THE ASTRONOMICAL JOURNAL VOLUME 76, NUMBER 2 MARCH 1971

Resonances in the Neptune–Pluto System

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(Received 10 August 1970; revised 12 October 1970)

Pluto's orbit has been integrated for 4.5 million years. The previously discovered libration of $3\lambda-2\lambda_N-\tilde{\omega}$, where λ and λ_N are the mean longitudes of Pluto and Neptune and $\tilde{\omega}$ is Pluto's longitude of perihelion, is confirmed and has an average period of 19 951 yr. It was also found that the argument of perihelion ω librates about 90° with an amplitude of 24° and a period of 3 955 000±20 000 yr. There is an indication that both the difference between the nodes and the difference between the longitudes of perihelia of Neptune and Pluto may be locked on to the ω libration. All of the above effects are found to improve the stability of the Neptune–Pluto system by increasing the minimum distance of approach between the two bodies.

A SIMPLE examination of the orbital periods of Uranus, Neptune, and Pluto shows that they are very nearly in the ratio of 1:2:3. It is well-known that such commensurabilities or near commensurabilities of the periods give rise to perturbations of the motion which have long periods and moderately large amplitudes. For example the phase $2\lambda_N-\lambda_U$, where λ_N and λ_U are the mean longitudes of Neptune and Uranus, takes 4230 yr to circulate through 360° and gives rise to terms in the longitudes of the two planets with $\sim 1°$ amplitudes. Such long-period perturbations in the motion of Pluto are expected from both Uranus and Neptune.

It is another peculiarity of Pluto's orbit that its perihelion distance is less than that of Neptune. This condition suggests that Neptune and Pluto could make a close approach to one another. Such considerations lead Cohen and Hubbard (1965) to perform a 120 000-yr simultaneous integration of the five outer planets. Their results showed that the angle

$$\theta_N = 3\lambda - 2\lambda_N - \tilde{\omega}, \qquad (1)$$

where λ and $\tilde{\omega}$ are Pluto's mean longitude and longitude of perihelion, librated about 180° with an amplitude of 76° and a period of 19 670 yr. As a consequence of this libration of θ_N, Pluto and Neptune can never approach one another in the vicinity of Pluto's perihelion. In fact the closest approach of Pluto to Neptune, 18 a.u., was found to occur near Pluto's aphelion. Two other minima of no less than 25 a.u. occured during the 500 yr it took for the two planets to return to nearly the same relative positions. A later study by Cohen, Hubbard, and Oesterwinter (1967) improved the elements for Pluto. A 300 000-yr integration with these new elements caused θ_N to librate with an amplitude of 80° and a period of 19 440 yr. Recently, Cohen, Hubbard, and Oesterwinter (1971) have extended their integration to 1 000 000 yr.

The term in the perturbations of Pluto by Neptune with phase θ_N arises from the large eccentricity of Pluto's orbit. Brouwer (1966) pointed out that the high inclination of Pluto's orbit should give rise to another important term having a phase of $2\theta_N'$, where

$$\theta_N' = 3\lambda - 2\lambda_N - \Omega, \qquad (2)$$

where Ω is the longitude of Pluto's node. He also remarked that, since the argument of perihelion $\omega=\theta_N'-\theta_N$ only moved 0.2° in Cohen and Hubbard's integration, it was unclear whether ω circulates or oscillates. There was some hint from Cohen and Hubbard's diagrams that the change in ω was nonlinear. A study by Hori and Giacaglia (1967) concluded that the argument of perihelion circulates with a period of 30 million years. Their results were based upon the perturbations which would be expected from Neptune alone. Neptune's orbit was taken as circular and unperturbed.

An attempt was made by Cohen and Peters (1970) to integrate Pluto using a circular, uninclined orbit for Neptune and ring potentials expended into zonal harmonics for Jupiter, Saturn, and Uranus. Only partial success was achieved in matching the numerical integrations. There seemed to be an incompatability in getting the proper precession rate of ω and the proper libration period of θ_N with a single set of starting conditions.

The 120 000-yr integration indicated that significant variations of Pluto's elements occur over a time scale of several million years. An integration from 2.1 million years B.C. to 2.4 million years A.D. was undertaken to determine what the variations were

FORMULATION

The integration of Pluto's orbit was made using a variation-of-parameters technique. The derivatives of the Keplerian elements of Pluto are given by the planetary equations (Brouwer and Clemence 1961)

$$\begin{aligned}\frac{da}{dt}&=\frac{2}{na}\frac{\partial R}{\partial M},\\ \frac{de}{dt}&=\frac{1-e^2}{na^2e}\frac{\partial R}{\partial M}-\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R}{\partial\omega},\\ \frac{di}{dt}&=\frac{1}{na^2(1-e)^{\frac{1}{2}}}\left[\cot i\frac{\partial R}{\partial\omega}-\frac{1}{\sin i}\frac{\partial R}{\partial\Omega}\right],\\ \frac{dM}{dt}&=-\frac{2}{na}\frac{\partial R}{\partial a}-\frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}+n,\\ \frac{d\omega}{dt}&=\frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R}{\partial e}-\frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial R}{\partial i},\\ \frac{d\Omega}{dt}&=\frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\frac{\partial R}{\partial i},\end{aligned}\tag{3}$$

where a is the semimajor axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of perihelion, and M is the mean anomaly. The mean motion n is related to a through $n^2a^3=\mu$, where μ is the product of the gravitational constant and the sum of the mass of the sun and the mass of Pluto. The disturbing function R is the sum of the individual disturbing functions R_j from the jth perturbing planet:

$$R_j=\mu_j\left(\frac{1}{\Delta_j}-\frac{r\cos S_j}{r_j^2}\right).\tag{4}$$

The distance of Pluto from the sun is r, the disturbing planet's distance from the sun is r_j, Δ_j is the distance between Pluto and the disturbing planet, S_j is the heliocentric angle between Pluto and the disturbing planet, and μ_j is the product of the gravitational constant and the disturbing planet's mass.

The planetary equations may be integrated numerically as they stand, but the integration step size would have to be small compared to the orbital periods. Though a 4.5-million-year integration would be feasible with such a step size, there are techniques which can cut the required computation time by more than an order of magnitude when the short-period oscillations of Pluto's orbit are not needed. Short period here means less than 1000 yr.

Gauss' method is a well-known device for isolating the secular terms in the planetary equations. The disturbing function R_j is averaged over the mean anomalies of the disturbed and disturbing bodies M and M_j while the other elements are held constant. R_j is then replaced by

$$\langle R_j\rangle=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}R_j\,dM_j\,dM,\tag{5}$$

where $\langle\ \rangle$ denotes the averaging. In practice it is the partial derivatives of R_j which are averaged since it is the partials which are used in the planetary equations. The averaged partials are identical with the partials of $\langle R_j\rangle$. Poisson's theorem allows $\langle\partial R_j/\partial M\rangle$ to be set equal to zero. The integration over M_j always causes the second term on the right-hand side of Eq. (4) to disappear. To avoid solving Kepler's equation, it is best to convert the integral over M to an integral over true anomaly f by means of

$$dM=\frac{r^2}{a^2(1-e^2)^{\frac{1}{2}}}df.\tag{6}$$

The integral over M_j can be performed analytically in terms of elliptic integrals while the integral over f must be done numerically. For a specific discussion see Plummer (1960) or Musen (1963). Gauss' method is not applicable if the short-period perturbations become large, as is the case with a close approach, or if the mean motions of the disturbed and disturbing bodies are commensurable.

For the case of commensurate mean motions there is another averaging procedure which may be used. If the ratio of the two mean motions n/n_j is approximately equal to the ratio of two integers N/N_j then the averaged disturbing function $\langle R_j\rangle$ is

$$\langle R_j\rangle=\frac{1}{2\pi N}\int_0^{2\pi N}R_j\,dM.\tag{7}$$

All of the elements other than M and M_j are held constant during the integration. M_j is constrained by

$$N_jM-NM_j=\text{const.}\tag{8}$$

This procedure has been used by Schubart (1964, 1968) to study commensurabilities. The two kinds of averaging procedures use mean elements, elements with the short-period terms averaged out.

For the integration of Pluto, several simplifying approximations were made. The effects of the four terrestrial planets were approximated by including their masses in μ. The orbits of the outer planets were considered completely known and unaffected by Pluto. Pluto's motion was integrated as though it were an infinitessimal mass point except that its mass (Duncombe, Klepczynski, and Seidelmann 1968) was included in μ. In Table I are given the adopted masses of the outer planets. For Jupiter, Saturn, Uranus, and Neptune the secular variations of the elements were modeled according to the calculations of Brouwer and van Woerkom (1950). The average semimajor axes for Jupiter and Saturn were taken from Clemence (1949). For Uranus and Neptune the average semimajor axes and mean longitudes were taken from Hill (1913) since Clemence's data are not free of the 4230-yr terms arising from the near 1:2 commensurability between

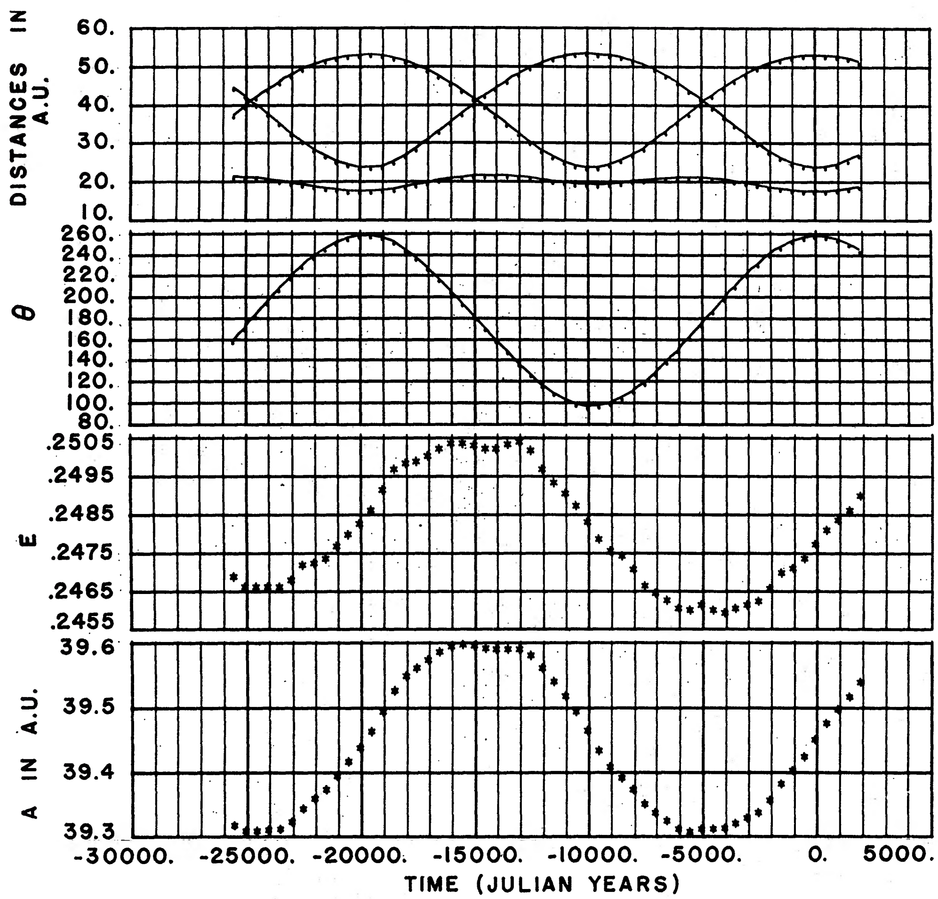

FIG. 1. 20 000-yr periodicities in the minimum distances of approach to Neptune, θ_N, eccentricity, and semimajor axis.

the two planets. The mean longitudes were considered to be linear with time.

INTEGRATIONS

Since the two averaging techniques require mean elements, the planetary equations were integrated over several centuries to get osculating elements for Pluto. These osculating elements were then averaged over the time span to obtain mean elements. The variations of the heliocentric elements were dominated by a 12-yr oscillation due to Jupiter. The period of Pluto also had a noticeable effect. It was decided to use the multiple of the Jovian periodicity which was nearest Pluto's orbital period as the averaging time. Thus the elements at yearly intervals were averaged over a 249-yr span forward and backward in time. Because Neptune may contribute significant 500-yr terms due to its commensurability with Pluto, the effects of Neptune were separately done over a ±500-yr span of time. The elements of Cohen, Hubbard, and Oesterwinter (1967) were used to start the integrations. For MJD 30000.0 the resulting heliocentric mean elements are referenced to the mean ecliptic and equinox of 1950.0:

$$a = 39.5380 \pm 8 \text{ a.u.}$$

$$e = 0.248975 \pm 7$$

$$i = 17^{\circ}\!.1431 \pm 1$$

$$\Omega = 109^{\circ}\!.6319 \pm 2$$

$$\omega = 113^{\circ}\!.7691 \pm 35$$

$$M = 289^{\circ}\!.1655 \pm 40$$

The errors are a combination of those given by Cohen, Hubbard, and Oesterwinter and an estimate of the errors resulting from the averaging of the osculating elements. This latter error was estimated by dividing the amplitude of the oscillations by the number of points (499). The error in the semimajor axis corresponds to a 4×10^{-4} arc sec/day error in the mean motion.

TABLE I. Inverse masses of the outer planets.

Planet	1/mass ($M_\odot$)
Jupiter	1047.35
Saturn	3501.6
Uranus	22869.0
Neptune	19314.0
Pluto	1812000.0

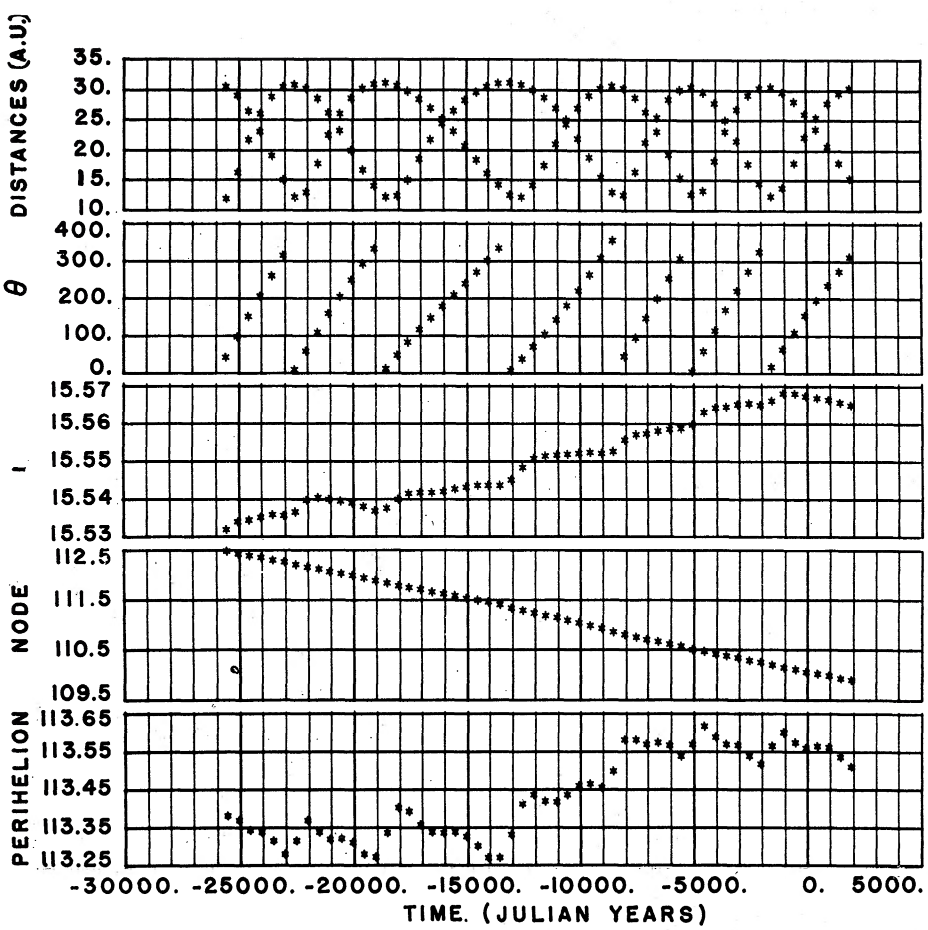

FIG. 2. Minimum distances of approach to Uranus, θ_U, inclination, node, and argument of perihelion.

The mean elements were referenced to the invariable plane and the orbit of Pluto was integrated forward to 2.4 million A.D. and backward to 2.1 million B.C. The Julian year was taken as the time unit. The planetary equations were integrated with a fourth-order, Runge–Kutta routine with a 500-yr integration step size. At each of the three times of a Runge–Kutta integration step the elements of the perturbing planets were evaluated. The eccentricities, inclinations, longitudes of perihelion, and nodes were calculated from Brouwer and van Woerkom's expressions, the semimajor axes were considered constants, and the linearized mean longitudes of Uranus and Neptune were calculated from Hill. Equations (5) and (7) were used to average the planetary equations for each of the four stages of a Runge–Kutta integration step. The averaging technique of Eq. (5) was used to calculate the pure secular perturbations from Jupiter and Saturn while Eq. (7), constrained by Eq. (8), was used to calculate the secular and commensurable perturbations from Uranus and Neptune. The outer integral of the partial derivatives of Eq. (5) and the integral of the partials of Eq. (7) were calculated by evaluating the integrand at 24 separate points, summing, and dividing by 24. The inner integrals of the partial derivatives of Eq. (5) were expressed in terms of complete elliptic integrals of the first and second kind (Plummer 1960) for each of the 24 points of the outer integral. The execution of the program required one minute of computer time per 100 000 yr on an IBM 360/91.

The integration step size was chosen by trying several values on 55 000-yr integrations. For step sizes around 500 yr the only significant difference, $\sim 0.1°$, was found for the mean anomaly. Presuming a propagation according to the $\frac{3}{2}$ power of the number of integration steps gives a 30° error after 2.4 million years. The uncertainty in the starting value of the mean motion causes a 100° uncertainty in the mean anomaly after an equal time. Errors, such as the above, have almost no effect on the libration amplitude of θ_N or the amplitudes of the other elements, the difference is made up by a very small shift in the libration period.

RESULTS

Figures 1 and 2 show the perturbations which take place on a time scale of a few thousand years. The first figure shows the minimum distances of approach between Neptune and Pluto, θ_N, e, and a. The minimum distances of approach are calculated for each 500-yr

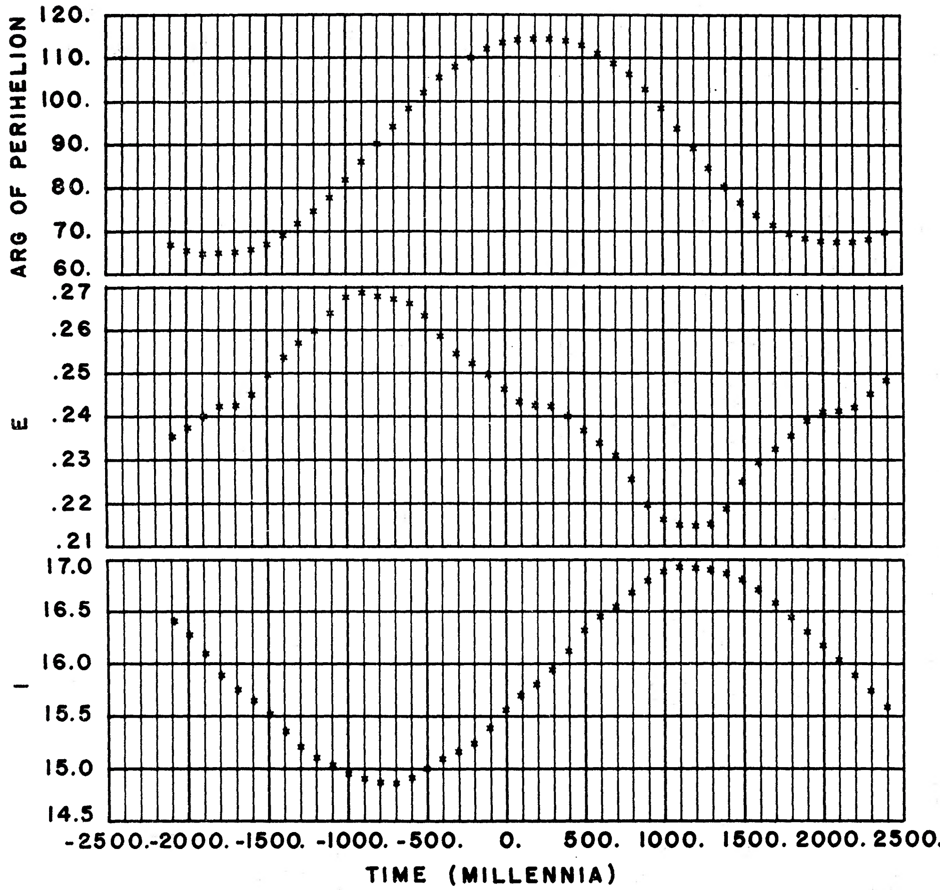

FIG. 3. Four-million-year periodicities in the argument of perihelion, eccentricity, and inclination.

average over the Neptune–Pluto commensurability. The smallest of the three curves is the encounter which takes place near the aphelion of Pluto, a result which holds true over the total span of the integration. The libration of θ_N and its resulting influence on the distances, e, and a are clearly visible. These effects were explained by Cohen and Hubbard. This libration, with an average period of 19 951 yr, was present throughout the full 4.5 million years.

The second figure shows the minimum distances of approach between Pluto and Uranus, θ_U, i, Ω, and ω. There are two minima in the distance between Pluto and Uranus for each 250-yr average over the near 1:3 commensurability. The phase of the commensurability, θ_U, is analagous to θ_N and is defined

$$\theta_U = 3\lambda - \lambda_U - 2\tilde{\omega}, \qquad (9)$$

where λ_U is the mean longitude of Uranus. Since θ_U circulates, the minimum distances occur close to $\theta_U = 0$, when Uranus and Pluto encounter one another near Pluto's perihelion. The average period of circulation is virtually the same as that of the 4230-yr near commensurability between Uranus and Neptune. It is strongly modulated by the libration of θ_N. The inclination and argument of perihelion show strong perturbations at each close encounter between Uranus and Pluto. Smaller perturbations are also visible in the semimajor axis and eccentricity. The periodicities seen in the first two figures are representative of the periodicities on the 1000–20 000-yr time scale throughout the full integration.

It was considered desirable to examine the behavior of Pluto's elements over the 4.5 million years without the distraction of the 20 000-yr periodicity. This was done by plotting the various parameters at times when $\theta_N = 180°$ and $d\theta_N/dt > 0$. The results are presented in Figs. 3, 4, and 5. The points are plotted for every fifth oscillation of θ_N. The semimajor axis showed no variation over these long time scales and was not plotted.

Figure 3 shows ω, e, and i. It can be seen that the argument of perihelion undergoes libration rather than circulation. Of the other natural bodies in the solar system only the asteroid (1373), Cincinnati (Kozai 1962; Marsden 1970) is known to show similar behavior. For Pluto, ω librates about 90° with an amplitude of about 24° and a period of $3.955 \pm 0.020 \times 10^6$ yr. There are large oscillations in e and i which are 90° out of phase with ω. The behavior of e, i, and ω is qualitatively similar to that described by Kozai (1962) for cases of librating ω under the influence of secular perturbations without commensurabilities. Some discussion of librating ω with commensurabilities is given by Hori and

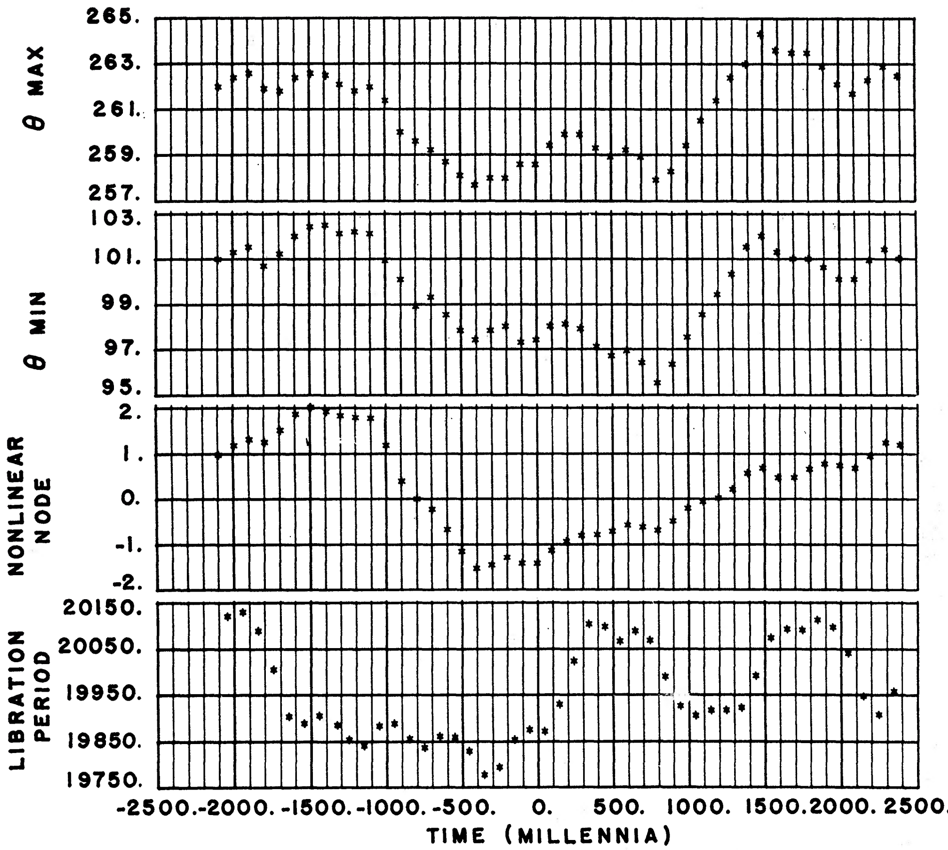

FIG. 4. Maximum and minimum θ_N, $\Omega-\bar{\Omega}$, and the period of libration.

Giacaglia (1967). For a given value of the semimajor axis, libration of ω would be expected if $(1-e^2)\cos^2 i$ is less than a critical value while circulation of ω is expected above this value. If Pluto were close to the boundary between libration and circulation then the amplitude of libration would be near 90°. Since the observed amplitude is only 24°, Pluto must lie well inside the libration region and the libration of ω should be quite stable. Since ω and θ_N both librate, $\theta_N' = \theta_N + \omega$ also librates. The interaction through the eccentricity apparently dominates the 2:3 commensurability but the inclination is also important.

Figure 4 shows the maximum and minimum values of θ_N during its libration, $\Omega-\bar{\Omega}$, and the libration period of θ_N. The maximum and minimum values of θ_N were smoothed since the changing value of θ_U can cause up to ±0.7° oscillations in these quantities. It is seen that θ_N only librates about 180° when $\omega = 90°$. The center of libration for θ_N gets up to 3.3° from 180° while the amplitude varies between 79.8° and 81.2°. Each one of the libration periods was calculated from five cycles of the libration. $\Omega-\bar{\Omega}$ is the node minus the linear term

$$\bar{\Omega} = 111.428° - 0.972091 \times 10^{-4}\ t°/\text{yr}, \tag{10}$$

where t is the Julian year. The linear term was chosen to make $\Omega-\bar{\Omega}$ zero at the two 90° crossings of ω. These crossings at −0.8026 and 1.1749 million years, were also used to calculate the libration period of ω.

The fact that ω cannot pass through 0° or 180° is significant in that it prevents Pluto's perihelion from getting close to the plane of the other planets. The result that the maximum eccentricity and the minimum inclination take place when $\omega = 90°$ rather than near an extremum of ω also helps keep Pluto's perihelion away from the other planets. Pluto's minimum radius at its nodes on the invariable plane was 33.56 a.u. at −1.55 million years. The minimum perihelion distance was 28.74 a.u. at −900 000 yr and the minimum approach to Uranus, 10.6 a.u., took place 100 000 yr earlier.

The closest approaches between Neptune and Pluto were 16.73 and 16.78 a.u. at 0.68 and 1.58 million years, respectively. These two minima can be seen in the first graph of Fig. 5 where Δ_N, the minimum distance of approach to Neptune during each 20 000-yr cycle, has been plotted. The minimum distance of approach during each libration of θ_N always takes place near the aphelion of Pluto at an extremum of θ_N. For $\omega > 90°$ this closest encounter to Neptune occurs when θ_N is at its maximum value, thus causing Pluto's encounter latitude to have its smallest negative value during the cycle. This effect can be seen in Fig. 1. For $\omega < 90°$ the closest encounter is at the minimum value of θ_N which is again the closest

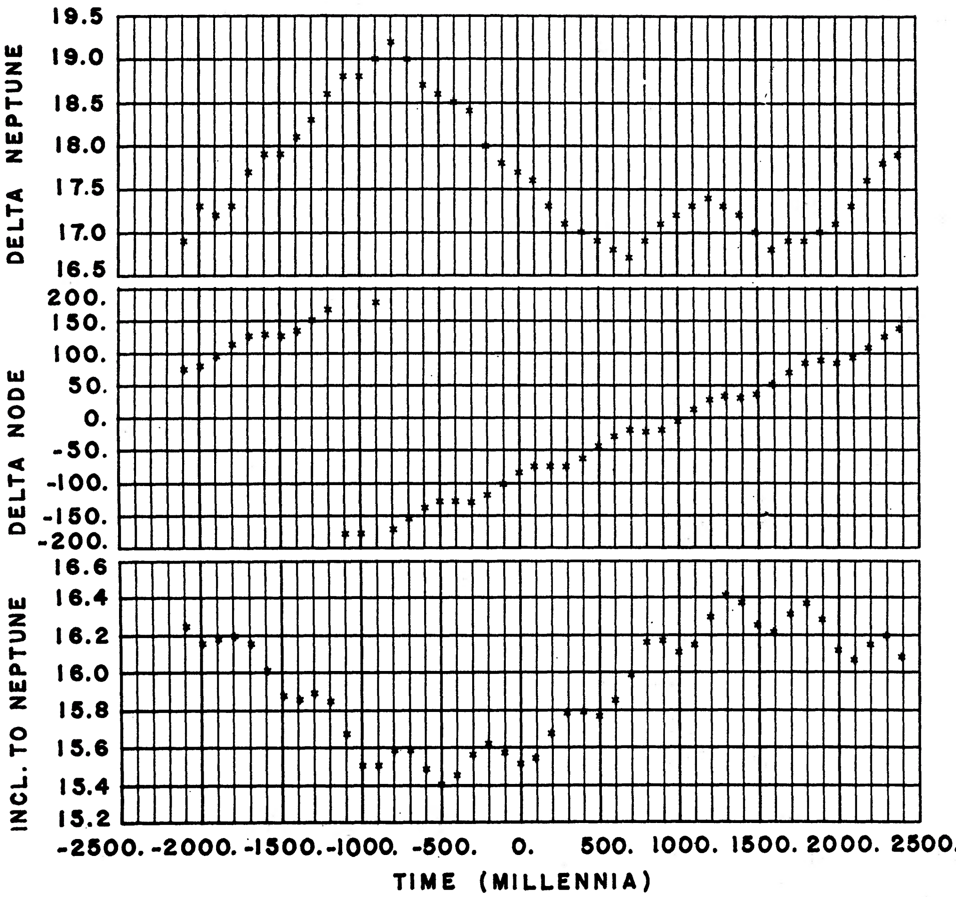

FIG. 5. Δ_N, $\Omega-\Omega_N$, and inclination to Neptune's orbit plane.

to the invariable plane. If the 4-million-year variation of ω is compared to that of θ_N it is seen that they have opposite signs. This means that the encounter at the θ_N extremum which is closest to the plane of the planets is also closer to the aphelion so that the small oscillations of the θ_N limits tend to increase Δ_N. The extremum which is closer to 180° has the closer encounter. Again this effect can be seen in Fig. 1 where the encounter distance is 17.7 a.u. when $\theta_N=259°$ and 19.5 a.u. when $\theta_N=97°$. The two minima of Δ_N bracket the minimum value of e. Since a minimum of e is a minimum of the aphelion and the encounter takes place at aphelion, this is expected. The small peak between the two minima results from ω passing through 90° and i passing through its maximum.

In addition to the 4-million-year periodicities associated with the libration of ω, most of the plots show oscillations with higher frequencies. These periodicities, called forced oscillations, arise from the finite eccentricities and inclinations of the orbits of the perturbing planets. It has long been known that the secular perturbations of the elements of the first eight planets of the solar system can be represented by a trigonometric series of the form

$$\begin{aligned}
e_j \sin \tilde{\omega}_j &= \sum_k M_{jk} \sin \nu_k, \\
e_j \cos \tilde{\omega}_j &= \sum_k M_{jk} \cos \nu_k, \\
\sin i_j \sin \Omega_j &= \sum_k N_{jk} \sin \nu_k', \\
\sin i_j \cos \Omega_j &= \sum_k N_{jk} \cos \nu_k',
\end{aligned} \tag{11}$$

where the subscript j refers to the jth planet and k refers to a frequency (Brouwer and Clemence 1961; Brouwer and van Woerkom 1950). The phases ν_k and ν_k' are linear functions of the time with rates $\dot{\nu}_k$ and $\dot{\nu}_k'$. There are as many independent frequencies as there are planets. The coefficients and frequencies for the eight inner planets are tabulated by Brouwer and van Woerkom and the frequencies are numbered here as they are in their paper but the signs of the frequencies are reversed. The series expressions are developed for objects with small eccentricities and inclinations, but Williams (1969) has shown that similar equations apply for an object of infinitessimal mass when its eccentricity and inclination are not small. In this latter case the amplitudes become periodic functions of the argument of perihelion and both $\dot{\nu}_k$ and $\dot{\nu}_k'$ frequencies

TABLE II. Frequencies and periods for the forced oscillations of Pluto.

k	$\frac{d\tilde{\omega}}{dt}-\frac{d\nu_k}{dt}$ (arc sec/yr)	Period (yr)	$\frac{d\Omega}{dt}-\frac{d\nu_k}{dt}$ (arc sec/yr)	Period (yr)
5	−4.64586	279 000		
6	−28.12402	46 000	25.38360	51 000
7	−3.06926	422 000	2.55271	508 000
8	−0.98327	1 318 000	0.32757	3 956 000

appear in each summation. Schubart's (1968) numerical integrations of the Hilda asteroids, whose mean motions exhibit a 3:2 commensurability with Jupiter, indicate that a similar characterization of the forced oscillations can be made for commensurable bodies.

As can be seen from Eq. (11), when one of the amplitudes is larger than the rest combined, then the corresponding phase may be identified with the average longitude of perihelion or average node for that planet. For example, the average rate of Neptune's node equals $\dot{\nu}_8'$ which is -0.67752 arc sec/yr. The two new frequencies which must be introduced for Pluto correspond to the average rates $d\tilde{\omega}/dt$ and $d\Omega/dt$ which both equal -0.34995 arc sec/yr. This value is from Eq. (10). The forced oscillations in e and $\tilde{\omega}$ should have frequencies

$$\frac{d\tilde{\omega}}{dt}-\frac{d\nu_k}{dt}$$

while the forced oscillations of i and Ω should have frequencies

$$\frac{d\Omega}{dt}-\frac{d\nu_k'}{dt}.$$

Table II shows the expected frequencies and periods for the $\dot{\nu}_k$ and $\dot{\nu}_k'$ which are important for the outer planets. Several of these forced oscillations are identifiable in Figs. 3, 4, and 5. The diagrams for the eccentricity and libration period show the 422 000- and 1 318 000-yr terms. The maxima of the 1.3-million-year term occur at 0.5 and 1.8 million A.D. in both diagrams. The flattened appearance of the extrema is due to the 422 000-yr term. The apparent lack of a maximum at 0.8 million B.C. may be due to the fact that Δ_N is largest here and $\dot{\nu}_8$ is the most important frequency for Neptune. A 508 000-yr term is weakly visible in the inclination plot. The 422 000-yr term seems to be an important effect for the node. If one characterizes the eccentricity and inclination oscillations by the amplitudes M_{jk} and N_{jk} then approximately $M_{97}=-0.001$, $M_{98}=0.003$, and $N_{97}=-0.0006$. These are of the same sign and approximately one-third the size of the corresponding terms for Neptune.

The frequency

$$\frac{d\Omega}{dt}-\frac{d\nu_8'}{dt}$$

presents a very special case. Its expected period of 3 956 000 yr almost exactly matches the 3 955 000-yr period of the oscillations associated with the argument of perihelion. In addition, 3 times the 1 318 000-yr period gives 3 954 000 yr. There should be a considerable amount of repetition in the values of the various parameters every 4 million years. It is the forced oscillations of ω, which are too small to be seen in Fig. 3, which result in the 20 000-yr uncertainty in the libration period for ω. The 4-million-year periods from the two forced oscillations have an uncertainty of about 2000 yr.

Resonances of the sort where

$$\frac{d\tilde{\omega}}{dt}-\frac{d\nu_k}{dt} \quad \text{or} \quad \frac{d\Omega}{dt}-\frac{d\nu_k'}{dt}$$

equals an integral multiple of the frequency associated with the ω oscillations have been described by Williams (1969) for secular perturbations. That work was done for nonlibrating ω but it should be extendable to the librating case by replacing the frequency of the secular perturbations, due to circulating ω, with the libration frequency. Then one gets that the argument $\tilde{\omega}-\nu_k$ or $\Omega-\nu_k'$ modulo 180°, when evaluated at the times when $\omega=90°$, will slowly oscillate about 0°. It is an assumption that the commensurability will not invalidate this result for Pluto. There seems to be a strong possibility that one or two of the arguments are resonating in this way for Pluto. For $\omega=90°$, $d\omega/dt>0$ one has $\tilde{\omega}-\nu_8$ modulo 180° = 171.947° and $\Omega-\nu_8'$ modulo 180° = 15.750° while for $\omega=90°$, $d\omega/dt<0$ the values are 171.844° and 15.683°, respectively. An integration exceeding 10 million years would be necessary to decide whether these angles are librating or slowly circulating. Such an integration is quite feasible.

There is a physical description for the resonances or near resonances

$$\frac{d\Omega}{dt}-\frac{d\nu_8'}{dt}=\frac{d\nu_F}{dt}$$

and

$$\frac{d\tilde{\omega}}{dt}-\frac{d\nu_8}{dt}=-3\frac{d\nu_F}{dt},$$

where $d\nu_F/dt$ is the libration frequency 0.32769 arc sec/yr. $\Omega-\nu_8'$ is the difference between the average nodes of Pluto and Neptune while $\tilde{\omega}-\nu_8$ is the difference between their average longitudes of perihelion. Neptune's perihelion rotates three times with respect to Pluto's and Neptune's node rotates once with respect to Pluto's for every 3 955 000 yr that it takes the argument of perihelion to librate once. The last two graphs of Fig. 5 show the difference between the real nodes for the two planets and the inclination between Pluto's orbit plane and Neptune's orbit plane. It can be seen that the two orbit planes rotate once with respect to one another in 4 million years. In Fig. 3 it can be seen that the inclination of Pluto varies by a total of 2.0° while in Fig. 5 the variation of the mutual inclination is 1.0° peak to peak or 0.8° if the 500 000-yr fluctuations are removed. The inclination of Neptune's orbit plane averages about 0.7° with oscillations of about 0.1°. If the argument $\Omega-\nu_8'$ is not really locked on to the phase of the oscillations of ω, the mutual inclination can eventually get as low as 14.0°. If the arguments are locked together, then the minimum of 15.4° from Fig. 5 should be near the absolute minimum. Since the closest approach of the two planets takes place when Pluto is far below Neptune's plane, a restriction on the variation of the mutual inclinations has the effect of increasing the minimum Δ_N. If the locking of the nodes holds up it will represent one more mechanism by which Pluto is held away from Neptune.

The two minima in the closest approach of Pluto to Neptune occur on either side of the $\omega=90°$, $d\omega/dt<0$ point. At this point $\tilde{\omega}-\nu_8=171.844°$ which means that Neptune's perihelion is nearly at the same longitude as Pluto's aphelion so that minimum Δ_N is increased somewhat. If $\tilde{\omega}-\nu_8$ is found to be locked on to the phase of the ω oscillation it will represent still another stabilizing influence between Neptune and Pluto. Actually, because the two minima are offset from the $\omega=90°$ point, the argument $\tilde{\omega}-\nu_8$ will have values of about 280° and 70° at the times of the minima. Thus it might be better to describe any possible locking of the value of $\tilde{\omega}-\nu_8$ to ω as an effect which prevents the aphelion of Neptune from becoming aligned with the aphelion of Pluto at the times of minimum Δ_N.

The three extrema of ω differed from 90° by 25.4°, 24.3°, and 22.7° in chronological order. This change in the amplitude may be due to the possible resonances of the node and longitude of perihelion with ω. Such resonances could give rise to periodic terms with periods in excess of 4 million years in all elements.

Because the time dependence of the elements of the perturbing planets was based on the calculations of Brouwer and van Woerkom, some discussion of the limitations of their results, due to the theory they used, is in order. The theory of the secular perturbations of the major planets used expansions in the planetary eccentricities and inclinations. Only the lowest-order terms from these expansions are kept in the solution Eq. (11) with the exception of a few higher-order terms for Jupiter and Saturn. This approximation should mainly degrade the results for those planets which can have the largest eccentricities and inclinations, i.e., Mercury and Mars which are not of interest here. So far as the outer planets are concerned, only small shifts in the frequencies $d\nu_k/dt$ and $d\nu_k'/dt$ would be expected. The theory also does not take into account terms arising from the near commensurabilities between the outer planets. This effect is difficult to assess but it is suspected that it would cause some change in the amplitudes M_{jk} and the initial values of the phases ν_k in Eq. (11). Finally, the solution for the secular perturbations of the planets did not include Pluto. This omission was justified on the basis of the small mass of Pluto and it was done because it appeared, since the Neptune-Pluto commensurability had not been discovered, that the distance between Neptune and Pluto could go to zero which would have prevented a solution. Properly including the effects of Pluto requires that the theory include both higher-order terms and commensurabilities. The effect would be to alter the frequencies, amplitudes, and phases of the known terms and to add an additional term to the series in Eq. (11). The most strongly affected terms of the series would be those which dominate the expressions for Uranus and Neptune, numbers 7 and 8 in both the eccentricity and inclination series. Since the eighth eccentricity and inclination frequency are both suspected of being involved in locked resonances with Pluto, the already uncertain condition of these two resonances becomes more uncertain. An improvement in the solar system model would be very desirable before extending the Pluto integration to longer times. It is very unlikely that an improvement in the model would effect either the θ_N libration or the ω libration beyond slight changes in the amplitudes and periods. Changes in the model would mainly show up as changes in Pluto's forced oscillations.

COMPARISON WITH OTHER WORK

The results of this work compare favorably with that of Cohen and Hubbard (1965) and Cohen, Hubbard, and Oesterwinter (1967, 1971). There is a 400-yr difference in the libration period of θ_N over the 300 000-yr interval but it seems traceable to the different values used for the mass of Pluto and the differences in the integration. They used a mass which is 5.4% of Neptune's while recent work (Duncombe, Klepczynski, and Seidelmann 1968) gives 1.1%. The integration by Cohen *et al.* included perturbations of Pluto on Neptune while this work did not. Since the frequency of libration involves terms approximately proportional to the square root of the masses, the difference should be in the vicinity of 2.7%. The actual difference was 2.0%. Presumably then inclusion of the perturbations of Pluto on Neptune in this work would have shortened the libration period of θ_N by about 80 yr. This influence of Pluto's mass on Neptune and the period of libration may explain the difficulties encountered by Cohen and Peters (1970) who also ignored perturbations by Pluto.

Since Hori and Giacaglia (1967) concluded that ω should circulate rather than librate, reasons for the discrepancy should be sought. Investigation shows that an erroneous value for Neptune's mass was used in their calculations. It was one of the assumptions of their paper that perturbations by planets other than Neptune could be ignored. This omission of the other planets should have almost no effect on the character of the

20 000-yr oscillations which are driven by Neptune alone, but it will have a profound influence on all longer-period effects. As one moves inward from Neptune the distance of the perturbing planets get larger but so do their masses. Thus the influence of each of the planets from Jupiter to Neptune is important. In fact, short integrations with less than all four perturbers indicate that Neptune tries to make ω regress while the other three planets try to make ω progress. This results in a near cancellation which is sometimes positive and sometimes negative. Hori and Giacaglia point out that, to the extent that the eccentricities and inclinations of the perturbing planets can be ignored, the quantity

$$H=(1-e^2)^{\frac{1}{2}}\cos i \tag{12}$$

should be conserved. In the above equation, e and i are understood to be freed from the 20 000-yr oscillations. A check on this quantity during the 4-million-year libration indicates that H is a poorly conserved quantity. This unexpected result is almost certainly due to the resonance, or near resonance, of the arguments $\Omega-\nu_8'$ and $\tilde{\omega}-\nu_8$ which arise from the finite eccentricities and inclinations of the perturbing planets. This latter effect was virtually unpredictable and indicates that the restricted problem, where these eccentricities and inclinations are ignored, is an insufficient approximation for the longest-period effects.

SUMMARY AND IMPLICATION

The 20 000-yr libration of θ_N is confirmed over 4.5 million years. This libration appears to be completely stable. The small variations in the mean value of θ_N which were observed are of such a nature as to increase the minimum distance of approach to Neptune and presumably increase the stability of the system. Taking this stability over the length of the integration as an indication of stability over the age of the solar system makes it seem unlikely that this libration started more recently than the stabilization of the planetary masses at their present values shortly after the formation of the solar system.

In addition to the libration of θ_N there are three more peculiarities of the Neptune–Pluto system. The argument of perihelion librates about 90° with a period of 3 955 000 yr and the angles Ω–Ω_N and $\tilde{\omega}$–$\tilde{\omega}_N$ are resonant or nearly resonant with the libration of ω. A longer integration would be needed to decide conclusively whether these last two arguments are resonant. All four of the resonances have the effect of increasing the minimum distance of approach between Pluto and Neptune and thus should be a stabilizing influence on the system.

As to why Pluto is in the θ_N resonance it seems that three possibilities should be considered. (i) The planet may have been maneuvered into the commensurability by the forces acting during the period immediately following the formation of the solar system. These forces would include the gravitational effect of the changing masses of the planets and the collision of gas and solid matter with proto-Pluto. (ii) The commensurability may have favored aggregation of the planet. (iii) There may have been more than one original object. Those objects which could come close to Neptune would have collided with Neptune or would have been ejected from the solar system by close encounters. The multiple bodies could have been of planet size or smaller. If they were small bodies, aggregation would have occured after removal of objects in unstable orbits and this would be a special case of the second possibility. The Hilda and Trojan asteroids are in commensurabilities which prevent close approaches to Jupiter, while similar noncommensurate orbits which would approach Jupiter closely are unoccupied. It is tempting to speculate that these asteroids represent an example of the third case without aggregation. This third case has the advantage that it might also explain why the other three stabilizing resonances occur. Of course all three possibilities may have operated in conjunction. That the θ_N libration might be a chance phenomena is rejected because of the uncanny affinity of all of the outer planets for commensurabilities. The evidence seems to indicate that commensurable motion is important somehow in planetary formation.

Improved stability through increased minimum distance of approach seems to be the unifying theme of the peculiarities of the Neptune–Pluto system. One cannot help but wonder whether this remarkable pair holds any further surprises.

ACKNOWLEDGMENTS

We wish to express our thanks to the Department of Planetary and Space Science of the University of California at Los Angeles and particularly to W. M. Kaula for the support given for this project.

A portion of this work was one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

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