PERIODS (seconds)				
MODEL NO.	$L/L_\odot$	$R/R_\odot$	$\rho_c/\bar{\rho}$	g_2	g_1	f	p_1	p_2
6N.......	5.27×10^{3}	1.12×10^{-2}	81.17	12.53	8.15	3.514	2.906	2.395
9N.......	1.16×10^{-2}	6.06×10^{-3}	14.02	62.96	45.98	2.703	1.493	1.009
11N.......	6.4×10^{-5}	5.99×10^{-3}	13.54	156.1	111.9	2.703	1.494	1.007

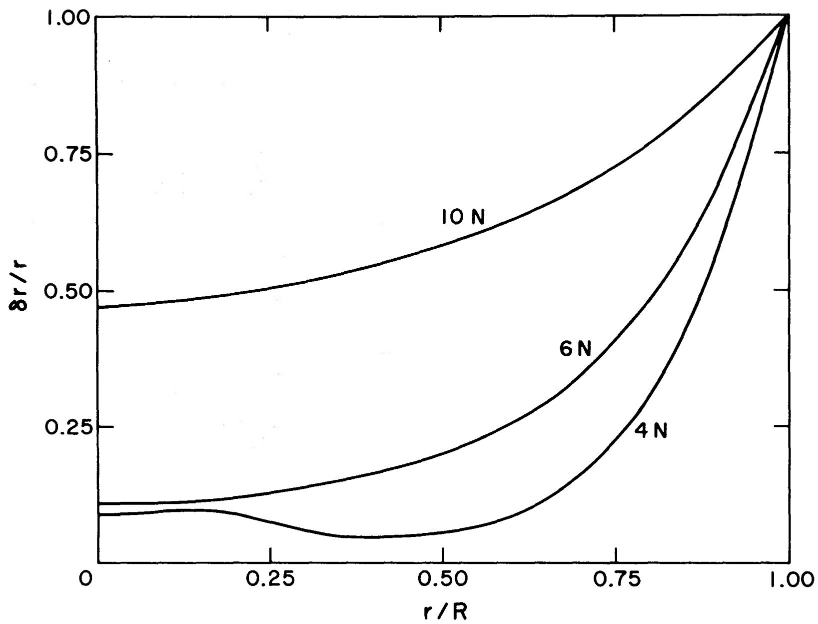

FIG. 3.—Eigenfunctions of the radial displacement $\delta r/r$ for the *f*-mode as a function of radial coordinate r/R for various 0.398 $M_\odot$ models.

if the stratification is nearly neutral. The quantity A is approximately given in the degenerate core of white dwarfs as

$$-A \propto c_v T \propto T^\alpha , \quad (26)$$

where $c_v = c_v(\text{ion}) + c_v(\text{electron})$ is the specific heat. If ions are the only source of thermal energy, then $\alpha = 1$; and if degenerate electrons dominate, then $\alpha = 2$. On the other hand, the luminosity of a white dwarf is related to the temperature at the edge of the degenerate core (Schwarzschild 1958) by

$$L \propto T^\beta ,$$

where $\beta = 3.5$ for Kramers opacity ($\kappa = \kappa_0 \rho T^{-3.5}$). By substituting equations (26) and (27) into equation (25), we obtain a period-luminosity relation

$$\log_{10} P(g\text{-mode}) = \text{const.} - (\alpha/2\beta) \log_{10} L . \quad (28)$$

The slope of the period-luminosity relation is then given by $\alpha/2\beta = 0.14$, if we put $\alpha = 1$ and $\beta = 3.5$. The difference between this slope and those of equations (23) and (24) is presumably attributable to the fact that the exponent of the luminosity-temperature relation is less than 3.5 in realistic white-dwarf models (Savedoff *et al.* 1969), and that the contribution of degenerate electrons to the specific heat is not negligible for low-mass white dwarfs (Van Horn 1971).

Eigenfunctions of radial displacement for the g_1-mode are shown in figure 4. It is seen that the position of the node moves toward the surface, as the white dwarf cools down. The amplitude of the eigenfunction in the degenerate core becomes so small in the highly evolved models that the g_1-mode looks like a surface oscillation.

Envelopes of all static models used in these calculations are radiative. However, it has been pointed out by Böhm and Cassinelli (1971) and Van Horn (1970) that cool white dwarfs have extensive outer convection zones, which appreciably reduce core

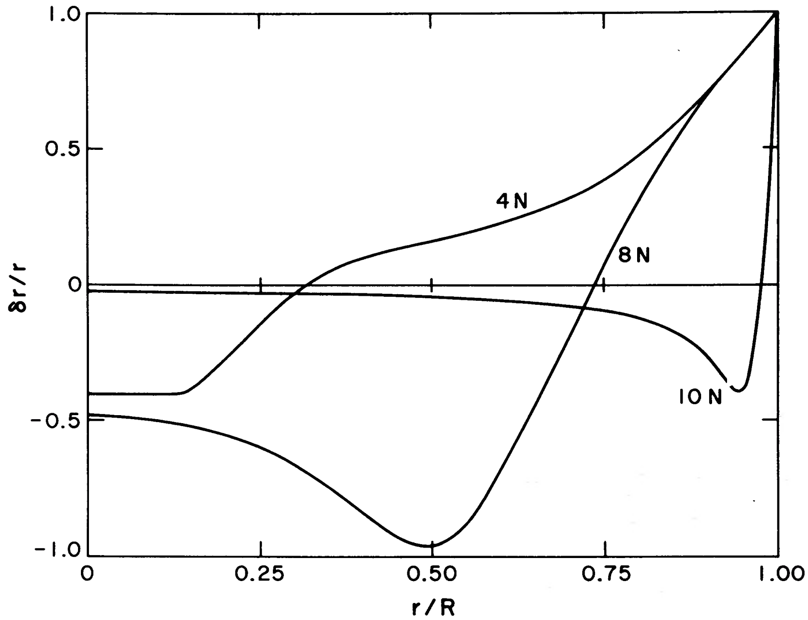

FIG. 4.—Same as fig. 3 but for the g_1-mode

temperatures. The existence of the outer convection zone in a cool white dwarf can affect g-mode oscillations in two ways: (i) by lowering the core temperature, and (ii) by forcing A of equation (5) to zero in the outer envelope. Both effects tend to reduce g-mode frequencies. For instance, if the presence of envelope convection reduces the core temperature by a factor of 4, the period of the g_1-mode will be increased by a factor of 2 as seen from equations (25) and (26) with $\alpha = 1$. In order to study the effect of (ii) on the frequencies of g-modes in an approximate way, we have computed periods of g-modes for the coolest white dwarf (11N model of 0.398 $M_\odot$) by *artificially* setting in the pulsation equations

$$A = 0 \quad \text{for} \quad r > r_f \,, \tag{29}$$

where r_f is the radius at the bottom of the assumed convective envelope. Results are shown in table 3, where columns (2), (3), and (4) indicate radius, temperature, and the degeneracy parameter of the bottom of the convective envelope, respectively. It is evident that periods of g-modes increase drastically if the bottom of the convection zone reaches the degenerate core. It should be pointed out, however, that there is a measure of inconsistency in this approach because modifications on the structure of the *static* envelope have not been taken into account. We do not believe our qualitative

TABLE 3

PERIODS OF g-MODES FOR 11N MODEL OF 0.398 $M_\odot$ WITH THE CONVECTIVE ENVELOPE

MODEL NO.	$x_f = r_f/R$	T_f/T_c	DEGENERACY χ_T	PERIODS (seconds) f	g_1	g_2
0.398 $M_\odot$, 11N........	(radiative)	...	...	16.97	209.8	291.1
Convective 1.........	0.970	0.803	0.472	16.97	264.9	411.5
Convective 2.........	0.852	0.963	0.117	16.95	527.5	...

conclusions would change if this were done, but what is eventually needed is better evolutionary models.

b) *Damping of Nonradial Oscillations*

Since the basic static models have no nuclear sources, we do not expect pulsational instabilities in these models and this has been confirmed by our numerical results. Damping rates of nonradial oscillations due to radiative heat leakage, neutrino losses, and gravitational radiation have been computed, and they are shown in table 4 for the 11N model of 0.398 $M_\odot$ and the 9N model of 1 $M_\odot$. The E_K in column (2) is the total kinetic energy of the oscillation, and $\mathscr{L}_{\rm ph}$, $\mathscr{L}_\nu$, and $\mathscr{L}_{\rm GW}$ are the time-averaged dissipation rates of oscillations due to radiative heat leakage, neutrino losses, and gravitational radiation, respectively, and they are given by

$$\mathscr{L}_{\rm ph} = \frac{1}{2}\int \frac{\delta T}{T}\,\delta\left(\frac{1}{\rho}\,\nabla\cdot\boldsymbol{F}\right)dM_r\,, \tag{30}$$

$$\mathscr{L}_\nu = -\frac{1}{2}\int \left(\frac{\delta T}{T}\right)\delta\epsilon_\nu dM_r\,, \tag{31}$$

and $\mathscr{L}_{\rm GW}$ by equation (21). The quantity $\tau(\text{ph} + \nu) = \kappa^{-1}$ in table 4 is the damping time due to radiation and neutrinos, and τ(total) is the total damping time, which includes the emission of gravitational radiation. It may be remarked here that the quantities E_K and $\mathscr{L}$ are normalized by the condition $\delta R/R = 1$ at the surface so that they are proportional to the square of the amplitude of the oscillations:

$$E_K = (E_K)_0 a^2 \qquad \text{and} \qquad \mathscr{L} = \mathscr{L}_0 a^2\,, \tag{32}$$

where

$$a \equiv \delta R/R\,.$$

From table 4 one finds that the emission of gravitational waves is the most efficient mechanism for damping of *f*- and *p*-modes of nonradial quadrupole oscillations, while radiative losses are the most important for damping of *g*-modes. Ordinary dissipation is smallest for the *f*-mode, and it is very large for *g*-modes (see cols. [3] and [7] of table 4). This is because *g*-mode eigenfunctions in a white dwarf have large amplitudes only near the surface so that the interior is practically stationary (as seen in fig. 4). Since the emission rate of gravitational waves is proportional to the sixth power of the oscillation frequency, it is important only for *f*- and *p*-modes. The *f*-mode emits gravitational waves most efficiently because the mass quadrupole moment is the highest for that mode. Neutrino losses are not important for the dissipation of oscillations even when they are the dominant energy loss mechanism in the static model. The emission of gravitational waves is so efficient that the damping time of the *f*-mode of the 1 $M_\odot$ model is as short as 40 years. We note that the damping times of nonradial oscillations of white dwarfs range from a few tens of years to some 10^5 years—times which are much shorter than the cooling times of white dwarfs. This may be contrasted with the damping times of radial pulsations of Paper I, which are close to the cooling times.

IV. COMPARISONS WITH OBSERVATIONS AND DISCUSSION

In table 5 we review some pertinent data for 10 ultrashort-period variables, most of which have been discovered recently. All of them are supposed to be white dwarfs either because white-dwarf spectra are seen or because they are components of nova-

TABLE 4

STABILITY OF NONRADIAL QUADRUPOLE OSCILLATIONS

Mode	E_K (ergs)	W/L	Energy Loss Rate (ergs s^{-1})			Damping Time (years)	
			$\mathscr{L}_{ph}$	$\mathscr{L}_{\nu}$	$\mathscr{L}_{GW}$	τ (ph + ν)	τ (total)
			0.398 $M_{\odot}$, 11N Model				
p_2..........	1.0×10^{48}	-8.0×10^{2}	1.1×10^{32}	2.8×10^{27}	5.7×10^{34}	5.7×10^{8}	1.1×10^{6}
p_1..........	3.2×10^{48}	-2.9×10^{2}	4.1×10^{31}	5.4×10^{27}	2.8×10^{36}	4.9×10^{9}	7.2×10^{4}
f............	1.4×10^{49}	$-32.$	4.6×10^{30}	8.8×10^{27}	4.0×10^{37}	1.9×10^{11}	2.2×10^{4}
g_1..........	1.8×10^{46}	-5.5×10^{4}	7.9×10^{33}	2.4×10^{26}	9.6×10^{22}	1.4×10^{5}	1.4×10^{5}
g_2..........	5.7×10^{45}	-1.8×10^{5}	2.6×10^{34}	3.7×10^{26}	7.4×10^{21}	1.4×10^{4}	1.4×10^{4}
			1.0 $M_{\odot}$, 9N Model				
p_2..........	1.0×10^{49}	-7.1×10^{2}	1.6×10^{34}	1.5×10^{32}	6.1×10^{39}	4.0×10^{7}	104.
p_1..........	2.8×10^{49}	-2.7×10^{2}	5.7×10^{33}	3.6×10^{32}	3.3×10^{40}	3.0×10^{8}	54.
f............	1.1×10^{50}	$-61.$	1.2×10^{33}	1.5×10^{32}	1.7×10^{41}	4.9×10^{9}	40.
g_1..........	1.3×10^{47}	-1.1×10^{5}	2.5×10^{36}	1.1×10^{31}	4.0×10^{25}	3.3×10^{3}	3.3×10^{3}
g_2..........	3.8×10^{46}	-3.4×10^{5}	8.0×10^{36}	1.9×10^{31}	5.7×10^{24}	3.0×10^{2}	3.0×10^{2}

TABLE 5

ULTRASHORT-PERIOD VARIABLES

Star	Sp. or Nature	Period (seconds)	Remarks	References
Z Cam.............	dwarf nova	17	variable period (16–19 s)	Robinson 1973
CN Ori............	dwarf nova	24		Warner and Robinson 1972
UX UMa..........	DA, nova-like binary	29	eclipsing binary (4^h43^m)	Warner and Robinson 1972
AH Her...........	dwarf nova	31		
DQ Her...........	nova	71	eclipsing binary (4^h39^m)	Warner *et al.* 1972
G61-29............	DB, nova-like?	105	eclipsing binary (6^h16^m)	Richer *et al.* 1973
HZ 29 (AM CVn)...	DBp	115 1015, orbital?		Warner and Robinson 1972
R548..............	DA	213 273		Lasker and Hesser 1971
G44-32............	DC	600 822 } orbital? 1638 }		Lasker and Hesser 1971
HL Tau............	DA	746	multiple periods (626, 661, 701, 992)	Warner and Robinson 1972

like binaries. Their periods range from 17 to 747 s if possible cases of orbital periods of binaries are excluded. Three groups may be distinguished from the observed periods: a short-period group ($P = \sim 17$–31 s), an intermediate-period group ($P = \sim 71$–213 s), and a long-period group ($P = 600$ and 747 s), although this classification is somewhat arbitrary. To date, several suggestions have been made to explain these periods. One of the most interesting of these is that what is observed is rotation periods of oblique magnetic white dwarfs (Ostriker and Hesser 1968). However, it seems difficult to explain multiple periods of R548, HL Tau 76 by this model. Let us examine, therefore, if these periods can be interpreted as those of non-radial oscillations as proposed by Warner and Robinson (1972). From tables 1 and 2, one finds that periods of g_1-modes for typical white dwarfs ($M = \sim 0.4$–$1\ M_\odot$ and $L = \sim 10^{-4}$–$10^{-2}\ L_\odot$) are ~ 50–200 s, which are consistent with those of the intermediate period group. Stars in the short-period group are all U Gem type stars except UX UMa, and regular light variations have been found only during the eruptions of their outburst cycles (Warner and Robinson 1972). Since amplitudes of outbursts of dwarf novae are ~ 3–5 mag and their absolute magnitude at minimum light is around $\langle M_V \rangle \sim +7.5$ (Kraft and Luyten 1965), the luminosity during the eruptions is supposed to be ~ 1–$10^2\ L_\odot$. Because of the period-luminosity relations given as equations (23) and (24), hot white dwarfs have shorter periods for g-modes than the ordinary white dwarfs, and it is possible to explain observed periods of the short-period group by those of g_1-modes, if masses of their white-dwarf components are larger than about 0.5 $M_\odot$. As for the two stars with long periods, it would be difficult to say they are g-mode oscillators (at least if they were white dwarfs of typical mass with radiative envelopes). However, if these stars have convective envelopes, it may not be impossible to have g_1-periods of the order of 10 min as already discussed in § III*a*. In fact, these two stars are located on the two-color diagram in regions where envelope convection becomes important (Lasker and Hesser 1971).

Since all models studied here are found to be pulsationally stable, one may wonder about the connection between the models and the observed variables. Several of the latter are novae and dwarf novae, and they need no special excitation mechanisms because oscillations can certainly be excited by their outbursts (Ostriker 1969). Some of them (UX UMa, HZ 29, and G44-32) are close binaries or suspected close binaries in which secondary components are thought to overflow their Roche limits, thereby causing accretion of hydrogen-rich material on their white-dwarf companions. The resulting hydrogen-shell burning is known to be thermally unstable. Defouw (1970) has shown that the thermal instability in such a situation tends to manifest itself as overstable convection, i.e., overstability of *g*-modes, if nonspherical perturbations are considered.

Finally we shall estimate the gravitational radiation from a typical nova outburst. The energy of a single outburst is estimated to be 10^{45} ergs. As a consequence the postnova star may undergo violent oscillations, whose kinetic energy is considered to be of the same order of magnitude as that of the outbursts. Suppose that one-tenth of this energy goes into the *f*-mode of quadrupole oscillation and that the postnova is a 1 $M_\odot$ white dwarf. We then find from equation (32) and table 4 that the amplitude of the *f*-mode oscillation thus excited would be $a^2 \sim 10^{-6}$ and that the emission rate of gravitational waves would be $\mathscr{L}_{\rm GW} \sim 10^{35}$ ergs $\rm s^{-1}$. However, this result depends strongly on the mass, and we have $\mathscr{L}_{\rm GW} \sim 3 \times 10^{32}$ ergs $\rm s^{-1}$ if $M = 0.398\ M_\odot$ is adopted. Although the emission rate of $\mathscr{L}_{\rm GW} \sim 10^{35}$ ergs $\rm s^{-1}$ is probably somewhat of an overestimate, we expect that the gravitational radiation from nova outbursts may be fairly large in any case. Since novae are much more frequent than supernovae, though much less energetic, we conclude that gravitational radiation from nova outbursts may be a common phenomenon in our Galaxy.

We should like to thank Dr. J. P. Cox for helpful discussions and Dr. H. M. Van Horn for sending us much of the white-dwarf data used here. This work was supported in part by NSF grant GP-36245 through the University of Colorado.

APPENDIX A

FOUR FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS OF NONRADIAL OSCILLATIONS

Following Dziembowski (1971), we adopt the four variables given in equation (8). We then obtain the four first-order linear differential equations describing adiabatic nonradial oscillations from equations (1)–(4). These are written as

$$r\frac{dy_1}{dr} = y_1(V/\Gamma_1 - 3) + y_2\left[\frac{l(l+1)}{c_1\omega^2} - V/\Gamma_1\right] + y_3(V/\Gamma_1)\,, \tag{A1}$$

$$r\frac{dy_2}{dr} = y_1(c_1\omega^2 + rA) + y_2(1 - U - rA) + y_3 rA\,, \tag{A2}$$

$$r\frac{dy_3}{dr} = y_3(1 - U) + y_4\,, \tag{A3}$$

and

$$r\frac{dy_4}{dr} = -y_1 UrA + y_2 UV/\Gamma_1 + y_3[l(l+1) - UV/\Gamma_1] - y_4 U\,, \tag{A4}$$

where

$$U = \frac{d \ln M_r}{d \ln r}\,, \qquad V = -\frac{d \ln P}{d \ln r} = \frac{g r \rho}{p}$$

and

$$c_1 = \left(\frac{r}{R}\right)^3 \frac{M}{M_r}\,.$$

The inner boundary conditions are given by

$$\frac{c_1 \omega^2}{l} y_1 - y_2 = 0\,, \qquad l y_3 - y_4 = 0\,, \tag{A5}$$

and the outer boundary conditions are

$$y_3(l + 1) + y_4 = 0\,, \tag{A6}$$

$$y_1\left\{1 + \left[\frac{l(l+1)}{c_1\omega^2} - 4 - c_1\omega^2\right]\Big/V\right\} - y_2 + y_3\left\{1 + \left[\frac{l(l+1)}{c_1\omega^2} - l - 1\right]\Big/V\right\} = 0\,.$$

(Note that the first of these conditions is the "discriminant" D of equation [11].) The normalization condition is given by equation (10). In the actual numerical computations, we have adopted $x = \ln(r/p)$ as an independent variable.

APPENDIX B

STABILITY INTEGRAL FOR NONRADIAL OSCILLATIONS

The Lagrangian variation of the energy equation, which appears in the stability integral of equation (12), is written as

$$\delta\left(\epsilon - \frac{1}{\rho}\nabla\cdot\boldsymbol{F}\right) = \delta\epsilon + \left(\frac{\rho'}{\rho}\tilde{\varepsilon} - \frac{d\tilde{\varepsilon}}{dr}\delta r\right) - \frac{1}{\rho}\nabla\cdot\boldsymbol{F}'\,, \tag{B1}$$

where $\tilde{\varepsilon} \equiv \rho^{-1}\nabla\cdot\boldsymbol{F} = dL_r/dM_r$ and $\boldsymbol{F}'$ is the Eulerian perturbation of the radiation flux, which is calculated from the transfer equation

$$\boldsymbol{F} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\nabla T\,. \tag{B2}$$

In what follows, a common factor $(N_l^m)^{-1/2} Y_l^m(\theta, \phi)$ is suppressed for simplicity. By separating $\boldsymbol{F}'$ into the radial and the horizontal components, we find

$$-\frac{1}{\rho}\nabla\cdot\boldsymbol{F}' = -\frac{1}{4\pi r^2\rho}\frac{dL_r'}{dr} + l(l+1)\frac{L_r}{4\pi r^3\rho}\frac{(T'/T)}{d\ln T/d\ln r}\,, \tag{B3}$$

where $L_r' = 4\pi r^2 F_r'$ and the second term of the right-hand side of equation (B3) represents the radiative heat leakage due to the horizontal temperature difference. We now introduce a nonradial analog δL_r to the Lagrangian perturbation of luminosity in radial pulsation problems, defined as

$$\delta L_r = L_r' + \frac{dL_r}{dr}\delta r\,. \tag{B4}$$

By using equations (B3), (B4), and the continuity equation (2), we finally obtain

$$\delta\left(\epsilon - \frac{1}{\rho}\nabla\cdot\boldsymbol{F}\right) = \delta\epsilon + l(l+1)\frac{L_r}{4\pi r^3\rho}\frac{T'}{T}\Big/\left(\frac{d\ln T}{d\ln r}\right) + \tilde{\varepsilon}\frac{l(l+1)}{c_1\omega^2}y_2 - \frac{1}{4\pi r^2\rho}\frac{d(\delta L_r)}{dr}\,, \tag{B5}$$

where δL_r is given by

$$\frac{\delta L_r}{L_r} = 4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta\kappa}{\kappa} + \frac{d(\delta T/T)}{d\ln r}\Big/\left(\frac{d\ln T}{d\ln r}\right) - \frac{l(l+1)}{c_1\omega^2}y_2\,. \tag{B6}$$

Therefore, the stability integral for nonradial oscillations is written as

$$W = W_1 + W_2 + W_3 \tag{B7}$$

with

$$W_1 = \int_0^R \left(\frac{\delta T}{T}\right)\delta\epsilon 4\pi r^2\rho dr\,, \tag{B8}$$

$$W_2 = \int_0^R \left(\frac{\delta T}{T}\right)l(l+1)L_r\left(\frac{T'}{T}\right)\Big/\left(\frac{d\ln T}{d\ln r}\right)\frac{dr}{r}\,, \tag{B9}$$

and

$$W_3 = \int_0^R \left(\frac{\delta T}{T}\right)\left[\frac{dL_r}{dr}\frac{l(l+1)}{c_1\omega^2}y_2 - \frac{d(\delta L_r)}{dr}\right]dr\,. \tag{B10}$$

The quantity W_1 represents a contribution to the stability integral due to thermonuclear sources or sinks (if any), which include the neutrino loss ($\epsilon_\nu < 0$). On the other hand, the effect of radiative heat leakage is represented by two quantities W_2 and W_3. We have separated this into two parts because, in nonradial oscillations, there exists a horizontal energy exchange represented by W_2 in addition to the usual energy dissipation in the radial direction (given by W_3). It is evident that we recover the ordinary expressions for the stability integral of radial pulsations if we set $l = 0$ in equations (B6)–(B10).

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