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ABSOLUTE MAGNITUDES OF CEPHEIDS. III. AMPLITUDE AS A FUNCTION OF POSITION IN THE INSTABILITY STRIP: A PERIOD-LUMINOSITY-AMPLITUDE RELATION

ALLAN SANDAGE AND G. A. TAMMANN*
Hale Observatories, Carnegie Institution of Washington, California Institute of Technology
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ABSTRACT

The amplitude of Cepheid variation is maximum at the blue edge of the instability strip, and decreases monotonically toward the red in the period range $0.40 < \log P < 0.86$. In this respect Cepheids resemble RR Lyrae stars where it is known that the largest-amplitude *c* and *ab* variables are bluest. For intermediate-period Cepheids ($0.86 < \log P < 1.3$) the trend is reversed; it returns to the original sense for stars with $\log P > 1.3$. The difference in behavior may be related to Christy's explanation of the Hertzsprung relation for stars with periods near 10 days.

Amplitude as a function of strip position permits formulation of a period-luminosity-amplitude relation which is equivalent to, and nearly as accurate as, the usual period-luminosity-color relation. Its advantage is that no colors are necessary.

What were previously considered fundamental differences between Cepheids in our Galaxy and those in the SMC have largely disappeared in the present formulation when the amplitude effect is considered. Revised moduli for the LMC $[(m - M)_{AB} = 18.91]$, the SMC $[(m - M)_{AB} = 19.35]$, and M31 $[(m - M)_{AB} = 24.76]$ are adopted. A principal conclusion is that Cepheids form an adequate foundation upon which to build the calibration of those brighter distance indicators which must be used to obtain an eventual value for the Hubble constant.

I. INTRODUCTION

Cepheid variables are still the chief distance indicators for late-type galaxies closer than $m - M \approx 28$. Galaxies within this range form the only sample from which brighter indicators can be found for use at the larger distances where determination of the Hubble expansion rate must be made [i.e., $(m - M) \geq 34$ beyond any local anisotropy].

The first steps of the calibration are now finished, and reports on the absolute magnitudes of brightest stars and the sizes of H II regions are in press. However, positive proof of the first assumption of this long chain, namely, that Cepheids obey the same period-luminosity-color relation from galaxy to galaxy, has not been available. This is despite the fact that it was shown in Papers I and II of this series (Sandage and Tammann 1968, 1969) that the slope of (1) the P-L relation and (2) lines of constant period in the H-R diagram, are the same from galaxy to galaxy to within both the observational errors and the error caused by nonuniform filling of the instability strip. Our Galaxy, M31, NGC 6822, and the two Magellanic Clouds were the test objects. These tests do not prove the constancy of absolute magnitudes of Cepheids in different galaxies.

On the contrary, it has been known for several decades that disturbing differences exist between Cepheids in our Galaxy and those in the Small Magellanic Cloud (SMC). (1) Shapley and McKibben (1940) first pointed out that the period-frequency relation differs fundamentally: the Small Cloud contains many Cepheids with periods between 1.5 and 4 days, whereas our Galaxy has very few. (2) More disturbing is Gascoigne's (1969, Figs. 9 and 12) convincing demonstration that the $B - V$ colors of SMC Cepheids are bluer than those in the Galaxy by about 0.1 mag for periods shorter than 10 days (also see Christy 1970, Fig. 11). (3) The amplitudes of these short-period, blue SMC Cepheids are very large, reaching 1.6 mag. It was argued by Arp and Kraft

* Permanent address: Astronomisches Institut der Universität Basel, Binningen, Switzerland.

(1961) that such large amplitudes proved that Cepheids in our Galaxy differ fundamentally from those in the SMC, because no large-amplitude Cepheids with periods less than 8 days were thought to exist in the Galaxy.

These three points taken together appeared as strong evidence that Cepheids differ, and that their use as precision distance indicators is suspect. We believe, however, that such is not the case, but rather that all three points are intimately related, and may be but different statements of a single special phenomenon, which is *unrelated to the form and zero point of the P-L-C relation*. For example, suppose that the position and width of the Cepheid instability strip in the ($M_{\rm bol}$, log T_e)-diagram is the same for all galaxies. Suppose further that the *filling* of this strip differs from galaxy to galaxy due to differences in the evolutionary tracks that thread the strip (Hofmeister 1965, 1967; Arp 1967). Those galaxies where the tracks penetrate more deeply into the strip (on the second passage: cf. Iben 1966; Kraft 1966; Sandage and Tammann 1969) will have bluer Cepheids on the average, as in the SMC compared with our Galaxy. Christy (1970) has suggested that such a difference in strip penetration can explain points 1 (the period-frequency difference) and 2 (the color difference). Point 3 would also follow naturally if the *amplitude of light variation were largest at the blue edge of the strip*, a region strongly occupied by SMC variables with $P < 10$ days, but relatively sparsely occupied in the Galaxy.

Such a picture seemed unlikely in the 1960's because the maximum amplitude was thought to occur in the *middle* of the instability strip. If so, the large amplitude of the SMC Cepheids near $P = 2$ days would not then be compatible with excessive blueness. However, if the maximum amplitude does in fact occur at the blue edge, all data would correlate properly.[1]

To test the point we have analyzed available data on amplitude as a function of strip-position. The result is interesting in several contexts. (1) If a unique relation exists, then the relative amplitude indicates where a given Cepheid lies in the intrinsic dispersion of the P-L relation, and the dispersion can be corrected for. In previous formulations of the problem (Sandage 1958; Kraft 1961*a*, *b;* Papers I and II; Gascoigne 1969), color was shown to be a second parameter leading to a P-L-C relation. If now relative amplitude is an equivalent second parameter, then a P-L-A relation is an equivalent description of the Cepheids, and one in which no knowledge of the reddening is necessary. (2) Reddening can be determined by noting the color that a Cepheid should have at the strip position implied by its observed amplitude. (3) Apparent distance moduli to galaxies can be obtained by observing Cepheid light curves in one color only, because a P-L-A relation is color independent. (4) Most important, a quantitative test of the excess blueness versus large amplitudes for SMC Cepheids can be made. If the two parameters obey the same correlation as in our Galaxy, then the chief objection to the use of Cepheids in different galaxies would be removed.

II. METHODS

We have used two methods, which although equivalent, depend on different types of data.

a) *Magnitude residuals from the P-L relation.*—The P-L relation has an intrinsic scatter that depends on the width of the instability strip in the H-R diagram. The upper envelope of the scatter is the trace of the blue edge of the instability strip in the (M, $B - V$)-plane, while the lower envelope is the red edge (Sandage 1958). Hence, amplitude as a function of strip position can be found by correlating amplitude with magnitude residuals from the centerline P-L relation for Cepheids in galaxies where the internal

[1] That the amplitude problem is not serious in the way suggested by Arp and Kraft (1961, Fig. 2) follows from Schaltenbrand and Tammann's (1970, Fig. 2) demonstration that the period-amplitude relation for the upper envelope is the *same* for Galactic and SMC Cepheids. Arp and Kraft's earlier different result was based on a smaller sample of Galactic Cepheids.

absorption is uniform (absorption gradients will produce artificial scatter in the P-L relation). The strength of the test is that no color information is needed (provided, of course, that the absorption is in fact uniform for those Cepheids which are used).

b) *Unreddened colors* of Cepheids have an intrinsic spread of $\sim$0.3 mag in $B - V$ at a given period. The spread is dictated by the width of the instability strip in $B - V$ of the H-R diagram. For any group of Cepheids whose $(B - V)^0$ colors are known, relative amplitudes can be correlated with *color differences* from the strip center. This maps the amplitude function through the strip.

But definition of the mean color-period relation from which the color differences are taken is difficult because of nonuniform filling of the strip in different galaxies. In Paper I we adopted different mean relations for the SMC and the Galactic Cepheids, but here, with a better understanding of the problem, we adopt the *same mean* for all galaxies, operationally defined in § III*c*, and show that relative amplitudes are well correlated with color differences from this mean.

The weakness of the method is that reddening-free colors must be known. Its strength is that we need no knowledge of *magnitude differences* as in method *a;* hence Cepheids in the sample need not be at the same distance, or have known absolute magnitudes. All Cepheids in the Galaxy with known intrinsic colors can be used.

Our problem for classical Cepheids is much more difficult than the analogous one for RR Lyrae stars where all that needs be done is to study the light curves of variables in a single appropriate globular cluster. Here the horizontal branch of the cluster cuts the instability strip, often with sufficient strength—as in M3, M15, and ω Cen—to produce RR Lyrae stars from the red to the blue edge of the strip (Roberts and Sandage 1955; Dickens and Saunders 1965). No such beautiful experiment is available for classical Cepheids, because no single cluster contains sufficient variables to map the strip well enough. Methods *a* and *b* do, however, combine the Cepheid data in an equivalent way.

III. BASIC DATA

a) *Magnitude Residuals*

For method *a*, only Cepheids from the following sources have been considered, all with log $P > 0.4$ as justified later.

i) The twelve Cepheids in open clusters and associations as listed in Paper II.

ii) Fifteen photoelectrically observed Cepheids in the LMC (Gascoigne 1969). The highly reddened Cepheid HV 2749 was omitted.

iii) Eighteen photoelectrically observed Cepheids in the SMC (Gascoigne 1969).

iv) Fifteen Cepheids in Field IV of M31 studied by Baade and Swope (1963). Field IV has sufficiently uniform absorption to be useful. The photometric accuracy of Baade and Swope's data appears to be exceedingly high. However, the very faintest stars at minimum light have suspiciously flat light curves, and their amplitudes may be affected. We restrict the sample to log $P > 0.5$, and further omit the faint and uncertain variable No. 46.

The four data groups contain sixty Cepheids. They are probably all for which the photometric amplitudes and absolute magnitudes are known with sufficient accuracy. The fainter Cepheids in IC 1613 (Sandage 1971) suffer from flat light curves at minimum ($B \approx 22.0$ observed with the Hooker 100-inch) which makes the amplitudes systematically suspect at about the 0.2 mag level. The Cepheids in NGC 6822 (Kayser 1967) fall outside the period range which we show in the next sections to be important.

The absolute magnitudes of the Galactic Cepheids (group i) were taken from Paper II (Table 5, col. [13]). For the extragalactic Cepheids, $M^0\langle_B\rangle$ was found using the published $\langle B\rangle$ values and the preliminary *adopted* distance moduli $(m - M)_{AB}$ summarized in Table 1. Table 2, column (3), lists all of the $M^0\langle_B\rangle$ values so derived.

The magnitude residuals for each Cepheid were determined by comparing these $M^0\langle_B\rangle$

TABLE 1

APPARENT BLUE DISTANCE MODULI FOR NEARBY GALAXIES

Galaxy	Sandage* Tammann (1968)	Gascoigne† (1969)	Tammann† (1969)	Adopted	Fig. 2b
LMC.	18.70	18.73	18.99	18.86	18.97
SMC.	19.10	19.36	19.44	19.40	19.32
M31.	24.84	. . .	24.80	24.80	24.72

* Corrected for the new zero point as in Paper II (Sandage and Tammann 1969). The LMC and SMC values in Paper I (1968) were based on figures given in Gascoigne and Kron (1965) which have been superseded by Gascoigne (1969).

† Both authors based their reduction on the same observations for the LMC and the SMC (Gascoigne 1969) and the same absolute standards (Paper II). The differences in $(m - M)_{AB}$ are due entirely to the reduction procedure.

values with the adopted centerline P-L relation defined by Table A1 of Paper I as corrected by 0.05 mag in Paper II. The residuals, denoted by R_B, are defined to be negative if the Cepheid is brighter than the centerline and vice versa.

b) Amplitudes

To account for the change of amplitude with period, we consider the *relative amplitude*, defined as

$$f_B = 10^{0.4(\Delta B - \Delta B_{\max})} \tag{1}$$

following Kraft (1960*b*) and called by him the amplitude defect. Here ΔB is the observed amplitude in magnitudes, and $\Delta B_{\max}$ is the maximum amplitude reached by any Cepheid at a given period, taken as the upper envelope of the period-amplitude relation derived by Schaltenbrand and Tammann (1970, Fig. 1). Their envelope has been adopted for periods greater than 4.5 days, but has been modified for shorter periods to be

$$\Delta B_{\max} = -0.392 \log P + 1.736 \tag{2}$$

for $0.4 < \log P \leq 0.67$ so as to encompass the photoelectrically observed Cepheids in the Magellanic Clouds. The relation is not well defined for $\log P < 0.4$, which is our reason for not considering shorter-period stars.

The f_B was calculated for each Cepheid in the sample. Photoelectric amplitudes for Galactic Cepheids were taken from the compilation of Schaltenbrand and Tammann (1971). Amplitudes for the extragalactic Cepheids were taken from the original sources listed in § III*a*. The results are listed in Table 2, and later in Table 3. Also given in Table 2 are the color residuals needed in method *b*.

The error in f_B depends on (1) the correctness of the upper-envelope curve and (2) the photometry of individual Cepheids. The second error should be smaller than $\sim\pm 0.06$ mag for Galactic Cepheids (only photoelectric data have been used). Observed amplitudes of extragalactic Cepheids may have errors as large as ± 0.15 mag, which introduces an error of $\sim$15 percent in f_B.

c) Color Residuals

The mean intrinsic color-period relation is required to define the line about which residuals are to be taken. Intrinsic colors of extragalactic Cepheids were obtained from the observed colors by adopting $E(B - V) = 0.08$ for the LMC (Gascoigne 1969), 0.02 for the SMC (Gascoigne 1969), and 0.16 mag for M31 (Baade and Swope 1963).

TABLE 2

Magnitude Residuals, Color Residuals, and Relative Amplitudes for the Adopted Cepheids

Name	log P	$M_{\langle B\rangle}$	R_B	$\langle B\rangle^\circ-\langle V\rangle^\circ$	$\delta(B-V)$	f_B
		GALAXY				
EV Sct	0.490	-2.05	+0.38	0.61	+0.16	0.373
CE Cas b	0.651	-2.64	+0.17	....	.....	0.661
CF Cas	0.687	-2.42	+0.48	0.59	+0.08	0.559
CE Cas a	0.711	-2.63	+0.33	....	.....	0.555
UY Per	0.730	-2.96	+0.04	....	.....	0.863
VY Per	0.743	-3.36	-0.32	....	.....	0.839
U Sgr	0.828	-3.37	-0.12	0.56	0.00	0.738
DL Cas	0.903	-3.14	+0.32	0.63	+0.02	0.705
S Nor	0.989	-3.30	+0.37	0.67	+0.04	0.759
VX Per	1.037	-3.70	+0.10	....	.....	0.673
SZ Cas	1.134	-4.10	-0.04	0.61	-0.06	0.316
RS Pup	1.617	-5.06	+0.19	0.83	+0.03	0.780
LMC $(m-M)_{AB}$ = 18.86 E(B-V) = 0.08						
ROB 44	0.408	-1.92	+0.32	0.50	+0.08	0.887
24	0.429	-2.13	+0.17	0.43	0.00	1.000
33	0.478	-1.98	+0.42	0.58	+0.14	0.429
25	0.529	-2.23	+0.30	0.52	+0.06	0.466
10	0.556	-2.54	+0.04	0.48	+0.01	0.470
29	0.569	-2.48	+0.13	0.50	+0.03	0.421
22	0.669	-2.61	+0.24	0.62	+0.11	0.964
HV2432	1.038	-4.11	-0.31	0.44	-0.19	0.501
886	1.380	-4.79	-0.12	0.69	-0.05	0.773
1003	1.387	-4.90	-0.20	0.63	-0.11	0.631
902	1.421	-4.94	-0.16	0.62	-0.13	0.863
2251	1.447	-5.01	-0.17	0.67	-0.09	0.625
1002	1.483	-5.21	-0.29	0.65	-0.12	0.973
2294	1.564	-5.40	-0.28	0.69	-0.11	0.863
909	1.575	-5.38	-0.23	0.69	-0.11	0.637
SMC $(m-M)_{AB}$ = 19.40 E(B-V) = 0.02						
HV11114	0.433	-2.50	-0.20	0.38	-0.05	0.895
2015	0.458	-2.47	-0.11	0.48	+0.04	0.698
HV1906	0.486	-2.66	-0.24	0.45	0.00	1.000
11216	0.494	-2.72	-0.27	0.39	-0.06	0.847
11113	0.507	-2.44	+0.04	0.42	-0.03	0.773
212	0.591	-3.13	-0.46	0.40	-0.08	0.832
214	0.624	-3.37	-0.62	0.35	-0.14	1.000
1425	0.658	-3.08	-0.25	0.37	-0.13	0.879
1492	0.799	-3.68	-0.51	0.44	-0.11	0.946
1400	0.823	-3.36	-0.12	0.50	-0.06	0.705
11112	0.826	-3.18	+0.07	0.55	-0.01	0.655
827	1.129	-4.37	-0.33	0.56	-0.09	0.655
1328	1.200	-4.74	-0.52	0.51	-0.17	0.350
1342	1.254	-4.48	-0.11	0.57	-0.13	0.340
817	1.276	-4.97	-0.55	0.59	-0.11	0.437
823	1.504	-4.62	+0.35	0.87	+0.09	0.738
2195	1.621	-5.67	-0.42	0.73	-0.08	0.895
HV837	1.629	-5.24	-0.01	0.86	+0.04	0.718
M31 $(m-M)_{AB}$ = 24.80 E(B-V) = 0.16						
IV-21	0.525	-2.55	-0.04	0.45	-0.01	0.625
48	0.532	-2.06	+0.48	....	.....	0.433
36	0.555	-2.29	+0.30	0.51	+0.04	0.535
13	0.580	-2.24	+0.41	0.68	+0.20	0.488
26	0.596	-2.67	+0.01	....	.....	0.437
2	0.640	-3.00	-0.22	0.40	-0.10	0.745
17	0.828	-2.98	+0.27	....	.....	0.731
9	0.930	-3.51	0.00	0.58	-0.01	0.895
8	0.984	-3.72	-0.05	0.55	-0.06	0.847
3	1.104	-3.57	+0.41	0.69	+0.04	0.871
5	1.109	-3.48	+0.51	0.81	+0.16	0.643
30	1.110	-3.42	+0.58	0.84	+0.19	0.597
31	1.125	-4.34	-0.31	0.49	-0.16	0.370
IV-15	1.328	-4.12	+0.43	0.72	0.00	0.535

TABLE 3

GALACTIC CEPHEIDS USED IN METHOD b*

Name	log P	$\langle B\rangle^0-\langle V\rangle^0$	$\delta(B-V)$	f_B	Name	log P	$\langle B\rangle^0-\langle V\rangle^0$	$\delta(B-V)$	f_B
EV Sct...	0.490	0.61	+0.16	0.373	S Nor....	0.989	0.67	+0.04	0.759
SS Sct...	0.565	0.59	+0.12	0.488	DD Cas..	0.992	0.67	+0.04	0.705
SY Cas..	0.610	0.51	+0.02	0.738	BZ Cyg..	1.006	0.59	−0.05	0.581
XY Cas..	0.653	0.63	+0.13	0.575	SY Aur..	1.006	0.60	−0.05	0.649
V482 Sco.	0.655	0.55	+0.05	0.649	ζ Gem...	1.007	0.68	+0.04	0.586
CF Cas...	0.687	0.59	+0.08	0.559	Y Sct....	1.015	0.66	+0.02	0.855
AP Sgr...	0.703	0.50	−0.02	0.824	Z Lac....	1.037	0.60	−0.04	0.964
V386 Cyg	0.720	0.54	+0.02	0.667	SV Per...	1.046	0.49	−0.16	0.705
SW Cas..	0.736	0.56	+0.03	0.649	RY Cas..	1.084	0.65	−0.01	0.809
X Lac....	0.736	0.55	+0.02	0.453	SZ Cas...	1.134	0.61	−0.06	0.316
FM Cas..	0.764	0.64	+0.10	0.581	TT Aql..	1.138	0.75	+0.08	0.955
RV Sco..	0.781	0.55	+0.01	0.871	TX Cyg..	1.167	0.64	−0.04	0.871
VV Cas..	0.793	0.57	+0.02	0.879	RW Cas..	1.170	0.79	+0.11	0.863
CR Cep..	0.795	0.57	+0.02	0.445	X Cyg...	1.214	0.72	+0.03	0.631
RR Lac..	0.808	0.57	+0.02	0.780	CD Cyg..	1.233	0.72	+0.02	0.766
U Sgr....	0.828	0.56	0.00	0.738	Y Oph...	1.234	0.65	−0.05	0.268
AP Cas...	0.836	0.53	−0.03	0.603	SZ Aql...	1.234	0.65	−0.05	0.955
η Aql....	0.856	0.62	+0.05	0.809	CP Cep..	1.252	0.71	+0.01	0.483
AK Cep..	0.859	0.49	−0.08	0.661	WZ Sgr..	1.339	0.79	+0.07	0.565
CD Cas..	0.892	0.59	−0.02	0.895	T Mon...	1.431	0.80	+0.05	0.535
VY Cyg..	0.895	0.58	−0.03	0.938	AQ Pup..	1.474	0.68	−0.08	0.817
RX Cam.	0.898	0.65	+0.04	0.879	RS Pup..	1.617	0.83	+0.03	0.780
DL Cas..	0.903	0.63	+0.02	0.705	SV Vul...	1.654	0.83	+0.02	0.847

* Intrinsic colors are mean of values listed in Paper I (Table A2) and those determined via equation (3) of the text.

For Galactic Cepheids, the Γ-index method of Kraft (1960*a*) yields intrinsic colors. An independent method using *UBV* colors alone is available (Tammann 1970), which is on the Kraft system in the mean, but which is useful to smooth the scatter of the spectroscopic Γ-method. Tammann shows that

$$E(B-V) = -0.129 \log P + 2.243(\langle B\rangle - \langle V\rangle) - 1.554(\langle U\rangle - \langle B\rangle) - 0.652\,. \quad (3)$$

Of all available Galactic Cepheids, we have used only those for which intrinsic colors are available from both methods, and the mean has been adopted. The small-amplitude Cepheid BY Cas was omitted because it is very blue, has a sinusoidal light curve (Malik 1965), and resembles in these respects the SMC Cepheids HV 1897 and HV 1779 (Gascoigne 1969), which probably pulsate in the first overtone. The adopted intrinsic colors are listed in Table 3, which with Table 2 defines the basic data.

The period-color relation from these data is shown in Figure 1. The large scatter is expected, since it is the width of the instability strip read along lines of constant period (cf. Fig. 7 of Paper II). Note Gascoigne's result that Cepheids in the SMC are significantly bluer than those in the Galaxy. A least-squares line through the points would not, even in principle, define the correct mean if the strip is incompletely populated in different ways from galaxy to galaxy. The proper relation depends on the relative number of Cepheids in each system, correlated with the degree of completeness of track penetration into the strip—knowledge which we do not now have in any complete way. We have fitted the data with a line by eye, taking into account the differences in the filling factor as a function of period between the Clouds and the Galaxy, to adopt

$$\langle B\rangle^0 - \langle V\rangle^0 = 0.323 \log P + 0.290\,. \quad (4)$$

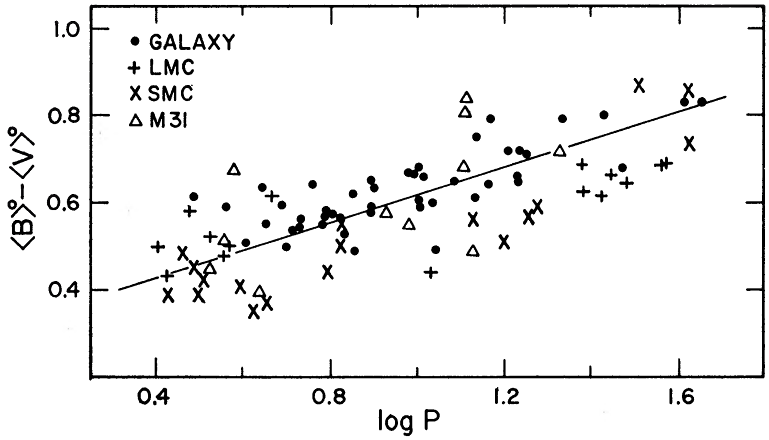

FIG. 1.—Intrinsic colors of Cepheids as a function of period. All data are from Tables 2 and 3. The scatter is expected from the finite width of the instability strip in the H-R diagram. The line is a compromise fit (eq. [4] of text) to all the galaxies, with the relative filling factor taken into account.

Note that equation (4) is about midway between the separate relations adopted in Paper I for the Galaxy and the Clouds (Sandage and Tammann 1968, eqs. [6] and [7]).

Color residuals $\delta(B - V)$ have been obtained for each Cepheid in the sample and are listed in Tables 2 and 3. The sign of $\delta(B - V)$ is taken negative if the observed color is bluer than given by equation (4). This convention is the same as in Papers I and II.

IV. RESULTS

a) Correlation of Magnitude Residuals and f_B

Upon analyzing the total material (Table 2), we found that Cepheids show quite different behavior in different period ranges. The three clearly separated intervals which emerged are $P < 8$ days, $8 \leq P \leq 20$ days, and $P > 20$ days. The correlations of f_B and R_B in each interval are shown in the upper panel of Figure 2, as plotted from data in Table 2.

It is evident that a strong correlation exists for the short-period Cepheids in the sense that those with the largest relative amplitudes (i.e., $f_B \rightarrow 1$) are brightest (i.e., negative R_B). This is clear evidence that Cepheids with periods less than $\sim$8 days *have largest amplitude at the blue edge of the instability strip*. The situation is less clear for the intermediate-period group, although an anticorrelation is suggested. Long-period Cepheids ($\log P > 1.3$) rather clearly define the same correlation as the first group, although with smaller confidence. We shall later return to the implications of the different behavior with period, but note here only that (1) the period boundary between groups 1 and 2 occurs where the conspicuous change in absolute amplitude occurs in the period-amplitude relation (Schaltenbrand and Tammann 1970) and (2) the middle-period interval is where the amplitudes are affected by a wave reflected from the stellar center which, upon reaching the surface, nearly coincides in phase with the principal maximum (Christy 1970). This appears to be a convincing theoretical explanation of the Hertzsprung (1926) progression of light-curve shapes. The effect of the reflected wave apparently confuses the basic underlying relation shown in the first- and third-period intervals. In all that follows, only the shorter-period interval will be considered, because therein are found the problems discussed in § I.

The dispersion in Figure 2*a* is considerable, and is *systematic* for each of the four galaxy groups. Taking the seven Cepheids of the Galactic system as standard gives the

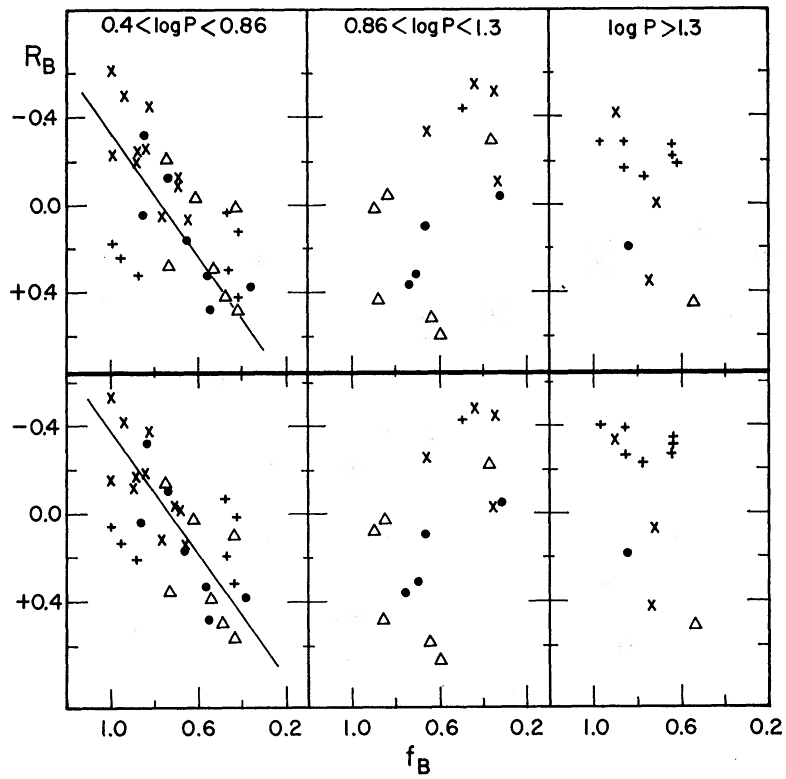

FIG. 2.—(*a*) (*Upper*) Correlation of magnitude residuals and relative amplitude in three period intervals, plotted from data in Table 2. Least-squares line through shortest-period data is $R_B = -1.46f_B + 1.09$ which excludes LMC points and forces the zero point to the seven Galactic Cepheids. Coding is same as Fig. 1. (*b*) (*Lower*) Same as upper, except that $(m - M)_{AB}$ for LMC, SMC, and M31 have been optimized for minimum scatter as listed in the final column of Table 1. Least-squares line is equation (5) of the text.

following systematic corrections to the distance moduli of the LMC, the SMC, and M31 so as to give zero systematic difference:

$$\text{LMC:} \quad \Delta(m - M)_{AB} = +0.11 \pm 0.16\,,$$

$$\text{SMC:} \quad \Delta(m - M)_{AB} = -0.08 \pm 0.04\,,$$

$$\text{M31:} \quad \Delta(m - M)_{AB} = -0.08 \pm 0.09\,.$$

The corrected moduli are listed in the final column of Table 1 and agree well with the "adopted" values of the penultimate column, which were obtained from entirely different considerations related to the P-L-C relation. Revised R_B values using these "*least amplitude-scatter moduli*" were calculated from Table 2, and the relations are replotted in Figure 2*b*.

A least-squares solution of data in Figure 2*b* (with the omission of the seven LMC stars because of their obvious large scatter away from an otherwise well defined relation) gives

$$R_B = -1.41f_B + 1.04\,, \tag{5}$$

where it is assumed that all the error is in the R_B. The correlation coefficient is very high at 0.833. Equation (5) is somewhat uncertain; it expresses the trend, but with considerable remaining scatter. The effect of scatter is seen from a second least-squares solution with f_B now as independent variable, which yields

$$R_B = -2.03 f_B + 1.48 , \tag{6}$$

or one which includes the LMC Cepheids with R_B as independent variable, giving

$$R_B = -0.90 f_B + 0.69 \tag{7}$$

with still a high correlation coefficient of 0.651 (the probability that such is the result of chance is less than 10^{-3} in a solution with 32 observations).

Much of the scatter in Figures 2*a* and 2*b* is due to observational errors. Consider first the errors of R_B. For the variables in the Galactic system, the error in the *difference* in moduli of open clusters containing Cepheids is of the order 0.1 mag. This, combined with allowance of $\Delta E(B - V) \approx 0.03$ mag for differential reddening across the clusters (giving $\Delta M^0{}_B \approx 0.12$), gives an rms error in R_B of 0.16 mag, which agrees well with the the observed mean deviation of 0.17 mag of the Galactic Cepheids relative to the adopted equation (5).

Small Magellanic Cloud.—Gascoigne (1969) estimates the error in his $\langle B \rangle$ values to be $\sim$0.1 mag. The back-to-front ratio is smaller than 0.04 mag on any reasonable geometry, and the reddening variations about the mean $E(B - V)$ of 0.02 mag should be negligible. The rms error in R_B should be close to 0.10 mag; the observed value of scatter in Figure 2*b* is $\sigma = 0.14$ mag for the SMC.

Large Magellanic Cloud.—Again the estimated error in $\langle B \rangle$ is $\sim$0.1 mag. The back-to-front ratio depends on the geometry. If the LMC is spherical, ΔR_B could be as high as 0.16 mag. An allowance of 0.08 mag permits some flattening, as is undoubtedly present in most Magellanic-type systems (cf. Hodge and Hitchcock 1966). Appreciable reddening gradients undoubtedly exist in the LMC, which makes Cepheids from this galaxy the most uncertain. A variation of $\Delta E(B - V) = \pm 0.06$ mag, or $\Delta R_B = \pm 0.24$ mag, seems possible. The errors combine to give an rms error in $R_B \simeq \pm 0.3$ mag. The observed scatter in R_B is $\sigma = 0.42$ mag.

M31.—The observed σ_B is 0.23 mag, which is about what is expected from the difficult photometry of both $\langle B \rangle$ and ΔB for stars fainter than $B \approx 21$ mag.

In all these systems, the error in f_B must also be considered. Typical errors of 0.1 mag in the observed amplitudes are expected which affects f_B by about 10 percent.

The conclusion is that much of the observed scatter about equation (5) (Fig. 2) is observational error. However, by analogy with RR Lyrae stars, it would be surprising if the intrinsic scatter were actually zero, since it is known that the period-amplitude relation for cluster variables in a given globular cluster has considerably larger scatter than the observational errors. We emphasize, however, that the data displayed in Figure 2 are both second-order terms, and the accuracy of original material must be very high even to establish the trends.

We believe the results of Figure 2 to be new. For Cepheids with $P \leq 8$ days, the largest amplitudes occur on the blue side of the strip (highest negative R_B). But other data which are accurate in a differential sense, such as the photographic material of Arp (1960), Woolley *et al.* (1962), and Dickens (1966), show the same trends. Furthermore, the data for NGC 1866 in the LMC (Hodge 1961; Arp and Thackeray 1967) do not contradict the result.

The earlier different conclusion that maximum amplitude occurs near the center of the strip is not primarily due to the less accurate earlier material. Rather, it is to be ascribed to the fundamental difference in the upper-envelope line of the period-amplitude relation which has been adopted here compared with that used in the 1960's.

b) Correlation of Color Differences and f_B

An independent test of the conclusions is available from Figure 3, plotted from data in Tables 2 and 3. The left panel (and possibly the right) shows that the *largest-amplitude Cepheids are bluest.* Note that the SMC variables are bluest of all and have the largest amplitudes.

The least-squares solution for the short-period group [where the LMC Cepheids are again omitted by noting that their $\delta(B - V)$ values are redder than average whereas at the same time their R_B values are fainter than average in Fig. 2, a condition suggesting differential reddening], assuming $\delta(B - V)$ as the independent variable, gives

$$\delta(B - V) = -0.333 f_B + 0.241 \,. \tag{8}$$

The thirty-four Cepheids used in the solution give a correlation coefficient of 0.686, which is highly significant (the probability of a chance result is $<10^{-4}$).

V. IS THERE A REMAINING SYSTEMATIC COLOR DIFFERENCE BETWEEN SMC AND GALACTIC CEPHEIDS?

We now return to the systematic color difference between Galactic Cepheids and those in the SMC. Figure 1 shows the raw $\langle B\rangle^0 - \langle V\rangle^0$ difference to be $\sim$0.1 mag in the interval $\log P < 0.86$ (see also Fig. 9 of Gascoigne 1969). However, Figure 3 shows that most of the effect is due to the larger $\langle f_B\rangle$ for the SMC than for the Galaxy. Can this explain the entire difference?

It is instructive to proceed semigraphically by combining Figure 1 and equation (8), to remove the amplitude effect from the data. We plot again in Figure 4*a* the data of Tables 2 and 3 for the raw color-period relation. This is Figure 1 exactly in a restricted period interval. The least-squares line is

$$[\langle B\rangle^0 - \langle V\rangle^0]_{\text{mean}} = 0.250 \log P + 0.346 \,, \tag{9}$$

valid only for $\log P < 0.86$. Because of this period restriction, equation (9) differs slightly from equation (4) which is valid for the entire range. The correlation coefficient in Figure 4*a* is only 0.383, and the rms deviation of a single observation is 0.079 mag.

By means of equation (8) we can calculate the intrinsic colors which a Cepheid of given f_B *would* have if it lay on the blue boundary ($f_B = 1$). This is done by forming $\langle B\rangle^0 - \langle V\rangle^0 - 0.333(1 - f_B)$ for all stars in Tables 2 and 3 with $\log P < 0.86$. The result is shown in Figure 4*b*. The least-squares line has the form

$$[\langle B\rangle^0 - \langle V\rangle^0]_{\text{blue edge}} = 0.241 \log P + 0.255 \,. \tag{10}$$

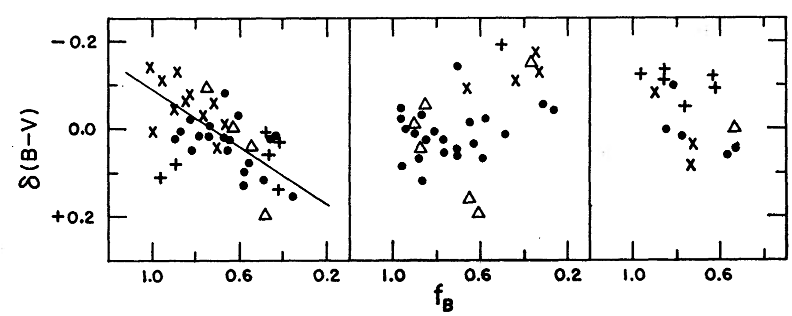

FIG. 3.—Correlation of color residuals and relative amplitude in the same period intervals as Fig. 2. Coding is same as in Figs. 1 and 2. Least-squares line through the short-period data is equation (8) of the text.

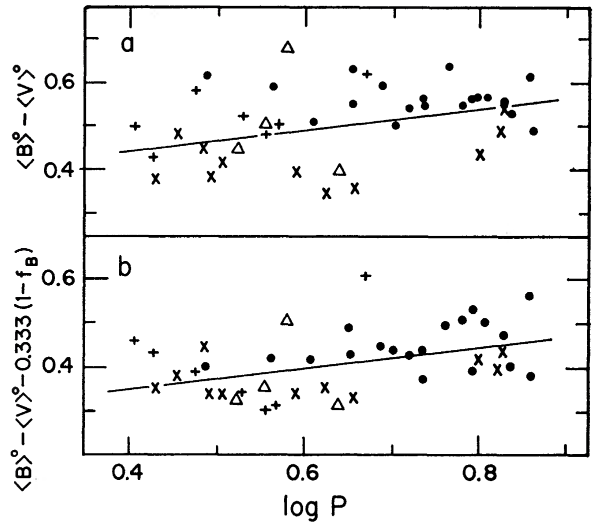

FIG. 4.—(*a*) (*Upper*) Period-color relation for Cepheids from Tables 2 and 3. Same as Fig. 1 in the restricted period range of log $P < 0.86$. Coding same as in Figs. 1–3. Least-squares line is equation (9) of text. (*b*) (*Lower*) Same as Fig. 4*a* after correcting for the mean amplitude effect of equation (8) (cf. Fig. 3) by subtracting $0.333(1 - f_B)$ from the observed intrinsic colors. This corrects the colors to the mean blue edge of the instability strip. Least-squares line is equation (10) of text.

There is an obvious improvement of Figure 4*b* over 4*a*, although some scatter remains. The correlation coefficient is increased to 0.488, and the rms deviation of a single observation is reduced to 0.057 mag, which is now quite close to the observational error.

But the major importance of Figure 4*b* is the large reduction of the *systematic* difference between the galaxies which is present in Figure 4*a*. Table 4 lists these systematic differences as read from the relevant least-squares lines. Without the amplitude correction, the difference between our Galaxy and the SMC is $0.036 + 0.067 = 0.103 \pm 0.030$ mag, which is significant. But taking the relative amplitude into account reduces the effect to $0.015 + 0.026 = 0.041 \pm 0.020$ mag, which is hardly significant. The small residual is now within the observational errors composed of (1) the reddening corrections of Galactic Cepheids, (2) photometric errors for the rather faint SMC Cepheids as measured with a 50-inch telescope, (3) systematic errors in the reduction of light curves to $\langle B\rangle - \langle V\rangle$, known to reach $\epsilon(B - V) \simeq 0.02$ mag, and (4) errors of 10 perçent in f_B due to uncertainties in observed amplitudes which introduce, via equation (8), color-correction errors of $\sim$0.02 mag.

TABLE 4

SYSTEMATIC INTRINSIC COLOR DIFFERENCES WITH AND WITHOUT THE AMPLITUDE CORRECTION IN FIGURE 4*

System	$\Delta(B-V)$, Fig. 4*a*	$\Delta(B-V)$, Fig. 4*b*
Galaxy......	$+0.036 \pm 0.015$	$+0.015 \pm 0.013$
SMC.......	-0.067 ± 0.027	-0.026 ± 0.015
LMC.......	$+0.041 \pm 0.025$	$+0.026 \pm 0.041$
M31........	$+0.005 \pm 0.064$	$+0.013 \pm 0.047$

* Valid only over the period range defined in Fig. 4: log $P < 0.86$.

The conclusion from Figures 3 and 4 and Table 4 is that *there is no evidence for intrinsic differences in the color properties of Cepheids in the four galaxies,* when the difference in amplitudes is considered.

VI. PERIOD-COLOR-AMPLITUDE RELATION: A METHOD TO FIND REDDENING

The intrinsic color is, by the definition of $\delta(B-V)$,

$$[\langle B\rangle^0 - \langle V\rangle^0]_{\text{true}} = [\langle B\rangle^0 - \langle V\rangle^0]_{\text{mean line}} + \delta(B-V) . \quad (11)$$

Combining equations (8) and (9) gives for an individual Cepheid (whose period is within $0.40 < \log P < 0.86$),

$$\langle B\rangle^0 - \langle V\rangle^0 = 0.250 \log P - 0.333 f_B + 0.587 . \quad (12)$$

Note that equation (10) for the blue boundary is recovered to within 0.005 mag (over the valid period range) by putting $f_B = 1$ in equation (12).

A more direct formulation of the P-C-f relation is by using the data of Tables 2 and 3 to obtain a formal least-squares solution directly, which gives, in the interval $0.40 < \log P < 0.86$,

$$\langle B\rangle^0 - \langle V\rangle^0 = 0.227 \log P - 0.311 f_B + 0.584 . \quad (13)$$

This, by definition, is the best representation of the data, and differs from equation (12) by less than 0.02 mag over the period range considered.

Equation (13) can be used to obtain reddenings of Cepheids on the color system of Table 3 (basically Kraft's). However the scatter is large, being the same as that shown in Figure 4*b* ($\sigma = 0.057$ mag). At least half of the scatter can result from errors in f_B [if $\sigma(\delta f_B/f_B) \approx 0.1$, then the error in $B - V$ will have $\sigma \simeq 0.022$ mag for a typical f_B of 0.7]. Additional scatter results from the reddening error of the calibrating stars in Table 3, which can hardly be less than $\sigma = 0.03$ mag. That remaining ($\sigma \simeq 0.04$) may be the true cosmic scatter (i.e., the failure of amplitude to be a perfect second parameter).

VII. THE PERIOD-LUMINOSITY-AMPLITUDE RELATION

It is evident from the preceding sections that there must exist a P-L-f relation which is equivalent to the usual formulation of the P-L-C relation as combined with some form of Figure 2 or equation (5). To obtain the relation explicitly we can proceed in three ways. (1) The P-L-C relation of Paper II (eqs. [19] and [20]) or of Tammann (1970) can be transformed by substitution of equation (13) for the color term. This has the logical disadvantage that the P-L-C relation of Paper II is valid for the entire period range of $\log P < 1.9$. For use in the smaller interval $0.40 < \log P < 0.86$, a different optimization would apply. (2) An adopted centerline P-L relation can be combined with the $R_B = g(f_B)$ of equation (5). (3) A least-squares solution for $M_B = f(\log P, f_B)$ can be made directly for the short-period Cepheids in Table 2. We consider here only methods (2) and (3).

Method 2.—To be consistent, we must use the same centerline P-L relation that was used to derive the R_B values leading to equation (5), i.e., Table A1 of Paper I, modified by 0.05 mag brighter in Paper II. The full nonlinear relation is well approximated over the interval $0.40 < \log P < 0.86$ by the centerline relation

$$M^0{}_{\langle B\rangle} = -2.35 \log P - 1.29 . \quad (14)$$

Each Cepheid can be corrected to this line for amplitude effect by noting from equation (5) that $f_B = 0.738$ when $R_B = 0$, or from equation (8) that $f_B = 0.725$ when $\delta(B - V) = 0$; each condition separately defines a mean line in either the P-L or P-C relation. We adopt $f_B = 0.731$ as the centerline condition. Equation (5) shows that the

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value of the systematic deviations due to amplitude is $\Delta M_B = 1.41(0.731 - f_B)$, which, when applied to equation (14) with the proper sign, gives

$$M^0\langle B\rangle = -2.350 \log P - 1.41 f_B - 0.26\,, \qquad (15)$$

for $0.40 < \log P < 0.86$. The constant is not well determined, containing as it does the considerable error of the zero point of equation (5) (note the scatter of Fig. 2). We therefore choose to define the constant by requiring *zero* systematic difference in M_B for the seven fundamental calibrating Galactic Cepheids (Table 2). This gives

$$M^0\langle B\rangle = -2.350 \log P - 1.41 f_B - 0.224\,. \qquad (16)$$

The rms deviation of the seven calibrating Cepheids from equation (16) is ± 0.17 mag.

Method 3.—The direct least-squares solution, if stars in Table 2 with $0.40 < \log P < 0.86$ are used (but with the LMC omitted for the previous reasons), and with the $M^0\langle B\rangle$ values modified to the "amplitude-minimized" moduli listed in the final column of Table 1, gives

$$M^0\langle B\rangle = -2.386 \log P - 1.406 f_B - 0.205\,, \qquad (17)$$

where again the constant term has been adjusted to the seven Galactic Cepheids alone. Equation (17) is in remarkable agreement with equation (16), and is adopted here as the final solution.

Figure 5 illustrates the reduction in intrinsic scatter when the amplitude correction is

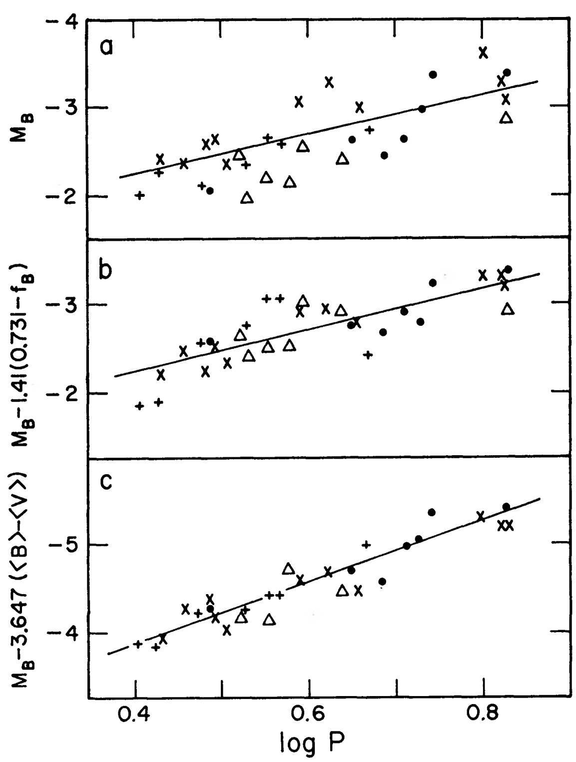

FIG. 5.—(*a*) The P-L relation from $0.40 < \log P < 0.86$ using data listed in Table 2 but revising $M^0\langle B\rangle$ values with moduli listed in the final column of Table 1. Coding is same as Figs. 1–4. The line is equation (14) of text. (*b*) Same as Fig. 5*a* but corrected for amplitude effect via equation (5) using $f_B = 0.731$ for the centerline. Line is the same as in Fig. 5*a*. The rms deviation is 0.24 mag including LMC, or 0.17 mag excluding LMC. (*c*) Equivalent to Fig. 5*b* but using colors in the P-L-C relation of equation (18) to correct for the finite strip width. The rms deviation is 0.17 mag. Line is equation (18) of the text.

made. The raw P-L relation is shown in the upper panel. The correlation coefficient is 0.739, and the rms deviation from the line of a single Cepheid (with the LMC omitted) is $\sigma = \pm 0.30$ mag. (This agrees with the known intrinsic width of the P-L relation of 1.2 mag [Paper I] since 95 percent of the points should be within $\pm 2\,\sigma$, or ± 0.60 mag, of the centerline.) Figure 5*b* shows the reduction in scatter by applying $1.41\,(0.731 - f_B)$ to the ordinate. The rms deviation is now reduced to 0.17 mag omitting the LMC, or to 0.24 mag including the LMC, and the correlation coefficient rises to 0.876. The scatter that is still present ($\sigma = 0.17$) is in part due to errors in $\langle B\rangle$ and f_B, but also to intrinsic noise of the amplitude effect itself.

Figure 5*c*, in comparison with Figure 5*b*, illustrates the relative merits of the P-L-C and P-L-f relations. The latest form of the P-L-C relation (Paper II and Tammann 1970) is

$$M^0\langle_B\rangle = -3.534 \log P + 3.647(\langle B\rangle^0 - \langle V\rangle^0) - 2.469\,. \tag{18}$$

The ordinate in Figure 5*c* is $M^0\langle_B\rangle - 3.647(\langle B\rangle^0 - \langle V\rangle^0)$, where the $M^0\langle_B\rangle$ values have been calculated using the moduli listed in column (4) of Table 1, i.e., labeled "Tammann 1969." These are based strictly on equation (18) using the whole of the available Cepheid material of all periods. These moduli reduce the scatter in Figure 5*c* in the same way as the "amplitude minimizing moduli" minimizes Figure 5*b*.

The scatter in Figure 5*c* is not appreciably smaller than that in Figure 5*b*—an indication that the P-L-C and P-L-f representations are of nearly equal accuracy.

VIII. DISTANCE MODULI OF THE LMC, THE SMC, AND M31

Equation (17) can be used to calculate absolute luminosities of all extragalactic Cepheids under consideration. Combining these with $\langle B\rangle$ gives $(m - M)_{AB}$ for each Cepheid. We have done this for seven Cepheids in the LMC giving $(m - M)_{AB} = 18.97 \pm 0.16$, eleven in the SMC for $(m - M)_{AB} = 19.30 \pm 0.04$, and seven in M31 giving $(m - M)_{AB} = 24.72 \pm 0.08$, where the errors are the rms deviations, reduced by $\sqrt{n}$ for the number of Cepheids. These moduli are practically identical with those listed in the final column of Table 1 which were derived graphically from Figure 2.

We list in Table 5 a comparison of these moduli with those determined from the P-L-C relation as applied to each individual Cepheid (i.e., eq. [18] as applied by Tammann 1969, averaged with Gascoigne's 1969 independent formulation). Column (3) has been taken from the penultimate column of Table 1. The agreement between the methods is highly satisfactory, and is by no means trivial. The two methods employ different parameters (color and f_B) which are fully independent. Furthermore, the *zero point* of the P-L-C relation rests on thirteen Galactic Cepheids (Paper II), whereas only seven could be used for equation (17). The number of Cepheids to which the methods could be applied also differed greatly; the P-L-C relation could be used for seventeen Cepheids in the LMC, twenty-four in the SMC, and seventeen in M31, whereas only seven, eleven, and seven, respectively, could be studied with equation (17).

TABLE 5

COMPARISON OF MODULI FROM P-L-f AND P-L-C RELATIONS

System (1)	$(m-M)_{AB}$ P-L-f (2)	$(m-M)_{AB}$ P-L-C (3)	$(m-M)_{AB}$ Mean (4)	$E(B-V)$ (5)	$(m-M)_0$ Adopted (6)
LMC........	18.97	18.86	18.91	0.08	18.59
SMC........	19.30	19.40	19.35	0.02	19.27
M31.........	24.72	24.80	24.76	0.16	24.12

It should be emphasized that use of the second parameter (either color or f_B) is crucial to obtain correct distances to galaxies like the SMC where larger than average amplitudes or bluer than average colors exist. Neglect of the second parameter by force-fitting a raw P-L relation for minimum scatter to Table A1 of Paper I can lead to errors as large as 0.4 mag in $m - M$.

IX. SMALL-AMPLITUDE CEPHEIDS: POLARIS

The blue boundary of the instability strip is well defined observationally by large-amplitude Cepheids, but not so the red due to the incomplete discovery of small-amplitude variables. The classic small-amplitude Cepheid is Polaris ($\log P = 0.599$) with $\Delta B = 0.27$, $f_B = 0.325$, and an observed color of $\langle B \rangle - \langle V \rangle = 0.59$. In Paper II we believed that Polaris lay on the centerline of the P-L relation (cf. Fig. 4 of Paper II) despite its small amplitude—in serious contradiction with the present correlations. The conclusion, however, depended on adopting $E(B - V) = 0.09$ (Fernie 1966).

It now seems most probable that the reddening is considerably smaller. Use of $\langle U \rangle = 2.95$, $\langle B \rangle = 2.58$, and $\langle V \rangle = 1.99$ (Schaltenbrand and Tammann 1971) in equation (3) gives $E(B - V) = +0.02$. Independent evidence is available from McNamara's (1968) high-weight determination of $E(B - V) = 0.00$ from multicolor photometry. With this value, Polaris now satisfies the P-C-f relation (eq. [13]). We consider this to be supportive evidence for the present point of view. Three conclusions can be drawn from the result.

1. The photometric effect of the unseen spectroscopic comparison of α UMi A is negligible. To be sure, this was expected from Roemer's (1965) spectroscopic orbit which requires $\Delta V > 5$ mag, but it is here likewise demanded to ensure that the small amplitude is real and not a result of photometric contamination. That the small amplitude *is* real also follows from the small radial-velocity range (Fernie 1966).

2. Polaris is a normal Cepheid of type I (Eggen 1965; Fernie 1966) which happens to lie near the red edge of the instability strip; hence the small amplitude.

3. The absolute magnitude is $M^0{}_{\langle B \rangle} = -2.10$ from the P-L-f relation (eq. [17]). The P-L-C relation (eq. [18]) gives $M^0{}_{\langle B \rangle} = -2.43$. A mean is $M^0{}_{\langle B \rangle} = -2.26$, $M^0{}_{\langle V \rangle} = -2.85$. McNamara (1968) has concluded that if α UMi B is a physical companion, then Polaris must have $M^0{}_{\langle V \rangle} = -3.7$. From the discrepancy we conclude that components A and B most likely form only an optical pair where B is considerably more distant.

Other small-amplitude Cepheids are known. In a special effort to find such stars, Stobie and Alexander (1970) discovered HR 4768 to be a Cepheid with $\Delta B = 0.29$, $\log P = 0.524$, and $f_B = 0.319$. If it is type I, it is the star with smallest f_B known in its class. The reddening is not accurately known, but equation (3) gives $E(B - V) = 0.08$, which, when combined with the observed color of $\langle B \rangle - \langle V \rangle = 0.63$, gives $\langle B \rangle^0 - \langle V \rangle^0 = 0.55$. If the variable follows the present precepts, the intrinsic color should agree with equation (13). It does to within 0.05 mag. Stated differently, $\delta(B - V) = +0.08$ as read from Figure 1, and this agrees with the correlation line of Figure 3 to within 0.05 mag. More accurate reddening values are desirable for this star to improve this comparison.

The case of HR 8157 is particularly interesting as it is a close visual binary (which has now disappeared near conjunction) and is a probable Cepheid (Millis 1969) with very small apparent amplitude ($\Delta B = 0.21$ mag). The observed color is $\langle B \rangle - \langle V \rangle = 0.50$, which with $E(B - V) = 0.12$ (Abt and Levy 1970) gives an intrinsic color of 0.38, whereas equation (13) requires $\langle B \rangle^0 - \langle V \rangle^0 = 0.61$ if $\log P = 0.523$ and $f_B = 0.299$. There is, however, evidence from double-star observers that the companion is equal in V to the primary, which would affect the f_B value; furthermore, one would conclude that the companion is probably very blue because the variable has the unique property of no apparent change in $U - B$ over the cycle. But the spectroscopic evidence of Abt and Levy is contradictory. The star is an obviously important case to understand.

Undoubtedly, many more small-amplitude Cepheids remain to be discovered, and these will be of considerable importance in extending the present results.

X. CONCLUSIONS AND SUMMARY

(1) Cepheids with $0.40 < \log P < 0.86$ have the largest amplitudes on the blue side of the strip. The amplitudes decrease monotonically as the strip is sampled toward the red. The trend is less clear for Cepheids with $0.86 < \log P < 1.3$, and the data suggest that the sense of the correlation is reversed (Figs. 2 and 3). Cepheids with $\log P > 1.3$ behave again like the shorter-period group. The behavior of the middle-period interval may be related to Christy's reflection phenomenon which produces interference with the fundamental amplitude for periods in a range near 10 days.

Of interest are Christy's (private communication) recent theoretical calculations which predict that the largest-amplitude Cepheids with $P \lesssim 8$ days should, in fact, occur at the blue edge. In this respect the stars behave similarily to the RR Lyrae stars, as is known observationally (Roberts and Sandage 1955, Figs. 5 and 6) and from theory (Christy 1966, Fig. 16).

(2) The Hertsprung progression of light-curve shapes is known to be nearly identical in our Galaxy and the Clouds (Payne-Gaposchkin and Gaposchkin 1966). Recent work by van Genderen (1970, Fig. 1) shows that the correspondence is indeed remarkably good. If the Hertzsprung-Christy phenomenon is the explanation for the three period panels in Figures 2 and 3, then, from van Genderen's demonstration, it is not so surprising that the four galaxies studied here follow the same amplitude behavior pattern of Figures 2 and 3.

(3) The amplitude, magnitude, and color data are compatible with the assumption that a unique instability strip exists in the (M_B, $B - V$)-plane, the population of which may vary systematically from galaxy to galaxy. In Paper I we believed the Magellanic Cepheids at $\log P = 0.6$ to be 0.21 mag bluer than those in the Galaxy and M31 (cf. eqs. [6] and [7] of Paper I). Gascoigne's (1969) new data removed the color difference at $\log P = 0.6$ for the LMC, and reduced the difference between the Galaxy and the SMC to $\sim$0.13 mag (cf. his Fig. 9). Now, we can explain this difference to within a systematic residual of only 0.04 ± 0.02 mag by noting that SMC Cepheids have unusually large amplitudes, which would require the excessive blueness that is observed. And the 0.04 mag is hardly significant when one considers the errors involved.

(4) The existence of the same type of correlation of R_B and of $\delta(B - V)$ with f_B for the four galaxies argues for a unique upper envelope to the period-amplitude relation, at least in the period interval $0.4 < \log P < 0.86$.

(5) In this same period interval, the centerlines of the P-L and P-C relations are represented by Cepheids with $f_B \simeq 0.731$. The blue boundary of the instability strip lies at $f_B = 1$, which, from equation (8), lies 0.09 mag blueward of the mean line. The smallest f_B in the present sample is 0.37 for EV Sct ($\log P < 0.86$). Again from equation (8), this star is expected to be 0.12 mag redder than the centerline color, a value agreeing well with the observed value of $\delta(B - V) = 0.16$ mag. The color of Polaris is now consistent with its small amplitude, provided that its reddening is negligible.

The smallest possible value of f_B is 0.23 ($\Delta B = 0$, $\Delta B_{\max} = 1.6$). If equation (8) applies at the red edge of the strip where $\Delta B = 0$, this edge would be 0.17 mag redder than the middle line, giving the strip a total width of $\Delta(B - V) = 0.26$ mag (for $0.40 < \log P < 0.86$).

(6) Reddening values can be found from the P-C-f relation of equation (13), with a *systematic* accuracy somewhat better than the scatter ($\sigma = 0.057$) shown in Figure 4*b* because of random errors of colors of the calibrating stars.

(7) Distance moduli can be found from the P-L-f relation (eq. [17]) with an accuracy that is not significantly different from that obtained with the P-L-C relation (eq. [18]).

Revised moduli to the LMC, the SMC, and M31 obtained by using both methods are listed in Table 5. Failure to take the second parameter (color or f_B) into account can result in errors up to 0.4 mag in final $(m - M)$ values.

(8) Extension of the present relations would be considerable if data for Cepheids covering the full range of f_B were available in at least one suitable galaxy. There may be no Galactic Cepheids of large amplitude with log $P \lesssim 0.65$. Accurate photometry of *small*-amplitude Cepheids in the SMC would therefore be very valuable. Such data could possibly settle the question whether the small (and only marginally significant) color difference of 0.04 ± 0.02 mag between the SMC and the Galaxy is real.

(9) There now seems to be no compelling observational evidence for a difference in the characteristics of Cepheids from galaxy to galaxy. These results appear to justify the use of Cepheids as the foundation upon which to build the distance-scale calibration of brighter indicators in local galaxies; calibrations which will eventually lead to the determination of the Hubble constant.

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